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ON THE REGULARITY OF THE ONE-SIDED HARDY-LITTLEWOOD MAXIMAL FUNCTIONS

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Abstract. In this paper we study the regularity properties of the one-dimensional one-sided Hardy-Littlewood maximal operators $\mathcal{M}^+$ and $\mathcal{M}^-$. More precisely, we prove that $\mathcal{M}^+$ and $\mathcal{M}^-$ map $W^{1,p}(\mathbb{R}) \to W^{1,p}(\mathbb{R})$ with $1 < p < \infty$, boundedly and continuously. In addition, we show that the discrete versions M^+ and M^- map $\mathrm{BV}(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$ boundedly and map $l^1(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$ continuously. Specially, we obtain the sharp variation inequalities of M^+ and M^-, that is,

$$\mathrm{Var}(M^+(f)) \leqslant \mathrm{Var}(f) \quad \text{and} \quad \mathrm{Var}(M^-(f)) \leqslant \mathrm{Var}(f)$$

if $f \in \mathrm{BV}(\mathbb{Z})$, where $\mathrm{Var}(f)$ is the total variation of f on $\mathbb{Z}$ and $\mathrm{BV}(\mathbb{Z})$ is the set of all functions $f\colon \mathbb{Z} \to \mathbb{R}$ satisfying $\mathrm{Var}(f) < \infty$.

Keywords: one-sided maximal operator; Sobolev space; bounded variation; continuity

MSC 2010: 42B25, 46E35

1. Introduction

Over the last years there has been considerable effort in understanding the behavior of differentiability under a maximal operator. The first work in this direction is due to Kinnunen [9] who showed that the centered Hardy-Littlewood maximal operator defined by

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \,\mathrm{d}y$$

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is bounded on the Sobolev spaces $W^{1,p}(\mathbb{R}^d)$ for all $p > 1$, where $d \geqslant 1$ and $B(x,r)$ is the ball in $\mathbb{R}^d$ centered at x with radius r and $|B(x,r)|$ denotes the volume of $B(x,r)$. Recall that the Sobolev spaces $W^{1,p}(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, are defined by

$$W^{1,p}(\mathbb{R}^d) := \{f\colon \mathbb{R}^d \to \mathbb{R}\colon \|f\|_{1,p} = \|f\|_{L^p(\mathbb{R}^d)} + \|\nabla(f)\|_{L^p(\mathbb{R}^d)} < \infty\},$$

where $\nabla(f)$ is the weak gradient of f. Subsequently, Kinnunen and Lindqvist in [10] gave a local version of the original boundedness on $W^{1,p}(\Omega)$, where Ω is an open set in $\mathbb{R}^d$. This paradigm that an L^p-bound implies a $W^{1,p}$-bound was later extended to a fractional version in [11], to a bilinear version in [5] and to a multisublinear version in [14]. Later on, the continuity of $\mathcal{M}\colon W^{1,p} \to W^{1,p}$ for $p > 1$ was established by Luiro in [15] and in [16] for its local version (continuity is not immediate from boundedness because of the lack of linearity).

The regularity at the endpoint case $p = 1$ seems to be a deeper issue. In this regard, one of the main questions was posed by Hajłasz and Onninen in [7], Question 1: is the operator $f \mapsto |\nabla(\mathcal{M}(f))|$ bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$? In 2002, Tanaka [19] first gave the affirmative answer to this question for the one-dimensional non-centered Hardy-Littlewood maximal function defined by

$$\widetilde{\mathcal{M}}(f)(x) = \sup_{s,t>0} \frac{1}{s+t} \int_{x-s}^{x+t} |f(y)|\,\mathrm{d}y.$$

Precisely, Tanaka showed that if $f \in W^{1,1}(\mathbb{R})$, then $\widetilde{\mathcal{M}}(f)$ has a weak derivative in $L^1(\mathbb{R})$ and

$$\|(\widetilde{\mathcal{M}}(f))'\|_{L^1(\mathbb{R})} \leqslant 2\|f'\|_{L^1(\mathbb{R})},$$

where f' is the distributional derivative of f. This result was later refined by Aldaz and Pérez-Lázaro in [1] who obtained, under the assumption that f is of bounded variation on $\mathbb{R}$, $\widetilde{\mathcal{M}}(f)$ is absolutely continuous and

$$\operatorname{Var}(\widetilde{\mathcal{M}}(f)) \leqslant \operatorname{Var}(f),$$

where $\operatorname{Var}(f)$ denotes the total variation of f. This implies that

$$\|(\widetilde{\mathcal{M}}(f))'\|_{L^1(\mathbb{R})} \leqslant \|f'\|_{L^1(\mathbb{R})}, \tag{1.1}$$

provided $f \in W^{1,1}(\mathbb{R})$. A simple proof of (1.1) was given by Liu et al. in [13] under the condition that $f \in W^{1,1}(\mathbb{R})$. More recently, in the remarkable work [12], Kurka showed that if f is of bounded variation on $\mathbb{R}$, then

$$\operatorname{Var}(\mathcal{M}(f)) \leqslant C \operatorname{Var}(f)$$

for a certain $C > 1$.

In this paper we focus on the action of one-sided Hardy-Littlewood maximal operator acting on $W^{1,p}(\mathbb{R})$ functions. For a locally integrable function f on $\mathbb{R}$, the one-sided Hardy-Littlewood maximal functions are defined as

$$\mathcal{M}^+(f)(x) = \sup_{s>0} \frac{1}{s} \int_x^{x+s} |f(y)|\,\mathrm{d}y \quad \text{and} \quad \mathcal{M}^-(f)(x) = \sup_{t>0} \frac{1}{t} \int_{x-t}^{x} |f(y)|\,\mathrm{d}y.$$

One can easily check that

$$\widetilde{\mathcal{M}}(f)(x) = \max\{\mathcal{M}^+(f)(x), \mathcal{M}^-(f)(x)\}, \tag{1.2}$$

$$\mathcal{M}^+(f)(x) = \mathcal{R}\mathcal{M}^-(\mathcal{R}f)(x) \tag{1.3}$$

for $x \in \mathbb{R}$, where $\mathcal{R}$ denotes the reflection operator, that is, $\mathcal{R}f(x) = f(-x)$ for any $x \in \mathbb{R}$.

