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SOME RELATIONS SATISFIED BY HERMITE-HERMITE MATRIX POLYNOMIALS

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Abstract. The classical Hermite-Hermite matrix polynomials for commutative matrices were first studied by Metwally et al. (2008). Our goal is to derive their basic properties including the orthogonality properties and Rodrigues formula. Furthermore, we define a new polynomial associated with the Hermite-Hermite matrix polynomials and establish the matrix differential equation associated with these polynomials. We give the addition theorems, multiplication theorems and summation formula for the Hermite-Hermite matrix polynomials. Finally, we establish general families and several new results concerning generalized Hermite-Hermite matrix polynomials.

Keywords: Hermite-Hermite polynomials; matrix generating functions; orthogonality property; Rodrigues formula; associated Hermite-Hermite polynomials; generalized Hermite-Hermite matrix polynomials

MSC 2010: 33C45, 34A25, 15A60, 44A45, 33C50, 33C80

1. Introduction

Orthogonal matrix polynomials constitute a promising field whose development leads to significant results both from the theoretical as well as the practical points of view. Some results in the theory of classical orthogonal polynomials have been extended to orthogonal matrix polynomials, for instance, see [3], [5], [8], [29]. Important connections between orthogonal matrix polynomials and matrix differential equations of the second order appear in [2], [8], [6], [7]. Extensions to the matrix framework of the classical families of Legendre, Laguerre, Hermite, Chebychev, Hermite-Hermite and Gegenbauer polynomials have been introduced in [1], [9], [10], [12], [14]–[31]. The interest in the family of Hermite polynomials is based on their intrinsic mathematical properties due to which these polynomials have found wide-ranging applications in physics.

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Our main aim in this paper is to consider a new system of Hermite-type matrix polynomials. The organization of this paper is as follows. In Section 2, we discuss the orthogonality properties and Rodrigues formula of the Hermite-Hermite matrix polynomials in a fairly direct way. In Section 3, we define a polynomial associated with the generalized Hermite-Hermite matrix polynomials and obtain the matrix differential equation associated with these polynomials. A class of polynomials associated with the generalized Hermite-Hermite polynomials is introduced and studied in Section 4. Section 5 deals with some relations involving the addition theorems of the Hermite-Hermite polynomials and the multiplication theorems discussed there will hopefully provide the matter of forthcoming investigations in this and related fields.

Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A. Furthermore, the identity matrix and the null matrix or zero matrix in $\mathbb{C}^{N\times N}$ will be denoted by I and 0, respectively. In this expression, $\Re(z)$ is the real part of the complex number z. If A is a matrix in $\mathbb{C}^{N\times N}$, its two-norm is denoted by $\|A\|_2$, and is defined as

$$\|A\|_2 = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where for a vector x in $\mathbb{C}^N$, $\|x\|_2 = (x^{\mathrm{T}}x)^{1/2}$ is the Euclidean norm of x.

Notation 1.1 (Dunford and Schwartz [4]). *If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z which are defined in an open set Ω of the complex plane, and A, B are matrices in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset\Omega$ and $\sigma(B)\subset\Omega$ such that $AB = BA$, then from the matrix functional calculus, it follows that*

$$f(A)g(B) = g(B)f(A).$$

Definition 1.1 (Jódar and Defez [9]). A matrix $A \in \mathbb{C}^{N\times N}$ is a positive stable matrix if it satisfies

$$\Re(z) > 0 \quad \text{for every eigenvalue } z \in \sigma(A). \tag{1.1}$$

Lemma 1.1. *If $A(k,n)$ and $B(k,n)$ are matrices in $\mathbb{C}^{N\times N}$ for $n \geqslant 0$, $k \geqslant 0$, it follows in an analogous way to the proof of Lemma 11 and 10 of Rainville [13] that*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A(k, n-2k), \tag{1.2}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} A(k,n-mk),$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n-k); \quad m\in\mathbb{N}.$$

Similarly, we can write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+2k), \tag{1.3}$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+mk),$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k); \quad m\in\mathbb{N}.$$

In the following, we recall the main relations and some properties of the Hermite-Hermite matrix polynomials mentioned in [11].

