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# RELATIVE GORENSTEIN INJECTIVE COVERS WITH RESPECT TO A SEMIDUALIZING MODULE

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*Abstract.* Let $R$ be a commutative Noetherian ring and let $C$ be a semidualizing $R$-module. We prove a result about the covering properties of the class of relative Gorenstein injective modules with respect to $C$ which is a generalization of Theorem 1 by Enochs and Iacob (2015). Specifically, we prove that if for every $G_C$-injective module $G$, the character module $G^+$ is $G_C$-flat, then the class $\mathcal{GI}_C(R)\cap\mathcal{A}_C(R)$ is closed under direct sums and direct limits. Also, it is proved that under the above hypotheses the class $\mathcal{GI}_C(R)\cap\mathcal{A}_C(R)$ is covering.

*Keywords*: semidualizing module; $G_C$-flat module; $G_C$-injective module; cover; envelope

*MSC 2010*: 13D05, 13D45, 18G20

## 1. Introduction

Throughout this paper $R$ is a commutative Noetherian ring and all modules are unital. The notion of a "semidualizing module" is one of the most central notions in the relative homological algebra. This notion was first introduced by Foxby in [9]. Then Vasconcelos in [23] and Golod in [10] rediscovered these modules using different terminology for different purposes. This notion has been investigated by many authors from different points of view; see for example [2], [3], [13] and [22]. Among various research areas on semidualizing modules, one basically focuses on extending the "absolute" classical notion of homological algebra to the "relative" setting with respect to a semidualizing module.

The notions of Gorenstein projective and Gorenstein injective modules have semidualizing counterpart. In [13], Holm and Jørgensen introduced the classes of $G_C$-projective and $G_C$-injective modules where $C$ is a semidualizing $R$-module. Note that when $C = R$, we recover the classes of Gorenstein projective and Gorenstein injective modules. Some properties of $G_C$-projective and $G_C$-injective modules are investigated in [13] and [24]. Recently, the existence of the Gorenstein injective, Gorenstein projective and Gorenstein flat covers and envelopes has been studied intensively in e.g. [7] and [14].

In [4], Enochs and Holm are concerned with covering and enveloping properties of $\mathcal{GP}(R)$, $\mathcal{GI}(R)$ and $\mathcal{GF}(R)$. Indeed, they proved that if $(R, \mathfrak{m}, k)$ is a Cohen-Macaulay local ring admitting a dualizing module, then the class of finite Gorenstein projective/flat dimension is covering and preenveloping, and the class of $R$-modules of finite Gorenstein injective dimension is preenveloping. In [18], the authors investigated the relative setting of this result with respect to a semidualizing $R$-module $C$. In particular, they showed that every module in $\mathcal{B}_C(R)$ with finite $G_C$-projective dimension admits a $G_C$-projective precover. Furthermore, in [14], Theorem 3.2, covering and enveloping properties are investigated for two important classes of $R$-modules, namely the Auslander class $\mathcal{A}_C(R)$ and the Bass class $\mathcal{B}_C(R)$. Indeed, it is proved that both the classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ are covering and preenveloping. Also Enochs and Iacob in [5], Theorem 1, proved that if the character modules of Gorenstein injective modules are Gorenstein flat, then the class of Gorenstein injective modules is closed under direct limits and is covering.

In this paper, we are concerned with covering properties of the class $\mathcal{GI}_C(R)$. Specifically, in Theorem 3.7 it is shown that if for every $G_C$-injective module $G$, the character module $G^+$ is $G_C$-flat, then for every $X \in \mathcal{A}_C(R)$ such that $X^+$ is $G_C$-flat, we have that $X$ is $G_C$-injective. Note that when $R$ has a dualizing complex the character module assumption is satisfied. Then, as the main result of this paper, in Theorem 3.8 it is proved that if for every $G_C$-injective module $G$, the character module $G^+$ is $G_C$-flat, then the class $\mathcal{GI}_C(R) \cap \mathcal{A}_C(R)$ is closed under direct sums and direct limits. This is a direct generalization of the aforementioned result of Enochs and Iacob. Also, it is proved that under the above hypotheses the class $\mathcal{GI}_C(R) \cap \mathcal{A}_C(R)$ is covering. Note that the class $\mathcal{GI}_C(R) \cap \mathcal{A}_C(R)$ is studied extensively in [20].