The study of the operator $\mathcal{M}^+$ started in the 1930s (see [8]). During the same years the basic results about the ergodic maximal operator were obtained. The ergodic maximal operator is defined by

$$\mathcal{M}_\tau(f)(x) = \sup_{h>0} \frac{1}{h} \int_0^h |f(\tau^t x)|\,\mathrm{d}t$$

for all measurable functions $f\colon X \to \mathbb{R}$, where $(X, \mathcal{F}, \mu)$ is a measure space and $\{\tau^t\colon t \in \mathbb{R}\}$ is a flow of measure-preserving transformations on X. Note that $\mathcal{M}^+$ is a particular case of the ergodic maximal operator when (X, μ) is $\mathbb{R}$ with the Lebesgue measure and $\tau^t(x) = x + t$. It follows from (1.2) that both $\mathcal{M}^+$ and $\mathcal{M}^-$ are of weak type $(1,1)$ and of type (p,p) for $p > 1$ (also see [17] for the weighted boundedness). By transference arguments, the boundedness for the general operator $\mathcal{M}_\tau$ can be obtained by using the corresponding results for the particular case $\mathcal{M}^+$ (see [18] for a recent exposition in the discrete case).

The investigation of the regularity of $\mathcal{M}^+$ and $\mathcal{M}^-$ began with Tanaka, see [19], who proved that if $f \in W^{1,1}(\mathbb{R})$, then the distributional derivatives of $\mathcal{M}^+(f)$ and $\mathcal{M}^-(f)$ are integrable functions, and

$$\|(\mathcal{M}^+(f))'\|_{L^1(\mathbb{R})} \leqslant \|f'\|_{L^1(\mathbb{R})}, \quad \|(\mathcal{M}^-(f))'\|_{L^1(\mathbb{R})} \leqslant \|f'\|_{L^1(\mathbb{R})}.$$

It is observed that $\mathcal{M}^+(f)$ and $\mathcal{M}^-(f)$ are also absolutely continuous on $\mathbb{R}$, which follows from a combination of arguments in [13] and [19]. Based on the above, it is natural to ask

Question A. Are the one-sided Hardy-Littlewood maximal operators $\mathcal{M}^+$ and $\mathcal{M}^-$ bounded and continuous from $W^{1,p}(\mathbb{R})$ to $W^{1,p}(\mathbb{R})$ for $p > 1$?

We will give some affirmative answers to the above question by the following

Theorem 1. *Let $1 < p < \infty$. Then both $\mathcal{M}^+$ and $\mathcal{M}^-$ map $W^{1,p}(\mathbb{R}) \to W^{1,p}(\mathbb{R})$ boundedly. Furthermore, if $f \in W^{1,p}(\mathbb{R})$, then*

$$|(\mathcal{M}^+(f))'(x)| \leqslant \mathcal{M}^+(f')(x), \quad |(\mathcal{M}^-(f))'(x)| \leqslant \mathcal{M}^-(f')(x)$$

for almost every $x \in \mathbb{R}$.

Theorem 2. *Let $1 < p < \infty$. Then both $\mathcal{M}^+$ and $\mathcal{M}^-$ map $W^{1,p}(\mathbb{R}) \to W^{1,p}(\mathbb{R})$ continuously.*

Remark 1. We remark that the one-sided maximal operators $\mathcal{M}^+$ and $\mathcal{M}^-$ map $W^{1,\infty}(\mathbb{R})$ into $W^{1,\infty}(\mathbb{R})$ boundedly, which follows from arguments similar to those in [9], Remark (iii).

On the other hand, the investigation of the regularity of maximal operators in discrete setting has attracted the attention of many authors (see [2], [4], [20] et al.). Recall that the total variation of f is the $\ell^1(\mathbb{Z})$-norm of the difference of f, i.e.

$$\text{Var}(f) = \|f'\|_{\ell^1(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |f(n+1) - f(n)|. \tag{1.4}$$

We denote by $\text{BV}(\mathbb{Z})$ the set of all functions $f\colon \mathbb{Z} \to \mathbb{R}$ satisfying $\text{Var}(f) < \infty$. We also write

$$\text{Var}(f; [a, b]) = \sum_{n=a}^{b-1} |f(n+1) - f(n)|$$

for the variation of f on the interval $[a, b] \subset \mathbb{Z}$. In 2012, Bober et al. in [2] initially studied the regularity of the discrete version of $\widetilde{\mathcal{M}}$ defined by

$$\widetilde{M}(f)(n) = \sup_{r,s\in\mathbb{N}} \frac{1}{r+s+1} \sum_{k=-r}^{s} |f(n+k)|,$$

and proved that if $f \in \text{BV}(\mathbb{Z})$, then

$$\text{Var}(\widetilde{M}(f)) \leqslant \text{Var}(f).$$

Here, $\mathbb{N} = \{0, 1, 2, \ldots\}$. Recently, Temur in [20] extended Bober et al's result to the centered version of $\widetilde{M}$ denoted by M. For general dimension $d \geqslant 1$, Carneiro and Hughes [4] established the endpoint regularity of the discrete Hardy-Littlewood maximal operator.

The second aim of this paper is to investigate the endpoint regularity of the discrete one-sided Hardy-Littlewood maximal operators

$$M^+(f)(n) = \sup_{N\in\mathbb{N}} \frac{1}{N+1}\sum_{i=0}^{N}|f(n+i)| \quad \text{and} \quad M^-(f)(n) = \sup_{N\in\mathbb{N}} \frac{1}{N+1}\sum_{i=0}^{N}|f(n-i)|$$

for $n \in \mathbb{Z}$. The operator M^+ arose first in Dunford and Schwartz's work [6] and was studied by Calderón in [3], who proved that M^+ is of weak type $(1,1)$ and of type (p,p) for $1 < p < \infty$. From this and the fact that $M^-(f) = \mathcal{R}M^+(\mathcal{R}f)$, one can conclude that M^- is of weak type $(1,1)$ and of type (p,p) for $1 < p < \infty$. In light of the aforementioned facts concerning the endpoint regularity of the discrete maximal functions, a natural question is the following

Question B. Are the operators M^+ and M^- bounded and continuous from $\ell^1(\mathbb{Z})$ to $\mathrm{BV}(\mathbb{Z})$?