Definition 1.2. In [11], the Hermite-Hermite matrix polynomials are defined by

$$ {}_HH_n(x,A) = n!\sum_{k=0}^{[n/2]} \frac{(-1)^k(\sqrt{2A})^{n-2k}H_{n-2k}(x,A)}{k!(n-2k)!}, \tag{1.4}$$

where A is a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying condition (1.1) and $H_n(x,A)$ are the Hermite matrix polynomials (see [6], [9])

$$H_n(x,A) = n!\sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!(n-2k)!}(x\sqrt{2A})^{n-2k}, \quad n\geqslant 0.$$

According to [11], we have

$$\sum_{n=0}^{\infty} \frac{{}_HH_n(x,A)t^n}{n!} = \exp\bigl(xt(\sqrt{2A})^2 - t^2(I+(\sqrt{2A})^2)\bigr). \tag{1.5}$$

Theorem 1.1. *Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying condition (1.1). The Hermite-Hermite matrix polynomials ${}_HH_n(x,A)$ satisfy the relation*

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\,{}_HH_n(x,A) = \frac{(\sqrt{2A})^{2r}n!}{(n-r)!}\,{}_HH_{n-r}(x,A)), \quad 0\leqslant r\leqslant n. \tag{1.6}$$

Theorem 1.2. *Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying condition* (1.1). *Then we have*

$$ {}_HH_n(x,A) = x(\sqrt{2A})^2\,{}_HH_{n-1}(x,A) - 2(n-1)\,{}_HH_{n-2}(x,A), \quad n \geqslant 2. \tag{1.7}$$

Corollary 1.1. *The Hermite-Hermite matrix polynomials are a solution of the matrix differential equation of the second order*

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}x^2}I - 2x\frac{\mathrm{d}}{\mathrm{d}x}A^2(I+2A)^{-1} + 2nA^2(I+2A)^{-1}\Big]\,{}_HH_n(x,A) = 0, \quad n \geqslant 0. \tag{1.8}$$

2. Orthogonality and Rodrigues formula of the Hermite-Hermite matrix polynomials

In this section, we will discuss the orthogonality properties of the Hermite-Hermite matrix polynomials.

From (1.8), we can write

$$\begin{aligned} &\mathrm{e}^{-A^2x^2(I+2A)^{-1}}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,{}_HH_n(x,A) \\ &\quad - \mathrm{e}^{-A^2x^2(I+2A)^{-1}}2xA^2(I+2A)^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_n(x,A) \\ &\quad + \mathrm{e}^{-2A^2x^2(I+2A)^{-1}}2nA^2(I+2A)^{-1}\,{}_HH_n(x,A) = 0 \\ &\Rightarrow \Big[\mathrm{e}^{-A^2x^2(I+2A)^{-1}}\frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_n(x,A)\Big]' \\ &\quad + \mathrm{e}^{-A^2x^2(I+2A)^{-1}}2nA^2(I+2A)^{-1}\,{}_HH_n(x,A) = 0. \end{aligned}\tag{2.1}$$

Replacing the index n by m in (2.1), we get

$$\begin{aligned} &\Big[\mathrm{e}^{-A^2x^2(I+2A)^{-1}}\frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_m(x,A)\Big]' \\ &\quad + \mathrm{e}^{-A^2x^2(I+2A)^{-1}}2mA^2(I+2A)^{-1}\,{}_HH_m(x,A) = 0. \end{aligned}\tag{2.2}$$

If we multiply (2.1) by ${}_HH_m(x,A)$ and (2.2) by ${}_HH_n(x,A)$, and subtract, we obtain

$$\begin{aligned} &\mathrm{e}^{-A^2x^2(I+2A)^{-1}}2A^2(I+2A)^{-1}(n-m)\,{}_HH_n(x,A)\,{}_HH_m(x,A) \\ &\quad = \Big[\mathrm{e}^{-A^2x^2(I+2A)^{-1}}\Big({}_HH_n(x,A)\frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_m(x,A) \\ &\qquad - {}_HH_m(x,A)\frac{\mathrm{d}}{\mathrm{d}x}\,{}_HH_n(x,A)\Big)\Big]'. \end{aligned}\tag{2.3}$$