## 2. Preliminaries

Throughout this paper $\mathcal{M}(R)$ denotes the category of $R$-modules. We use the term "subcategory" to mean a "full, additive subcategory $\mathcal{X} \subseteq \mathcal{M}(R)$ such that, for all $R$-modules $M$ and $N$, if $M \cong N$ and $M \in \mathcal{X}$, then $N \in \mathcal{X}$". Write $\mathcal{P}(R)$ and $\mathcal{I}(R)$ for the subcategories of all projective and injective $R$-modules, respectively.

An $R$-complex is a sequence

$$X = \dots \xrightarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \xrightarrow{\partial_{n-1}^X} \dots$$

of $R$-modules and $R$-homomorphisms such that $\partial_{n-1}^X \partial_n^X = 0$ for each integer $n$.

**Definition 2.1.** Let $\mathcal{X}$ be a subcategory of $\mathcal{M}(R)$ and let $M$ be an $R$-module. A homomorphism $\varphi\colon X \to M$ with $X \in \mathcal{X}$ is an *$\mathcal{X}$-precover* of $M$ if for every homomorphism $\psi\colon Y \to M$ with $Y \in \mathcal{X}$ there exists a homomorphism $f\colon Y \to X$ such that $\varphi f = \psi$. If every $R$-module admits an $\mathcal{X}$-precover, then we say that the class $\mathcal{X}$ is *precovering*.

An $\mathcal{X}$-cover of $M$ is an $\mathcal{X}$-precover $\varphi\colon X \to M$ with the additional property that any endomorphism $f\colon X \to X$ with $\varphi = \varphi f$ must be an automorphism. If every $R$-module admits an $\mathcal{X}$-cover, then we say that the class $\mathcal{X}$ is covering.

Preenvelopes and envelopes are defined dually, see [6], Chapters 5 and 6, for further details.

**Definition 2.2.** Let $\mathcal{X}$ be a class of $R$-modules and let $M$ be an $R$-module. An *$\mathcal{X}$-resolution* of $M$ is a complex of $R$-modules in $\mathcal{X}$ of the form

$$X = \ldots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \longrightarrow 0$$

such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for $n \geqslant 1$. The *$\mathcal{X}$-projective dimension* of $M$ is the quantity

$$\mathcal{X}\text{-pd}_R(M) = \inf\{\sup\{n\colon\ X_n \neq 0\}\colon\ X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

In particular, $\mathcal{X}\text{-pd}_R(0) = -\infty$. The modules of $\mathcal{X}$-projective dimension zero are the nonzero modules in $\mathcal{X}$.

Dually, an *$\mathcal{X}$-coresolution* of $M$ is a complex of $R$-modules in $\mathcal{X}$ of the form

$$X = 0 \longrightarrow X_0 \xrightarrow{\partial_0^X} X_{-1} \xrightarrow{\partial_{-1}^X} \ldots$$

such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for $n \leqslant -1$. The *$\mathcal{X}$-injective dimension* of $M$ is the quantity

$$\mathcal{X}\text{-id}_R(M) = \inf\{\sup\{n\colon\ X_n \neq 0\}\colon\ X \text{ is an } \mathcal{X}\text{-coresolution of } M\}.$$

In particular, $\mathcal{X}\text{-id}_R(0) = -\infty$. The modules of $\mathcal{X}$-injective dimension zero are the nonzero modules in $\mathcal{X}$.

When $\mathcal{X}$ is the class of projective $R$-modules we write $\mathrm{pd}_R(M)$ for the associated homological dimension and call it the projective dimension of $M$. Similarly, the injective dimension of $M$ is denoted $\mathrm{id}_R(M)$.

The notion of semidualizing modules, defined next, goes back at least to Foxby [9], but was rediscovered by others.

**Definition 2.3.** A finitely generated $R$-module $C$ is called *semidualizing* if the natural homothety homomorphism $\chi_C^R\colon R \to \operatorname{Hom}_R(C,C)$ is an isomorphism and $\operatorname{Ext}_R^{\geqslant 1}(C,C)=0$. An $R$-module $D$ is called *dualizing* if it is semidualizing and has finite injective dimension.