This question will be addressed by the next results.

Theorem 3. *Both M^+ and M^- map $\mathrm{BV}(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$ boundedly. Moreover, if $f \in \mathrm{BV}(\mathbb{Z})$, then*

$$\mathrm{Var}(M^+(f)) \leqslant \mathrm{Var}(f) \quad \textit{and} \quad \mathrm{Var}(M^-(f)) \leqslant \mathrm{Var}(f).$$

Theorem 4. *Both M^+ and M^- map $\ell^1(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$ continuously.*

Remark 2. We remark that our method applies to other maximal operators as well. In particular, employing the method in the proof of Theorem 4, one can obtain that both $\widetilde{M}$ and M map $\ell^1(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$ continuously.

The rest of paper is organized as follows. In Section 2 we present the proof of Theorems 1 and 2. The proofs of Theorems 3 and 4 will be given in Section 3. We would like to remark that the main ideas employed in this paper follow from [2], [4], [9], [15], but some new methods and techniques are necessary. Especially, the proof of [4], Theorem 2, is highly dependent on two discrete versions of Luiro's lemma (see Lemmas 3 and 4 in [4]), but similar lemmas are unnecessary in the proof of Theorem 4. Moreover, our method is very simple.

2. Proofs of Theorems 1 and 2

This section is devoted to the proofs of Theorems 1 and 2. Let us begin with the proof of Theorem 1.

Proof of Theorem 1. We only prove Theorem 1 for the operator $\mathcal{M}^+$ since the other case is analogous. One can easily check that $\mathcal{M}^+$ is a sub-linear operator which commutes with translations and is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. From this and Theorem 1 in [7] we obtain that $\mathcal{M}^+$ maps $W^{1,p}(\mathbb{R}) \to W^{1,p}(\mathbb{R})$ boundedly with $1 < p < \infty$. Let $\{s_k\}_{k \geqslant 1}$ be an enumeration of positive rational numbers. We can write

$$\mathcal{M}^+(f)(x) = \sup_{k \geqslant 1} \frac{1}{s_k} \int_x^{x+s_k} |f(y)| \, \mathrm{d}y.$$

Define the family of operators $\{T_k\}_{k \geqslant 1}$ by

$$T_k(f)(x) = \max_{1 \leqslant i \leqslant k} \frac{1}{s_i} \int_x^{x+s_i} |f(y)| \, \mathrm{d}y.$$

Obviously, $T_k(f)$ converges to $\mathcal{M}^+(f)$ pointwise. On the other hand, one can easily check that

$$|(T_k(f))'(x)| \leqslant \mathcal{M}^+(f')(x) \tag{2.1}$$

for almost every $x \in \mathbb{R}$. Combining this with the boundedness of $\mathcal{M}^+$ implies that $\{T_k(f)\}$ is an increasing sequence of functions in $W^{1,p}(\mathbb{R})$, and

$$\|T_k(f)\|_{1,p} \leqslant \|\mathcal{M}^+(f)\|_{L^p(\mathbb{R})} + \|\mathcal{M}^+(f')\|_{L^p(\mathbb{R})} \leqslant C_p \|f\|_{1,p}.$$

The weak compactness of Sobolev implies $\mathcal{M}^+(f) \in W^{1,p}(\mathbb{R})$, $T_k(f)$ converges to $\mathcal{M}^+(f)$ in $L^p(\mathbb{R})$ and $(T_k(f))'$ converges to $(\mathcal{M}^+(f))'$ weakly in $L^p(\mathbb{R})$, which together with (2.1) leads to

$$|(\mathcal{M}^+(f))'(x)| \leqslant \mathcal{M}^+(f')(x)$$

for almost every $x \in \mathbb{R}$. This proves Theorem 1. $\square$

Before presenting the proof of Theorem 2, we shall give some notation and lemmas. If $A \subset \mathbb{R}$ and $r \in \mathbb{R}$, we define

$$d(r, A) := \inf_{a \in A} |r - a| \quad \text{and} \quad A_{(\lambda)} := \{x \in \mathbb{R} \colon d(x, A) \leqslant \lambda\} \quad \text{for } \lambda \geqslant 0.$$

Denote by $\|f\|_{p,A}$ the L^p-norm of $f\chi_A$ for all measurable sets $A \subset \mathbb{R}$. Fix $f \in L^p(\mathbb{R})$ with $1 \leqslant p < \infty$ and $x \in \mathbb{R}$, define the sets $\mathcal{A}^+(f)(x)$ and $\mathcal{A}^-(f)(x)$ by

$$\mathcal{A}^+(f)(x) := \Big\{r \geqslant 0\colon\ \mathcal{M}^+(f)(x) = \limsup_{k\to\infty} \frac{1}{r_k}\int_x^{x+r_k} |f(y)|\,\mathrm{d}y \ \text{ for } r_k > 0,\ r_k \to r\Big\}$$

and

$$\mathcal{A}^-(f)(x) := \Big\{r \geqslant 0\colon\ \mathcal{M}^-(f)(x) = \limsup_{k\to\infty} \frac{1}{t_k}\int_{x-t_k}^{x} |f(y)|\,\mathrm{d}y \ \text{ for } t_k > 0,\ t_k \to r\Big\}.$$

We also define $u_{x,f}\colon [0,\infty) \mapsto \mathbb{R}$ by

$$u_{x,f}(0) = |f(x)| \quad \text{and} \quad u_{x,f}(r) = \frac{1}{r}\int_x^{x+r} |f(y)|\,\mathrm{d}y \quad \text{for } r \in (0,\infty).$$

We notice that the following facts are valid: (i) $u_{x,f}$ are continuous on $(0,\infty)$ for all $x \in \mathbb{R}$ and at $r = 0$ for almost every $x \in \mathbb{R}$; (ii) $\lim\limits_{r\to\infty} u_{x,f}(r) = 0$ since $u_{x,f}(r) \leqslant \|f\|_{L^p(\mathbb{R})} r^{-1/p}$; (iii) the set $\mathcal{A}(f)(x)$ is nonempty and closed for any $x \in \mathbb{R}$; (iv) almost every point is a Lebesgue point. Thus we have

$$\begin{gathered}\mathcal{M}^+(f)(x) = u_{x,f}(r) \quad \text{if } \ 0 < r \in \mathcal{A}(f)(x), \quad x \in \mathbb{R},\\ \mathcal{M}^+(f)(x) = |f(x)| \quad \text{for almost every } x \in \mathbb{R} \text{ such that } 0 \in \mathcal{A}(f)(x).\end{gathered}$$

We refer now to [15] for the ideas of the proofs for the next lemmata.