Integrating (2.3) over the interval $[a,b]$, one gets

$$\begin{aligned}(2.4)\quad &\int_a^b \mathrm{e}^{-2A(I+2A)^{-1}x^2} 2A^2(I+2A)^{-1}(n-m)\,{}_HH_n(x,A)\,{}_HH_m(x,A)\,\mathrm{d}x \\ &= \Big[\mathrm{e}^{-A^2x^2(I+2A)^{-1}}\Big({}_HH_n(x,A)\frac{\mathrm{d}}{\mathrm{d}x}{}_HH_m(x,A) \;-\; {}_HH_m(x,A)\frac{\mathrm{d}}{\mathrm{d}x}{}_HH_n(x,A)\Big)\Big]\Big|_a^b.\end{aligned}$$

Since the multiple of any polynomial in x by $\mathrm{e}^{-A^2x^2(I+2A)^{-1}} \to 0$ as $x\to\infty$ or $x\to-\infty$, and taking limits in (2.4) as $a\to-\infty$, $b\to\infty$, it follows that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-A^2x^2(I+2A)^{-1}}\,{}_HH_n(x,A)\,{}_HH_m(x,A)\,\mathrm{d}x = 0;\quad m\neq n.$$

We see therefore that the Hermite-Hermite matrix polynomials are an orthogonal set over the interval $(-\infty,\infty)$ with weight function $\mathrm{e}^{-A^2x^2(I+2A)^{-1}}$.

From (1.2) and (1.5), we obtain

$$(2.5)\qquad \sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{{}_HH_k(x,A)\,{}_HH_{m-k}(x,A)t^{m-k}}{(m-k)!k!} = \exp\big(4Axt-2(I+2A)t^2\big),$$

so

$$\begin{aligned}(2.6)\qquad &\int_{-\infty}^{\infty}\mathrm{e}^{-A^2x^2(I+2A)^{-1}}\exp\big(4Axt-2(I+2A)t^2\big)\,\mathrm{d}x \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\int_{-\infty}^{\infty}\mathrm{e}^{-A^2x^2(I+2A)^{-1}}\frac{{}_HH_k(x,A)\,{}_HH_{m-k}(x,A)t^{m-k}}{(m-k)!k!}\,\mathrm{d}x.\end{aligned}$$

Equation (2.6) is

$$\begin{aligned}(2.7)\qquad &\int_{-\infty}^{\infty}\mathrm{e}^{-A^2x^2(I+2A)^{-1}}\exp\big(4Axt-2(I+2A)t^2\big)\,\mathrm{d}x \\ &= \mathrm{e}^{2t^2(I+2A)}\int_{-\infty}^{\infty}\mathrm{e}^{-(xA(\sqrt{I+2A})^{-1}-2\sqrt{I+2A}t)^2}\,\mathrm{d}x \\ &= \sqrt{\pi}A^{-1}\sqrt{I+2A}\mathrm{e}^{2t^2(I+2A)} \\ &= \sqrt{\pi}A^{-1}\sqrt{I+2A}\sum_{n=0}^{\infty}\frac{2^nt^{2n}}{n!}(I+2A)^n,\end{aligned}$$

which can be written

$$\begin{aligned}(2.8)\qquad &\sqrt{\pi}A^{-1}\sqrt{I+2A}\sum_{n=0}^{\infty}\frac{2^nt^{2n}}{n!}(I+2A)^n \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\int_{-\infty}^{\infty}\mathrm{e}^{-A^2x^2(I+2A)^{-1}}\frac{{}_HH_k(x,A)\,{}_HH_{m-k}(x,A)t^{m-k}}{(m-k)!k!}\,\mathrm{d}x.\end{aligned}$$

Then m must be even, $m = 2n$ and $k = n$. Therefore we have

$$\frac{2^n}{n!}\sqrt{\pi}A^{-1}\sqrt{I+2A}(I+2A)^n = \int_{-\infty}^{\infty} \mathrm{e}^{-A^2x^2(I+2A)^{-1}} \frac{{}_HH_n(x,A)\,{}_HH_n(x,A)}{n!n!}\,\mathrm{d}x,$$

which yields

$$\int_{-\infty}^{\infty} \mathrm{e}^{-A^2x^2(I+2A)^{-1}}\,{}_HH_n^2(x,A)\,\mathrm{d}x = 2^n n!\sqrt{\pi}A^{-1}\sqrt{I+2A}(I+2A)^n.$$

Therefore, the following result has been established.