For a semidualizing $R$-module $C$, we set

$$\begin{aligned}
\mathcal{P}_C(R) &= \{P\otimes_R C\colon\ P \text{ is a projective } R\text{-module}\},\\
\mathcal{F}_C(R) &= \{F\otimes_R C\colon\ F \text{ is a flat } R\text{-module}\},\\
\mathcal{I}_C(R) &= \{\operatorname{Hom}_R(C,I)\colon\ I \text{ is an injective } R\text{-module}\}.
\end{aligned}$$

The $R$-modules in $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$ and $\mathcal{I}_C(R)$ are called *$C$-projective*, *$C$-flat* and *$C$-injective*, respectively.

The next two classes were also introduced by Foxby in [9].

**Definition 2.4.** Let $C$ be a semidualizing $R$-module. The *Auslander class* with respect to $C$ is the class $\mathcal{A}_C(R)$ of $R$-modules $M$ such that:

(i) $\operatorname{Tor}_i^R(C,M)=0=\operatorname{Ext}_R^i(C,C\otimes_R M)$ for all $i\geqslant 1$, and
(ii) the natural map $\gamma_C^M\colon\ M\to \operatorname{Hom}_R(C,C\otimes_R M)$ is an isomorphism.

The *Bass class* with respect to $C$ is the class $\mathcal{B}_C(R)$ of $R$-modules $M$ such that:

(i) $\operatorname{Ext}_R^i(C,M)=0=\operatorname{Tor}_i^R(C,\operatorname{Hom}_R(C,M))$ for all $i\geqslant 1$, and
(ii) the natural evaluation map $\xi_M^C\colon\ C\otimes_R \operatorname{Hom}_R(C,M)\to M$ is an isomorphism.

The next definition is due to Holm and Jørgensen [13].

**Definition 2.5.** Let $C$ be a semidualizing $R$-module. A *complete $\mathcal{I}_C\mathcal{I}$-resolution* is a complex $Y$ of $R$-modules satisfying the following conditions:

(i) $Y$ is exact and $\operatorname{Hom}_R(I,Y)$ is exact for each $I\in\mathcal{I}_C(R)$, and
(ii) $Y_i\in\mathcal{I}_C(R)$ for all $i\geqslant 0$ and $Y_i$ is injective for all $i<0$.

An $R$-module $M$ is *$G_C$-injective* if there exists a complete $\mathcal{I}_C\mathcal{I}$-resolution $Y$ such that $M\cong\operatorname{Coker}(\partial_1^Y)$; in this case $Y$ is a *complete $\mathcal{I}_C\mathcal{I}$-resolution* of $M$. The class of all $G_C$-injective $R$-modules is denoted by $\mathcal{GI}_C(R)$. In the case $C=R$, we use the more common terminology "complete injective resolution" and "Gorenstein injective module" and the notation $\mathcal{GI}(R)$.

A *complete $\mathcal{PP}_C$-resolution* is a complex $X$ of $R$-modules such that:

(i) $X$ is exact and $\operatorname{Hom}_R(X,P)$ is exact for each $P\in\mathcal{P}_C(R)$, and
(ii) $X_i$ is projective for all $i\geqslant 0$ and $X_i\in\mathcal{P}_C(R)$ for all $i<0$.

An $R$-module $M$ is *$G_C$-projective* if there exists a complete $\mathcal{PP}_C$-resolution $X$ such that $M\cong\operatorname{Coker}(\partial_1^X)$; in this case $X$ is a *complete $\mathcal{PP}_C$-resolution* of $M$. The class of all $G_C$-projective $R$-modules is denoted by $\mathcal{GP}_C(R)$. In the case $C = R$, we use the more common terminology "complete projective resolution" and "Gorenstein projective module" and the notation $\mathcal{GP}(R)$.

A *complete $\mathcal{FF}_C$-resolution* is a complex $Z$ of $R$-modules such that:

(i) $Z$ is exact and $Z \otimes_R I$ is exact for each $I \in \mathcal{I}_C(R)$, and
(ii) $Z_i$ is flat for all $i \geqslant 0$ and $Z_i \in \mathcal{F}_C(R)$ for all $i < 0$.