Lemma 1 ([15], Lemma 2.2)**.** *Let $1 \leqslant p < \infty$. Suppose $f_j \to f$ in $L^p(\mathbb{R})$ when $j \to \infty$. Then for all $R > 0$ and $\lambda > 0$ we have*

$$\begin{aligned}\lim_{j\to\infty} |\{x \in (-R,R)\colon\ \mathcal{A}^+(f_j)(x) \nsubseteq \mathcal{A}^+(f)_{(\lambda)}\}| = 0,\\ \lim_{j\to\infty} |\{x \in (-R,R)\colon\ \mathcal{A}^-(f_j)(x) \nsubseteq \mathcal{A}^-(f)_{(\lambda)}\}| = 0.\end{aligned}$$

The Hausdorff distance between two sets A and B is defined as

$$\pi(A,B) := \inf\{\delta > 0\colon\ A \subset B_{(\delta)} \text{ and } B \subset A_{(\delta)}\}.$$

By Lemma 1 and an argument similar to that in the proof of [15], Corollary 2.3, we have

Lemma 2. *Let $1 < p < \infty$ and $f \in L^p(\mathbb{R})$. Then for all $\lambda > 0$ and $R > 0$, we have*

$$\begin{aligned}\lim_{h\to 0} |\{x \in (-R,R)\colon\ \pi(\mathcal{A}^+(f)(x),\ \mathcal{A}^+(f)(x+h)) > \lambda\}| = 0,\\ \lim_{h\to 0} |\{x \in (-R,R)\colon\ \pi(\mathcal{A}^-(f)(x),\ \mathcal{A}^-(f)(x+h)) > \lambda\}| = 0.\end{aligned}$$

Below we present two formulas for the derivatives of the one-sided maximal operators $\mathcal{M}^+$ and $\mathcal{M}^-$, which will play key roles in the proof of Theorem 2.

Lemma 3. *Let $f \in W^{1,p}(\mathbb{R})$ with $1 < p < \infty$. Then for almost all $x \in \mathbb{R}$, we have*

$$(\mathcal{M}^+(f))'(x) = \frac{1}{r}\int_x^{x+r} |f|'(y)\,\mathrm{d}y, \quad 0 < r \in \mathcal{A}^+(f)(x),$$
$$(\mathcal{M}^+(f))'(x) = |f|'(x) \quad \text{if } 0 \in \mathcal{A}^+(f)(x);$$
$$(\mathcal{M}^-(f))'(x) = \frac{1}{r}\int_{x-r}^{x} |f|'(y)\,\mathrm{d}y, \quad 0 < r \in \mathcal{A}^-(f)(x),$$
$$(\mathcal{M}^-(f))'(x) = |f|'(x) \quad \text{if } 0 \in \mathcal{A}^-(f)(x).$$

Proof. We only prove Lemma 3 for the operator $\mathcal{M}^+$ since the other case is analogous. Without loss of generality we may assume that $f \geqslant 0$, since $|f| \in W^{1,p}(\mathbb{R})$ if $f \in W^{1,p}(\mathbb{R})$ with $1 < p < \infty$. It follows from Theorem 1 that $\mathcal{M}^+(f) \in W^{1,p}(\mathbb{R})$. By Lemma 2 we can choose a sequence $\{s_k\}_{k=1}^{\infty}$, $s_k > 0$ such that $\lim\limits_{k\to\infty} s_k = 0$ and $\lim\limits_{k\to\infty} \pi(\mathcal{A}^+(f)(x), \mathcal{A}^+(f)(x+s_k)) = 0$ for almost every $x \in (-R, R)$. Let

$$f_{s_k}(x) = \frac{f_{\tau(s_k)}(x) - f(x)}{s_k} \quad \text{with } f_{\tau(s_k)}(x) = f(x + s_k).$$

Then we have

$$\|f_{\tau(s_k)} - f\|_{L^p(\mathbb{R})} \to 0 \quad \text{as } k \to \infty,$$
$$\|f_{s_k} - f'\|_{L^p(\mathbb{R})} \to 0 \quad \text{as } k \to \infty,$$
$$\|\mathcal{M}^+(f_{s_k} - f')\|_{L^p(\mathbb{R})} \to 0 \quad \text{as } k \to \infty,$$
$$\|(\mathcal{M}^+(f))_{s_k} - (\mathcal{M}^+(f))'\|_{L^p(\mathbb{R})} \to 0 \quad \text{as } k \to \infty.$$

Furthermore, there exists a subsequence $\{h_k\}_{k=1}^{\infty}$ of $\{s_k\}_{k=1}^{\infty}$ and a measurable set $A_1 \subset (-R, R)$ satisfying $|(-R, R) \setminus A_1| = 0$ such that

(i) $f_{\tau(h_k)}(x) \to f(x)$, $f_{h_k}(x) \to f'(x)$, $\mathcal{M}^+(f_{h_k} - f')(x) \to 0$ and $(\mathcal{M}^+(f))_{h_k}(x) \to (\mathcal{M}^+(f))'(x)$ when $k \to \infty$ for any $x \in A_1$;

(ii) $\lim\limits_{k\to\infty} \pi(\mathcal{A}^+(f)(x), \mathcal{A}^+(f)(x + h_k)) = 0$ for any $x \in A_1$.