Theorem 2.1. *Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying condition* (1.1)*, then the Hermite-Hermite matrix polynomials satisfy the following orthogonality formula:*

$$\int_{-\infty}^{\infty} \mathrm{e}^{-A^2x^2(I+2A)^{-1}}\,{}_HH_n(x,A)\,{}_HH_m(x,A)\,\mathrm{d}x = \begin{cases} 0, & m \neq n; \\ 2^n n!\sqrt{\pi}A^{-1}(I+2A)^{n+1/2}, & m = n. \end{cases} \tag{2.9}$$

In the following theorem, we provide a Rodrigues formula for the Hermite-Hermite matrix polynomials.

Theorem 2.2. *Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying condition* (1.1)*, then the Hermite-Hermite matrix polynomials ${}_HH_n(x,A)$ satisfy the Rodrigues formula:*

$$\begin{aligned}{}_HH_n(x,A) &= (-1)^n((I+2A)A^{-1})^n \exp\bigl(x^2A^2(I+2A)^{-1}\bigr) \\ &\quad\times \frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp\bigl(-x^2A^2(I+2A)^{-1}\bigr).\end{aligned} \tag{2.10}$$

Proof. We use (1.5) and Taylor's theorem, which states that

$$F(t) = \sum_{n=0}^{\infty} \frac{\mathrm{d}^n F(t)}{\mathrm{d}t^n}\bigg|_{t=0} \frac{t^n}{n!}. \tag{2.11}$$

The matrix generating function (2.11) with the aid of the Taylor's theorem gives

$$\begin{aligned}{}_HH_n(x,A) &= \Bigl[\frac{\partial^n}{\partial t^n}\exp\bigl(xt(\sqrt{2A})^2 - t^2(I+(\sqrt{2A})^2)\bigr)\Bigr]\Big|_{t=0} \\ &= \exp\bigl(x^2A^2(I+2A)^{-1}\bigr) \\ &\quad\times \Bigl[\frac{\partial^n}{\partial t^n}\exp\bigl(-\bigl[xA(\sqrt{I+2A})^{-1} - t\sqrt{I+2A}\bigr]^2\bigr)\Bigr]\Big|_{t=0}.\end{aligned} \tag{2.12}$$

Setting $f(x,t,A) = \exp\bigl(-[xA(\sqrt{I+2A})^{-1} - t\sqrt{I+2A}]^2\bigr)$ in (2.12), we have

$$\frac{\partial}{\partial t} f(x,t,A) = -(I+2A)A^{-1}\frac{\partial}{\partial x} f(x,t,A), \tag{2.13}$$
$$\frac{\partial^n}{\partial t^n} f(x,t,A) = (-1)^n((I+2A)A^{-1})^n\frac{\partial^n}{\partial x^n} f(x,t,A).$$

So

$$\begin{aligned}\frac{\partial^n}{\partial t^n}\bigl[\exp\bigl(-[xA(\sqrt{I+2A})^{-1} - t\sqrt{I+2A}]^2\bigr)\bigr]\big|_{t=0} &= (-1)^n((I+2A)A^{-1})^n \\ &\times \frac{\partial^n}{\partial x^n}\bigl[\exp\bigl(-[xA(\sqrt{I+2A})^{-1} - t\sqrt{I+2A}]^2\bigr)\bigr]\big|_{t=0},\end{aligned} \tag{2.14}$$

and we have

$$\begin{aligned}{}_HH_n(x,A) &= (-1)^n((I+2A)A^{-1})^n\exp\bigl(x^2A^2(I+2A)^{-1}\bigr) \\ &\times \Bigl[\frac{\partial^n}{\partial x^n}\exp\Bigl(-\bigl[xA(\sqrt{I+2A})^{-1} - t\sqrt{I+2A}\bigr]^2\Bigr)\Bigr]\Big|_{t=0}.\end{aligned} \tag{2.15}$$

Therefore, the result is established. □

3. Associated Hermite-Hermite matrix polynomials

The object of this section is to introduce an associated polynomial with the Hermite-Hermite matrix polynomials. From (1.6), we can write

$$D^p\,{}_HH_n(x,A) = (\sqrt{2A})^{2p}n!\sum_{k=0}^{[(n-p)/2]}\frac{(-1)^k(\sqrt{2A})^{n-p-2k}H_{n-p-2k}(x,A)}{k!(n-p-2k)!}; \quad 0 \leqslant p \leqslant n, \tag{3.1}$$

where p is non-negative integer.