An $R$-module $M$ is *$G_C$-flat* if there exists a complete $\mathcal{FF}_C$-resolution $Z$ such that $M \cong \mathrm{Coker}(\partial_1^Z)$; in this case $Z$ is a *complete $\mathcal{FF}_C$-resolution* of $M$. The class of all $G_C$-flat $R$-modules is denoted by $\mathcal{GF}_C(R)$. In the case $C = R$, we use the more common terminology "complete flat resolution" and "Gorenstein flat module" and the notation $\mathcal{GF}(R)$.

**Remark 2.6.** We recall the notion of trivial extension of the ring $R$ by an $R$-module. If $M$ is an $R$-module, then the direct sum $R \oplus M$ can be equipped with the product:

$$(a, m)(a', m') = (aa', am' + a'm),$$

where $a, a' \in R$ and $m, m' \in M$. This turns $R \oplus M$ into a ring which is called the trivial extension of $R$ by $M$ and denoted $R \ltimes M$. This construction goes back to Nagata, see [16], who called it the idealization of $M$. There are canonical ring homomorphisms $R \rightleftarrows R \ltimes M$, which enable us to view $R$-modules as $(R \ltimes M)$-modules and vice versa. Reiten and Foxby in [17] and [9] proved that $R \ltimes M$ is Gorenstein if and only if $R$ is Cohen-Macaulay and $M$ is a dualizing module.

In [13], it is shown that the relative Gorenstein dimensions with respect to a semidualizing module have the following nice properties.

**Fact 2.7.** Let $C$ be a semidualizing $R$-module. The following statements hold for every $R$-module $M$:

(i) $\mathcal{GI}_C\text{-id}_R(M) = \mathrm{Gid}_{R\ltimes C}(M)$.
(ii) $\mathcal{GP}_C\text{-pd}_R(M) = \mathrm{Gpd}_{R\ltimes C}(M)$.
(iii) $\mathcal{GF}_C\text{-pd}_R(M) = \mathrm{Gfd}_{R\ltimes C}(M)$.

**Definition 2.8.** Let $\mathcal{X}$ be a class of $R$-modules.

(1) We say that $\mathcal{X}$ is *closed under extensions*, if for every short exact sequence of $R$-modules $0 \to A \to B \to C \to 0$, the condition $A, C \in \mathcal{X}$ implies that $B \in \mathcal{X}$.
(2) We say that $\mathcal{X}$ is *projectively resolving*, if it contains all projective $R$-modules, and for every short exact sequence of $R$-modules $0 \to A \to B \to C \to 0$ with $C \in \mathcal{X}$, the conditions $A \in \mathcal{X}$ and $B \in \mathcal{X}$ are equivalent. This definition is from Auslander and Bridger, see [1].

(3) We say that $\mathcal{X}$ is *injectively resolving*, if it contains all injective $R$-modules, and for every short exact sequence of $R$-modules $0 \to A \to B \to C \to 0$ with $A \in \mathcal{X}$, the conditions $B \in \mathcal{X}$ and $C \in \mathcal{X}$ are equivalent.

Recall that given an $R$-module $M$, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by $M^+$.

The following definition is from [14].

**Definition 2.9.** Let $\mathcal{X}$ and $\mathcal{Y}$ be two clases of $R$-modules. We say that a pair $(\mathcal{X}, \mathcal{Y})$ is a *duality pair* if the following conditions hold:

(i) For every $R$-module $M$, one has $M \in \mathcal{X}$ if and only if $M^+ \in \mathcal{Y}$.

(ii) The class $\mathcal{Y}$ is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{X}, \mathcal{Y})$ is called coproduct-closed if the class $\mathcal{X}$ is closed under coproducts.

## 3. Relative Gorenstein injective covers and envelopes

In [4], Corollary 3.13, Enochs and Holm are concerned with covering and enveloping properties of $\mathcal{GP}(R)$, $\mathcal{GI}(R)$ and $\mathcal{GF}(R)$. Indeed, they proved that if $(R, \mathfrak{m}, k)$ is a Cohen-Macaulay local ring admitting a dualizing module, then the class of finite Gorenstein projective/flat dimension is covering and preenveloping, and the class of $R$-modules of finite Gorenstein injective dimension is preenveloping. In [18], Theorem 3.3, the authors investigated the relative setting of this result with respect to a semidualizing $R$-module $C$. In particular, they showed that every module in $\mathcal{B}_C(R)$ with finite $G_C$-projective dimension admits an $G_C$-projective precover. Note that the dual of this result is also correct, as follows.