Let

$$A_2 := \bigcap_{k=1}^{\infty} \{x \in \mathbb{R} \colon \mathcal{M}^+(f)(x + h_k) = f(x + h_k) \text{ if } 0 \in \mathcal{A}^+(f)(x + h_k)\},$$
$$A_3 := \bigcap_{k=1}^{\infty} \{x \in \mathbb{R} \colon \mathcal{M}^+(f)(x + h_k) \geqslant f(x + h_k)\},$$
$$A_4 := \{x \in \mathbb{R} \colon \mathcal{M}^+(f)(x) = f(x) \text{ if } 0 \in \mathcal{A}^+(f)(x)\}.$$

Note that $|(-R,R)\setminus A_i| = 0$ for $i = 2,3,4$. Let $x \in A_1 \cap A_2 \cap A_3 \cap A_4$ be a Lebesgue point of f'. For any fixed $r \in \mathcal{A}^+(f)(x)$, there exist radii $r_k \in \mathcal{A}^+(f)(x+h_k)$ such that $\lim_{k\to\infty} r_k = r$. We consider the following two cases:

Case A: $r > 0$. We may assume that $r_k > 0$ for all k.

$$
\begin{aligned}
(\mathcal{M}^+(f))'(x) &= \lim_{k\to\infty} \frac{1}{h_k}(\mathcal{M}^+(f)(x+h_k) - \mathcal{M}^+(f)(x)) \\
&\leqslant \lim_{k\to\infty} \frac{1}{h_k}\left(\frac{1}{r_k}\int_{x+h_k}^{x+h_k+r_k} f(y)\,\mathrm{d}y - \frac{1}{r_k}\int_x^{x+r_k} f(y)\,\mathrm{d}y\right) \\
&= \lim_{k\to\infty} \frac{1}{r_k}\int_x^{x+r_k} \frac{f(y+h_k)-f(y)}{h_k}\,\mathrm{d}y \\
&= \frac{1}{r}\int_x^{x+r} f'(y)\,\mathrm{d}y.
\end{aligned} \tag{2.2}
$$

The last equation holds, because $f_{h_k}\chi_{(x,x+r_k)} \to f'\chi_{(x,x+r)}$ in $L^1(\mathbb{R})$ as $k \to \infty$. On the other hand,

$$
\begin{aligned}
(\mathcal{M}^+(f))'(x) &= \lim_{k\to\infty} \frac{1}{h_k}(\mathcal{M}^+(f)(x+h_k) - \mathcal{M}^+(f)(x)) \\
&\geqslant \lim_{k\to\infty} \frac{1}{h_k}\left(\frac{1}{r}\int_{x+h_k}^{x+h_k+r} f(y)\,\mathrm{d}y - \frac{1}{r}\int_x^{x+r} f(y)\,\mathrm{d}y\right) \\
&= \lim_{k\to\infty} \frac{1}{r}\int_x^{x+r} \frac{f(y+h_k)-f(y)}{h_k}\,\mathrm{d}y \\
&= \frac{1}{r}\int_x^{x+r} f'(y)\,\mathrm{d}y.
\end{aligned} \tag{2.3}
$$

Combining (2.2) with (2.3) yields

$$
(\mathcal{M}^+(f))'(x) = \frac{1}{r}\int_x^{x+r} f'(y)\,\mathrm{d}y \quad \text{whenever } 0 < r \in \mathcal{A}^+(f)(x).
$$

Case B: $r = 0$. First we estimate the lower bound of $(\mathcal{M}^+(f))'(x)$. We can write

$$
\begin{aligned}
(\mathcal{M}^+(f))'(x) &= \lim_{k\to\infty} \frac{1}{h_k}(\mathcal{M}^+(f)(x+h_k) - \mathcal{M}^+(f)(x)) \\
&\geqslant \lim_{k\to\infty} \frac{1}{h_k}(f(x+h_k) - f(x)) = f'(x).
\end{aligned} \tag{2.4}
$$

Below we estimate the upper bound of $(\mathcal{M}^+(f))'(x)$. If we have $r_k = 0$ for infinitely many k, we can obtain that

$$
\begin{aligned}
(\mathcal{M}^+(f))'(x) &= \lim_{k\to\infty} \frac{1}{h_k}(\mathcal{M}^+(f)(x+h_k) - \mathcal{M}^+(f)(x)) \\
&= \lim_{k\to\infty} \frac{1}{h_k}(f(x+h_k) - f(x)) = f'(x).
\end{aligned} \tag{2.5}
$$

If there exists $k_0 \in \mathbb{N} \setminus \{0\}$ such that $r_k > 0$ when $k \geqslant k_0$, then

$$\begin{aligned}
(\mathcal{M}^+(f))'(x) &= \lim_{k\to\infty} \frac{1}{h_k}(\mathcal{M}^+(f)(x+h_k) - \mathcal{M}^+(f)(x)) \\
&\leqslant \lim_{k\to\infty} \frac{1}{h_k}\biggl(\frac{1}{r_k}\int_{x+h_k}^{x+h_k+r_k} f(y)\,\mathrm{d}y - \frac{1}{r_k}\int_{x}^{x+r_k} f(y)\,\mathrm{d}y\biggr) \\
&= \lim_{k\to\infty} \frac{1}{r_k}\int_{x}^{x+r_k} \frac{f(y+h_k) - f(y)}{h_k}\,\mathrm{d}y \\
&\leqslant \lim_{k\to\infty} \mathcal{M}^+(f_{h_k} - f')(x) + \lim_{k\to\infty} \frac{1}{r_k}\int_{x}^{x+r_k} f'(y)\,\mathrm{d}y \\
&\leqslant f'(x),
\end{aligned}$$

which together (2.4) with (2.5) implies that

$$(\mathcal{M}^+(f))'(x) = f'(x) \quad \text{whenever } r = 0 \in \mathcal{A}^+(f)(x).$$

Now we have shown the claim in the interval $(-R, R)$. Since R was arbitrary, this completes the proof of Lemma 3. □

Now we are in the position of proving Theorem 2.