The associated Hermite-Hermite matrix polynomials can be defined as

$${}_H\Phi_n^p(x,A) = \frac{(\sqrt{2A})^{-2p}(n-p)!}{n!}(x^q-1)^{p(n-p)}D^p\,{}_HH_n(x,A), \tag{3.2}$$

which can be written with the aid of (3.1) in the form

$$\begin{aligned}{}_H\Phi_n^p(x,A) &= (n-p)!(x^q-1)^{p(n-p)} \\ &\times \sum_{k=0}^{[(n-p)/2]}\frac{(-1)^k(\sqrt{2A})^{n-p-2k}H_{n-p-2k}(x,A)}{k!(n-p-2k)!}.\end{aligned} \tag{3.3}$$

Moreover, we have

$$ {}_H\Phi^p_{n+p}(x,A) = (x^q-1)^{pn} n! \sum_{k=0}^{[n/2]} \frac{(-1)^k(\sqrt{2A})^{n-2k} H_{n-2k}(x,A)}{k!(n-2k)!}, \tag{3.4} $$

i.e.,

$$ {}_H\Phi^p_{n+p}(x,A) = (x^q-1)^{pn}\, {}_H H_n(x,A). \tag{3.5} $$

Clearly, if $p = 0$, ${}_H\Phi^p_{n+p}(x,A)$ reduces to the Hermite-Hermite matrix polynomials ${}_H H_n(x,A)$.

Next, we obtain the matrix generating function for the associated Hermite-Hermite matrix polynomials. From (3.4), (1.3) and (1.4), we have

$$ \begin{aligned} \sum_{n=0}^{\infty} \frac{{}_H\Phi^p_{n+p}(x,A)t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{(-1)^k(\sqrt{2A})^{n-2k}}{k!(n-2k)!} t^n (x^q-1)^{pn} H_{n-2k}(x,A) \\ &= \exp\big(((x^q-1)^p xt\sqrt{2A})^2 - (x^q-1)^{2p}t^2(\sqrt{2A})^2\big) \\ &\quad \times \exp\big(-(x^q-1)^{2p}t^2 I\big). \end{aligned} $$

Therefore, the matrix generating function concerning the associated Hermite-Hermite matrix polynomials is given in the form

$$ \sum_{n=0}^{\infty} \frac{{}_H\Phi^p_{n+p}(x,A)t^n}{n!} = \exp\big((x^q-1)^p xt(\sqrt{2A})^2 - (x^q-1)^{2p}t^2(I+(\sqrt{2A})^2)\big). \tag{3.6} $$

To derive the matrix differential equation satisfied by the associated Hermite-Hermite matrix polynomials, let us take

$$ {}_H H_n(x,A) = (x^q-1)^{-pn}\, {}_H\Phi^p_{n+p}(x,A). \tag{3.7} $$

Differentiating with respect to x, we get

$$ \frac{\mathrm{d}}{\mathrm{d}x}{}_H H_n(x,A) = (x^q-1)^{-pn}\frac{\mathrm{d}}{\mathrm{d}x}{}_H\Phi^p_{n+p}(x,A) - pqnx^{q-1}(x^q-1)^{-pn-1}\,{}_H\Phi^p_{n+p}(x,A) $$

and

$$ \begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}x^2}{}_H H_n(x,A) &= (x^q-1)^{-pn}\frac{\mathrm{d}^2}{\mathrm{d}x^2}{}_H\Phi^p_{n+p}(x,A) \\ &\quad - 2pqnx^{q-1}(x^q-1)^{-pn-1}\frac{\mathrm{d}}{\mathrm{d}x}{}_H\Phi^p_{n+p}(x,A) \\ &\quad - pnq(q-1)x^{q-2}(x^q-1)^{-pn-1}\,{}_H\Phi^p_{n+p}(x,A) \\ &\quad + q^2pn(pn+1)x^{2q-2}(x^q-1)^{-pn-2}\,{}_H\Phi^p_{n+p}(x,A) \end{aligned} $$