**Theorem 3.1** ([18], dual of Theorem 3.3). *Let $M \in \mathcal{A}_C(R)$ be such that $\mathcal{GI}_C\text{-}\mathrm{id}_R(M) = n$, where $n$ is an integer. Then $M$ admits an injective $G_C$-injective preenvelop $\varphi\colon M \to G$, where $K = \mathrm{Coker}(\varphi)$ satisfies $\mathcal{I}_C\text{-}\mathrm{id}_R(K) = n - 1$.*

The main goal of this paper is Theorem 3.8, in which we investigate the covering properties of the class of $\mathcal{GI}_C(R)$. First, we prove the following results which will be used later.

**Proposition 3.2.** *Let $C$ be a semidualizing $R$-module. Then the class $\mathcal{GF}_C(R)$ is closed under extensions and is projectively resolving.*

Proof. The assertion follows from [12], Theorem 3.7, and Fact 2.7 (iii). □

**Lemma 3.3.** *Let $C$ be a semidualizing $R$-module. Then the class $\mathcal{GI}_C(R)$ is injectively resolving.*

Proof. This result follows easily from [12], Theorem 2.6, because of Fact 2.7 (i). $\square$

**Proposition 3.4.** *Let $C$ be a semidualizing $R$-module and let $N \in \mathcal{A}_C(R)$ be such that $\mathcal{GI}_C\text{-}\mathrm{id}_R(N) < \infty$. Then there exists a short exact sequence $0 \to B \to H \to N \to 0$ of $R$-modules, where $B$ is $G_C$-injective and $\mathcal{I}_C\text{-}\mathrm{id}_R(H) = \mathcal{GI}_C\text{-}\mathrm{id}_R(N)$.*

Proof. The assertion follows from results in [11] in the same way as [21], Corollary 5.10 (c). $\square$

The following lemma is in [21], Lemma 2.4 (b).

**Lemma 3.5.** *Let $C$ be a semidualizing $R$-module and let $M$ be an $R$-module. Then $\mathcal{I}_C\text{-}\mathrm{id}_R(M) = \mathcal{F}_C\text{-}\mathrm{pd}_R(M^+)$.*

Recall that for the class of $R$-modules $\mathcal{X}$, we will denote the right orthogonal class of $\mathcal{X}$ by $\mathcal{X}^{\perp}$, which is the class of all $R$-modules $M$ such that $\mathrm{Ext}^1_R(X, M) = 0$ for every $X \in \mathcal{X}$. The left orthogonal class of $\mathcal{X}$, denoted ${}^{\perp}\mathcal{X}$, is the class of all $R$-modules $N$ such that $\mathrm{Ext}^1_R(N, X) = 0$ for every $X \in \mathcal{X}$.

**Lemma 3.6.** *Let $C$ be a semidualizing $R$-module and let $M$ be an $R$-module such that $M \in \mathcal{GF}_C(R) \cap \mathcal{GF}_C(R)^{\perp}$. Then $M$ is $C$-flat.*

Proof. This result follows directly from parts (e) and (g) of [21], Corollary 5.10. $\square$

The following theorem is a generalization of [5], Lemma 1, with a similar proof.

**Theorem 3.7.** *Let $C$ be a semidualizing $R$-module and assume that for every $G_C$-injective module $G$, the character module $G^+$ is $G_C$-flat. If $X \in \mathcal{A}_C(R)$ is such that $X^+$ is $G_C$-flat, then $X$ is $G_C$-injective.*

Proof. Let $X \in \mathcal{A}_C(R)$ be such that $X^+ \in \mathcal{GF}_C(R)$. By [15], 7.12, there exists an exact sequence $(*)$: $0 \to X \to G \to L \to 0$ with $G \in \mathcal{GI}_C(R)$ and $L \in {}^{\perp}\mathcal{GI}_C(R)$, since $R$ is Noetherian. Then the exact sequence $(*)$ provides the exact sequence $0 \to L^+ \to G^+ \to X^+ \to 0$ with $X^+, G^+ \in \mathcal{GF}_C(R)$. Furthermore, Proposition 3.2 implies that $L^+ \in \mathcal{GF}_C(R)$.