P r o o f of Theorem 2. We only prove Theorem 2 for $\mathcal{M}^+$ by employing the idea in [15], since the other case is analogous. Let $f_j \to f$ in $W^{1,p}(\mathbb{R})$ when $j \to \infty$. We shall prove $\|\mathcal{M}^+(f_j) - \mathcal{M}^+(f)\|_{1,p} \to 0$ when $j \to \infty$. Since $\|\mathcal{M}^+(f_j) - \mathcal{M}^+(f)\|_{L^p(\mathbb{R})} \to 0$ when $j \to \infty$ because of the sublinearity of $\mathcal{M}^+$, it suffices to prove that $\|(\mathcal{M}^+(f_j))' - (\mathcal{M}^+(f))'\|_{L^p(\mathbb{R})} \to 0$ when $j \to \infty$. We may assume that the functions f_j and f satisfy $f_j \geqslant 0$ and $f \geqslant 0$. For any fixed $\varepsilon > 0$, there exists $j_0 \in \mathbb{N} \setminus \{0\}$ such that $\|f_j' - f'\|_{L^p(\mathbb{R})} < \varepsilon$ for any $j \geqslant j_0$. Let us choose $R > 0$ such that $\|\mathcal{M}^+(f')\|_{p,B_1} < \varepsilon$ with $B_1 = (-\infty, -R) \cup (R, \infty)$. By the absolute continuity, there exists $\eta > 0$ such that $\|\mathcal{M}^+(f')\|_{p,B} < \varepsilon$ for any measurable subset B of $(-R, R)$ satisfying $|B| < \eta$. As already observed, for almost every $x \in \mathbb{R}$, the function $u_{x,f'}$ is uniformly continuous on $[0, \infty)$ and we can find $\delta(x) > 0$ such that

$$|u_{x,f'}(r_1) - u_{x,f'}(r_2)| < R^{-1/p}\varepsilon \quad \text{if } |r_1 - r_2| < \delta(x).$$

We can write $(-R, R)$ as

$$(-R, R) = \biggl(\bigcup_{k=1}^{\infty} \Bigl\{x \in (-R, R)\colon\ \delta(x) > \frac{1}{k}\Bigr\}\biggr) \cup \mathcal{N},$$

where $\mathcal{N}$ is a zero set. From this we can choose $\delta > 0$ such that

$$\begin{aligned}
|\{x \in (-R, R)\colon\ &|u_{x,f'}(r_1) - u_{x,f'}(r_2)| \geqslant R^{-1/p}\varepsilon \\
&\text{for some } r_1, r_2, |r_1 - r_2| < \delta\}| =: |B_2| < \frac{\eta}{2}.
\end{aligned}$$

By Lemma 1 there exists $j_1 \in \mathbb{N} \setminus \{0\}$ such that

$$|\{x \in (-R, R)\colon\ \mathcal{A}^+(f_j)(x) \not\subseteq \mathcal{A}^+(f)(x)_{(\delta)}\}| =: |B^j| < \frac{\eta}{2} \quad \text{if } j \geqslant j_1.$$

Invoking Lemma 3 we have for almost every $x \in \mathbb{R}$ and fixed $j \geqslant j_0$,

$$\begin{aligned} |(\mathcal{M}^+(f_j))'(x) - (\mathcal{M}^+(f))'(x)| &= |u_{x,f_j'}(r_1) - u_{x,f'}(r_2)| \\ &\leqslant |u_{x,f_j'}(r_1) - u_{x,f'}(r_1)| + |u_{x,f'}(r_1) - u_{x,f'}(r_2)| \\ &\leqslant \mathcal{M}^+(f_j' - f')(x) + |u_{x,f'}(r_1) - u_{x,f'}(r_2)| \end{aligned}$$

for any $r_1 \in \mathcal{A}^+(f_j)(x)$ and $r_2 \in \mathcal{A}^+(f)(x)$. If $x \notin B_1 \cup B_2 \cup B^j$, we can choose $r_1 \in \mathcal{A}^+(f_j)(x)$ and $r_2 \in \mathcal{A}^+(f)(x)$ such that $|r_1 - r_2| < \delta$ and

$$|u_{x,f'}(r_1) - u_{x,f'}(r_2)| < R^{-1/p}\varepsilon.$$

On the other hand, for any $r_1 \in \mathcal{A}^+(f_j)(x)$ and $r_2 \in \mathcal{A}^+(f)(x)$, we have

$$|u_{x,f'}(r_1) - u_{x,f'}(r_2)| \leqslant 2\mathcal{M}^+(f')(x).$$

Note that $|B_2 \cup B^j| < \eta$ for all $j \geqslant j_1$. Thus we have

$$\begin{aligned} \|(\mathcal{M}^+(f_j))' - (\mathcal{M}^+(f))'\|_p &\leqslant \|\mathcal{M}^+(f_j' - f')\|_{L^p(\mathbb{R})} + 2\|\mathcal{M}^+(f')\|_{p,B_1} \\ &\quad + 2\|\mathcal{M}^+(f')\|_{p,B_2 \cup B^j} + \|R^{-1/p}\varepsilon\|_{p,(-R,R)} \leqslant C\varepsilon, \end{aligned}$$

for any $j \geqslant \max\{j_0, j_1\}$, which implies that $(\mathcal{M}^+(f_j))' \to (\mathcal{M}^+(f))'$ in $L^p(\mathbb{R})$ when $j \to \infty$. This completes the proof of Theorem 2. □

3. Proofs of Theorems 3 and 4

In this section we will prove Theorems 3 and 4. Let us begin with some notation.

Definition 1. We say that a point n is a local maximum of $f\colon\ \mathbb{Z} \to \mathbb{R}$ if

$$f(n-1) \leqslant f(n) \quad \text{and} \quad f(n) > f(n+1).$$

Lemma 4. *Let* $f \in \mathrm{BV}(\mathbb{Z})$.

(i) *If* n *is a local maximum of* $M^+(f)$, *then* $M^+(f)(n) = |f(n)|$.