Therefore, the following result is established:

Theorem 3.1. *Let p be a non-negative integer and let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying condition* (1.1)*. Then the associated Hermite-Hermite matrix polynomials, ${}_H\Phi^p_{n+p}(x,A)$, are a solution of the matrix differential equation of the second order in the form*

$$\begin{aligned}(3.8)\quad &(x^q-1)^2(2A+I)\frac{\mathrm{d}^2}{\mathrm{d}x^2}{}_H\Phi^p_{n+p}(x,A)\\ &\quad-(2pqnx^{q-1}(x^q-1)(2A+I)+2x(x^q-1)^2A^2)\frac{\mathrm{d}}{\mathrm{d}x}{}_H\Phi^p_{n+p}(x,A)\\ &\quad+\Big[pqnx^{q-2}(x^q-1)(2x^2A^2-(q-1)(2A+I))\\ &\quad+q^2pn(pn+1)x^{2q-2}(2A+I)+2n(x^q-1)^2A^2\Big]{}_H\Phi^p_{n+p}(x,A)=0.\end{aligned}$$

Note that when $p=0$, the above matrix differential equation reduces to the Hermite-Hermite matrix differential equation (1.8).

4. Some relations on Hermite-Hermite matrix polynomials

Here, let us recall some important properties of Hermite-Hermite matrix polynomials, namely the addition theorems and multiplication theorems, which will be used in this section.

Theorem 4.1. *For a positive stable matrix A in $\mathbb{C}^{N\times N}$ satisfying condition* (1.1)*, the following addition formulas for the Hermite-Hermite matrix polynomials hold true:*

$$(4.1)\qquad {}_HH_n(x+y,A)=n!\sum_{k=0}^{n}\frac{(y(\sqrt{2A})^2)^{n-k}\,{}_HH_k(x,A)}{k!(n-k)!}$$

and

$$(4.2)\qquad {}_HH_n(x+y,A)=n!\sum_{k=0}^{n}\frac{{}_HH_k(x\sqrt{2},A)\,{}_HH_{n-k}(y\sqrt{2},A)}{k!(n-k)!}\Big(\frac{1}{\sqrt{2}}\Big)^n.$$

Proof. From (1.5) and (1.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{{}_HH_n(x+y)t^n}{n!}&=\exp\big(xt(\sqrt{2A})^2-t^2(I+(\sqrt{2A})^2)\big)\mathrm{e}^{yt(\sqrt{2A})^2}\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{{}_HH_k(x,A)t^k}{k!}\frac{(yt(\sqrt{2A})^2)^n}{n!}\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{{}_HH_k(x,A)t^k}{k!}\frac{(yt(\sqrt{2A})^2)^{n-k}}{(n-k)!}.\end{aligned}$$

Comparing the coefficients of t^n, we obtain (4.1). From (1.5) and (1.2), we find

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{{}_H H_n(x+y) t^n}{n!} &= \exp\big((x+y)t(\sqrt{2A})^2 - t^2(I + (\sqrt{2A})^2)\big) \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{{}_H H_k(x\sqrt{2}, A)}{k!} \frac{{}_H H_n(y\sqrt{2}, A)}{n!} \Big(\frac{t}{\sqrt{2}}\Big)^{n+k} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{{}_H H_k(x\sqrt{2}, A)}{k!} \frac{{}_H H_{n-k}(y\sqrt{2}, A)}{(n-k)!} \Big(\frac{t}{\sqrt{2}}\Big)^{n},
\end{aligned}$$

by comparing the coefficients of t^n, we get (4.2). The proof is completed. $\square$

Theorem 4.2. *For a positive stable matrix A in $\mathbb{C}^{N\times N}$ satisfying condition* (1.1), *the following multiplication formulas for the Hermite-Hermite matrix polynomials hold true:*

$$(4.3) \qquad {}_H H_n(\mu x, A) = n! \sum_{k=0}^{n} \frac{1}{k!(n-k)!} (x(\sqrt{2A})^2)^k (\mu - 1)^k \, {}_H H_{n-k}(x, A)$$

and

$$(4.4) \quad {}_H H_n(\mu x, A) = n! \mu^n \sum_{k=0}^{[n/2]} \frac{1}{k!(n-2k)!} (I + (\sqrt{2A})^2)^k \Big(1 - \frac{1}{\mu^2}\Big)^k \, {}_H H_{n-2k}(x, A).$$