We claim that $L^+$ is $C$-flat. To see this, let $B \in \mathcal{GF}_C(R)$. Then $B^+ \in \mathcal{GI}_C(R)$ by the proof of [13], Proposition 2.15, and therefore $0 = \mathrm{Ext}^1_R(L, B^+) \cong (\mathrm{Tor}^R_1(L, B))^+$. This means that $\mathrm{Tor}^R_1(L, B) = 0$. We also have that

$$\mathrm{Ext}^1_R(B, L^+) \cong (\mathrm{Tor}^R_1(L, B))^+ = 0.$$

Thus $L^+ \in \mathcal{GF}_C(R) \cap \mathcal{GF}_C(R)^{\perp}$, which implies that $L^+$ is $C$-flat by Lemma 3.6. So, $L$ is $C$-injective by Lemma 3.5. Thus, in the short exact sequence $(*)$, $G \in \mathcal{GI}_C(R)$ and $L \in \mathcal{I}_C(R)$. Consequently, $\mathcal{GI}_C\text{-id}_R(X) < \infty$. By Proposition 3.4, there exists an exact sequence $0 \to B \to H \to X \to 0$, with $B \in \mathcal{GI}_C(R)$ and $\mathcal{I}_C\text{-id}_R(H) = \mathcal{GI}_C\text{-id}_R(X) < \infty$. Lemma 3.5 implies that $\mathcal{F}_C\text{-pd}_R(H^+) < \infty$. In the exact sequence $0 \to X^+ \to H^+ \to B^+ \to 0$ the modules $X^+, B^+$ belong to $\mathcal{GF}_C(R)$ by our assumption, and hence $H^+ \in \mathcal{GF}_C(R)$ as well. Since a $G_C$-flat module with finite $C$-flat dimension is $C$-flat, the module $H^+$ is $C$-flat. Hence $H$ is $C$-injective by Lemma 3.5, and since $\mathcal{GI}_C\text{-id}_R(X) = \mathcal{I}_C\text{-id}_R(H)$, we conclude that $X$ is $G_C$-injective. $\square$

The following theorem is a generalization of [5], Theorem 1, with a similar proof.

**Theorem 3.8.** *Let $C$ be a semidualizing $R$-module and suppose that for every $G_C$-injective module $G$, the character module $G^+$ is $G_C$-flat. Then the class $\mathcal{GI}_C(R) \cap \mathcal{A}_C(R)$ is covering and closed under direct sums and direct limits.*

Proof. By Theorem 3.7 and [19], Proposition 3.3.1, $M \in \mathcal{GI}_C(R) \cap \mathcal{A}_C(R)$ if and only if $M^+ \in \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$. By [13], Lemma 5.10, and [19], Proposition 3.1.6, the class $\mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$ is closed under finite direct sums and direct summands. Therefore, $(\mathcal{GI}_C(R) \cap \mathcal{A}_C(R), \mathcal{GF}_C(R) \cap \mathcal{B}_C(R))$ is a duality pair and it follows from [14], Theorem 3.1, that $\mathcal{GI}_C(R) \cap \mathcal{A}_C(R)$ is closed under pure submodules and pure quotients. By the proof of [13], Theorem 5.6, $\mathcal{GI}_C(R)$ is closed under products and by [8], Theorem 2.5, $\mathcal{A}_C(R)$ is closed under products too. Now [5], Lemma 2, implies that $\mathcal{GI}_C(R) \cap \mathcal{A}_C(R)$ is closed under direct sums and direct limits. Hence the duality pair $(\mathcal{GI}_C(R) \cap \mathcal{A}_C(R), \mathcal{GF}_C(R) \cap \mathcal{B}_C(R))$ is coproduct-closed. Therefore, [14], Theorem 3.1, yields that $\mathcal{GI}_C(R) \cap \mathcal{A}_C(R)$ is covering. $\square$

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