(ii) *If* n *is a local maximum of* $M^-(f)$, *then* $M^-(f)(n) = |f(n)|$.

P r o o f. We only prove the result for M^+ since the argument for M^- is analogous. We assume that $M^+(f)(n) > |f(n)|$ and need to prove that n is not a local maximum of $M^+(f)$. Below we consider the following two cases:

Case 1. $M^+(f)(n)$ is not attained for any $N \in \mathbb{N}$. Let $\{r_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integer numbers satisfying $\lim_{k\to\infty} r_k = \infty$. By our assumption, we can write

$$M^+(f)(n) = \sup_{\substack{N\in\mathbb{N}\\ N\geqslant r_k}} \frac{1}{N+1}\sum_{i=0}^{N} |f(n+i)|, \quad k \geqslant 1. \tag{3.1}$$

Then for any $k \geqslant 1$ and $N \geqslant r_k$ we have

$$\begin{aligned}\frac{1}{N+1}\sum_{i=0}^{N}|f(n+i)| &= \frac{1}{N+1}\left(\sum_{i=0}^{N}|f(n+1+i)| + |f(n)| - |f(n+1+N)|\right)\\ &\leqslant M^+(f)(n+1) + \frac{1}{r_k+1}\mathrm{Var}(f),\end{aligned}$$

which together with (3.1) implies that

$$M^+(f)(n) \leqslant M^+(f)(n+1) + \frac{1}{r_k+1}\mathrm{Var}(f), \quad k \geqslant 1. \tag{3.2}$$

Letting $k \to \infty$, (3.2) implies that $M^+(f)(n) \leqslant M^+(f)(n+1)$. Thus n is not a local maximum of $M^+(f)$.

Case 2. $M^+(f)(n)$ is attained for some $N \in \mathbb{N}$. By our assumption, there exists $N_0 \in \mathbb{N} \setminus \{0\}$ such that

$$M^+(f)(n) = \frac{1}{N_0+1}\sum_{i=0}^{N_0}|f(n+i)|.$$

It follows from our assumption $|f(n)| < M^+(f)(n)$ that

$$\begin{aligned}M^+(f)(n) &= \frac{1}{N_0+1}\left(\sum_{i=0}^{N_0-1}|f(n+1+i)| + |f(n)|\right)\\ &\leqslant \frac{1}{N_0+1}(N_0 M^+(f)(n+1) + |f(n)|)\\ &< \frac{1}{N_0+1}(N_0 M^+(f)(n+1) + M^+(f)(n)),\end{aligned}$$

which leads to $M^+(f)(n) < M^+(f)(n+1)$. Thus n is not a local maximum of $M^+(f)$. Lemma 4 is proved. □

Applying Lemma 4, we will establish the variation inequalities of the discrete one-sided Hardy-Littlewood maximal functions on an arbitrary interval $[a,b] \subset \mathbb{Z}$.

Lemma 5. *Let $[a,b]$ be an interval with a, b being integers (or possibly ∞ or $-\infty$) and $f \in \mathrm{BV}(\mathbb{Z})$. Then*

$$\begin{aligned} \mathrm{Var}(M^+(f);[a,b]) &\leqslant \mathrm{Var}(f;[a,b]);\\ \mathrm{Var}(M^-(f);[a,b]) &\leqslant \mathrm{Var}(f;[a,b]). \end{aligned}$$

P r o o f. We only prove the result for M^+, since the result of M^- can be obtained by the facts that $\mathrm{Var}(f;[a,b]) = \mathrm{Var}(\mathcal{R}f;[-b,-a])$ and $M^-(f) = \mathcal{R}M^+(\mathcal{R}f)$. We only consider the bounded interval $[a,b]$, since the assertion of Lemma 5 for unbounded intervals $[a,b]$ follows easily from this and the fact that $\mathrm{Var}(M^+(f);[a,b])$ is the supremum of $\mathrm{Var}(M^+(f);[a',b'])$ over bounded subintervals $[a',b'] \subset [a,b]$. Without loss of generality we may assume that $f \geqslant 0$. Let $-\infty < a < b < \infty$. We may assume without loss of generality that a_1 or a_l, $l \geqslant 1$, is respectively the first or last local maximum of $M^+(f)$. It follows from Lemma 4 that $M^+(f)(a_k) = f(a_k)$. Then

$$\begin{aligned} \mathrm{Var}(M^+(f);[a,b]) &= \mathrm{Var}(M^+(f);[a,a_1]) + \mathrm{Var}(M^+(f);[a_l,b]) \\ &\qquad + \sum_{k=1}^{l-1} \mathrm{Var}(M^+(f);[a_k,a_{k+1}]) \\ &\leqslant M^+(f)(a_1) - M^+(f)(a) + M^+(f)(a_l) - M^+(f)(b) \\ &\quad + \sum_{k=1}^{l-1} (M^+(f)(a_k) - M^+(f)(b_{k+1}) \\ &\qquad\quad + M^+(f)(a_{k+1}) - M^+(f)(b_{k+1})) \\ &\leqslant f(a_1) - f(a) + f(a_l) - f(b) \\ &\quad + \sum_{k=1}^{l-1} (f(a_k) - f(b_{k+1}) + f(a_{k+1}) - f(b_{k+1})) \\ &\leqslant \mathrm{Var}(f;[a,a_1]) + \mathrm{Var}(f;[a_l,b]) \\ &\quad + \sum_{k=1}^{l-1} (\mathrm{Var}(f;[a_k,b_{k+1}]) + \mathrm{Var}(f;[b_{k+1},a_{k+1}])) \\ &\leqslant \mathrm{Var}(f;[a,b]). \end{aligned}$$

This completes the proof of Lemma 5. □

P r o o f of Theorem 3. Theorem 3 can be seen as a special case of Lemma 5. □

P r o o f of Theorem 4. One can easily check that $\operatorname{Var}(f) = \operatorname{Var}(\mathcal{R}f)$ and $\|f\|_{\ell^1(\mathbb{Z})} = \|\mathcal{R}f\|_{\ell^1(\mathbb{Z})}$. Thus we only prove Theorem 4 for M^+. Let $f_k \to f$ in $\ell^1(\mathbb{Z})$ when $k \to \infty$. By (1.4), we need to prove that

$$\lim_{k\to\infty} \|(M^+(f_k))' - (M^+(f))'\|_{\ell^1(\mathbb{Z})} = 0. \tag{3.3}$$