Proof. Using (1.2) and (1.5), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{{}_H H_n(\mu x) t^n}{n!} &= \exp\big(\mu x t(\sqrt{2A})^2 - t^2(I + (\sqrt{2A})^2)\big) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{k! n!} (x(\sqrt{2A})^2)^k (\mu - 1)^k t^{n+k} \, {}_H H_n(x, A) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k!(n-k)!} (x(\sqrt{2A})^2)^k (\mu - 1)^k t^n \, {}_H H_{n-k}(x, A).
\end{aligned}$$

Therefore, (4.3) follows. From (1.2) and (1.5), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{{}_HH_n(\mu x, A)}{n!}\Big(\frac{t}{\mu}\Big)^n \\
&\quad = \exp\Big(xt(\sqrt{2A})^2 - \Big(\frac{t}{\mu}\Big)^2 (I + (\sqrt{2A})^2)\Big) \\
&\quad = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{k!n!}(I + (\sqrt{2A})^2)^k \Big(1 - \frac{1}{\mu^2}\Big)^k t^{n+2k}\, {}_HH_n(x, A) \\
&\quad = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{1}{k!(n-2k)!}(I + (\sqrt{2A})^2)^k \Big(1 - \frac{1}{\mu^2}\Big)^k t^n\, {}_HH_{n-2k}(x, A).
\end{aligned}$$

Therefore, the expression (4.4) is established and the proof is completed. □

Theorem 4.3. *Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying condition* (1.1). *Then the Hermite-Hermite matrix polynomials satisfy the following relations:*

$$ {}_HH_n(\alpha x, A) = \frac{n!}{\sqrt{2^n}} \sum_{k=0}^{n} \frac{{}_HH_k(\alpha x/\sqrt{2}, A)\, {}_HH_{n-k}(\alpha x/\sqrt{2}, A)}{k!(n-k)!} \tag{4.5}$$

and

$$ {}_HH_n(\alpha x + \beta y, A) = \frac{n!}{\sqrt{2^n}} \sum_{k=0}^{n} \frac{{}_HH_{n-k}(\beta y\sqrt{2}, A)\, {}_HH_k(\alpha x\sqrt{2}, A)}{k!(n-k)!}, \tag{4.6}$$

where α and β are constants.

P r o o f. From (1.2) and (1.5), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{{}_HH_{n-k}(\alpha x/\sqrt{2}, A)\, {}_HH_k(\alpha x/\sqrt{2}, A) t^n}{k!(n-k)!} \\
&\quad = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{{}_HH_n(\alpha x/\sqrt{2}, A)\, {}_HH_k(\alpha x/\sqrt{2}, A) t^{n+k}}{k!n!} \\
&\quad = \exp\Big(\frac{2\alpha}{\sqrt{2}} xt(\sqrt{2A})^2 - 2t^2(I + (\sqrt{2A})^2)\Big) \\
&\quad = \exp\big(\alpha x(t\sqrt{2})(\sqrt{2A})^2 - (t\sqrt{2})^2(I + (\sqrt{2A})^2)\big) \\
&\quad = \sum_{n=0}^{\infty} \frac{{}_HH_n(\alpha x, A)}{n!}(t\sqrt{2})^n.
\end{aligned}$$

Comparing the coefficients of t^n, we obtain (4.5). The series can be given as

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{n} &\frac{{}_H H_{n-k}(\beta y\sqrt{2},A)\,{}_H H_k(\alpha x\sqrt{2},A)t^n}{k!(n-k)!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{{}_H H_n(\beta y\sqrt{2},A)\,{}_H H_k(\alpha x\sqrt{2},A)t^{n+k}}{k!n!} \\
&= \exp\big((\alpha x+\beta y)(t\sqrt{2})(\sqrt{2A})^2-(t\sqrt{2})^2(I+(\sqrt{2A})^2)\big) \\
&= \sum_{n=0}^{\infty} \frac{{}_H H_n(\alpha x+\beta y,A)}{n!}(t\sqrt{2})^n.
\end{aligned}$$

By comparing the coefficients of t^n, we get (4.6). The proof is completed. □

In the following corollary, we obtain the properties Hermite-Hermite matrix polynomials as follows.