Since $\big||f_k| - |f|\big| \leqslant |f_k - f|$, we may assume without loss of generality that $f_k \geqslant 0$ for all $k \in \mathbb{Z}$ and $f \geqslant 0$. Since $f_k \to f$ in $\ell^1(\mathbb{Z})$, hence for any fixed $\varepsilon > 0$ there exists $K_0 = K_0(\varepsilon) \in \mathbb{N} \setminus \{0\}$ such that $\|f_k - f\|_{\ell^\infty(\mathbb{Z})} \leqslant \|f_k - f\|_{\ell^1(\mathbb{Z})} < \varepsilon$ for any $k \geqslant K_0$. Thus for any fixed $n \in \mathbb{Z}$ and $k \geqslant K_0$, we have

$$|M^+(f_k)(n) - M^+(f)(n)| \leqslant M^+(f_k - f)(n) \leqslant \|f_k - f\|_{\ell^\infty(\mathbb{Z})} < \varepsilon, \quad n \in \mathbb{Z}, \tag{3.4}$$

which implies $M^+(f_k)(n) \to M^+(f)(n)$ as $k \to \infty$ for any $n \in \mathbb{Z}$. This leads to

$$(M^+(f_k))'(n) \to (M^+(f))'(n) \quad \text{as } k \to \infty \tag{3.5}$$

for any $n \in \mathbb{Z}$. It follows from Theorem 3 that $\operatorname{Var}(M^+(f)) \leqslant \operatorname{Var}(f) \leqslant 2\|f\|_{\ell^1(\mathbb{Z})}$. Observe that

$$\big||(M^+(f_k))'(n)| - |(M^+(f_k))'(n) - (M^+(f))'(n)|\big| \leqslant |(M^+(f))'(n)|, \quad n \in \mathbb{Z}.$$

By the dominated convergence theorem and (3.5),

$$\lim_{k\to\infty} (\|(M^+(f_k))'\|_{\ell^1(\mathbb{Z})} - \|(M^+(f_k))' - (M^+(f))'\|_{\ell^1(\mathbb{Z})}) = \|(M^+(f))'\|_{\ell^1(\mathbb{Z})}.$$

Therefore, to prove (3.3), it suffices to prove

$$\lim_{k\to\infty} \|(M^+(f_k))'\|_{\ell^1(\mathbb{Z})} = \|(M^+(f))'\|_{\ell^1(\mathbb{Z})}. \tag{3.6}$$

It follows from (3.5) and Fatou's lemma that

$$\|(M^+(f))'\|_{\ell^1(\mathbb{Z})} \leqslant \liminf_{k\to\infty} \|(M^+(f_k))'\|_{\ell^1(\mathbb{Z})}. \tag{3.7}$$

Thus, to prove (3.6), we want to show that

$$\limsup_{k\to\infty} \|(M^+(f_k))'\|_{\ell^1(\mathbb{Z})} \leqslant \|(M^+(f))'\|_{\ell^1(\mathbb{Z})}. \tag{3.8}$$

We now prove (3.8). Since $\|f\|_{\ell^1(\mathbb{Z})} < \infty$, so for every $\varepsilon > 0$ there exists a sufficiently large integer radius $R = R(\varepsilon)$ such that

$$\sum_{\substack{|n|\geqslant R\\ n\in\mathbb{Z}}} f(n) < \varepsilon. \tag{3.9}$$

On the other hand, by (3.4), there exists $K_1 = K_1(\varepsilon, R) \in \mathbb{N} \setminus \{0\}$ such that

$$|(M^+(f_k))'(n) - (M^+(f))'(n)| \leqslant \frac{\varepsilon}{2R+1} \tag{3.10}$$

for any $k \geqslant K_1$ and $n \in [-R, R] \cap \mathbb{Z}$. Write then

$$\|(M^+(f_k))'\|_{\ell^1(\mathbb{Z})} = \sum_{\substack{|n|>R \\ n\in\mathbb{Z}}} |(M^+(f_k))'(n)| + \sum_{\substack{|n|\leqslant R \\ n\in\mathbb{Z}}} |(M^+(f_k))'(n)| =: S_1 + S_2. \tag{3.11}$$

Below we estimate S_1. It follows from (3.9) and Lemma 5 that

$$\begin{aligned} S_1 &\leqslant \operatorname{Var}(M^+(f_k); [R,\infty)) + \operatorname{Var}(M^+(f_k); (-\infty,-R]) \\ &\leqslant \operatorname{Var}(f_k; [R,\infty)) + \operatorname{Var}(f_k; (-\infty,-R]) \\ &\leqslant \operatorname{Var}(f_k - f; [R,\infty)) + \operatorname{Var}(f_k - f; (-\infty,-R]) \\ &\quad + \operatorname{Var}(f; (-\infty,-R] \cup [R,\infty)) \\ &\leqslant 2\|f_k - f\|_{\ell^1} + 2\sum_{\substack{|n|\geqslant R \\ n\in\mathbb{Z}}} f(n) \leqslant 4\varepsilon \end{aligned} \tag{3.12}$$

for any $k \geqslant K_0$. On the other hand, we get from (3.10) that

$$S_2 \leqslant \sum_{\substack{|n|\leqslant R \\ n\in\mathbb{Z}}} |(M^+(f))'(n)| + \varepsilon \leqslant \|(M^+(f))'\|_{\ell^1(\mathbb{Z})} + \varepsilon \tag{3.13}$$

for any $k \geqslant K_1$. From (3.12) and (3.13) we have

$$\|(M^+(f_k))'\|_{\ell^1(\mathbb{Z})} \leqslant \|(M^+(f))'\|_{\ell^1(\mathbb{Z})} + 5\varepsilon$$

for any $k \geqslant \max\{K_0, K_1\}$. This implies (3.8) and hence Theorem 4 is proved. □

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