Corollary 4.1. *For a positive stable matrix A in $\mathbb{C}^{N\times N}$ satisfying condition* (1.1), *the following relation for the Hermite-Hermite matrix polynomials holds true:*

$$(4.7)\qquad {}_H H_n\Big(\frac{x+y}{\sqrt{2}},A\Big)=\frac{n!}{\sqrt{2^n}}\sum_{k=0}^{n}\frac{{}_H H_k(y,A)\,{}_H H_{n-k}(x,A)}{k!(n-k)!}.$$

P r o o f. From (1.3) and (2.1), we can write

$$\begin{aligned}
\sum_{n=0}^{\infty} &\frac{(t\sqrt{2})^n}{n!}\,{}_H H_n\Big(\frac{x+y}{\sqrt{2}},A\Big) \\
&= \exp\Big(\frac{x+y}{\sqrt{2}}t\sqrt{2}(\sqrt{2A})^2-(t\sqrt{2})^2(I+(\sqrt{2A})^2)\Big) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n!k!}\,{}_H H_n(x,A)\,{}_H H_k(y,A)t^{n+k} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\,{}_H H_{n-k}(x,A)\,{}_H H_k(y,A)t^n.
\end{aligned}$$

By comparing the coefficients of t^n, we get (4.7) and the proof is completed. □

Theorem 4.4. *Let $k\in\mathbb{N}$ and A be a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying condition* (1.1). *Then the Hermite-Hermite matrix polynomials satisfy the following* *relation:*

$$(4.8)\qquad \sum_{s=0}^{[n/2]} \frac{(-k(I+(\sqrt{2A})^2))^s(kx(\sqrt{2A})^2)^{n-2s}}{s!(n-2s)!} = \sum_{n_1+n_2+\ldots+n_k=n} \frac{{}_HH_{n_1}(x,A)\,{}_HH_{n_2}(x,A)\ldots{}_HH_{n_k}(x,A)}{n_1!n_2!\ldots n_k!}.$$

P r o o f. From (2.2), we can find that

$$\sum_{n=0}^{\infty}\frac{(t\sqrt{k})^n}{n!}\,{}_HH_n(x\sqrt{k},A) = \exp\bigl(kxt(\sqrt{2A})^2 - kt^2(I+(\sqrt{2A})^2)\bigr) = \exp\bigl(kxt(\sqrt{2A})^2\bigr)\exp\bigl(-kt^2(I+(\sqrt{2A})^2)\bigr).$$

Using (1.2), we have

$$(4.9)\qquad \sum_{n=0}^{\infty}\frac{(kxt(\sqrt{2A})^2)^n}{n!}\sum_{s=0}^{\infty}\frac{(-kt^2(I+(\sqrt{2A})^2))^s}{s!} = \sum_{n=0}^{\infty}\sum_{s=0}^{[n/2]}\frac{(-k(I+(\sqrt{2A})^2))^s(kx(\sqrt{2A})^2)^{n-2s}}{s!(n-2s)!}t^n.$$

On the other hand, we get

$$(4.10)\qquad \exp\bigl(kxt(\sqrt{2A})^2 - kt^2(I+(\sqrt{2A})^2)\bigr) = \bigl[\exp\bigl(xt(\sqrt{2A})^2 - t^2(I+(\sqrt{2A})^2)\bigr)\bigr]^k = \Bigl[\sum_{n=0}^{\infty}\frac{t^n}{n!}\,{}_HH_n(x,A)\Bigr]^k = \sum_{n=0}^{\infty}\Bigl[\sum_{n_1+n_2+\ldots+n_k=n}\frac{{}_HH_{n_1}(x,A)\,{}_HH_{n_2}(x,A)\ldots{}_HH_{n_k}(x,A)}{n_1!n_2!\ldots n_k!}\Bigr]t^n.$$

If we combine (4.9) and (4.10), we obtain (4.8), the proof is completed. $\square$

Theorem 4.5. *For $k\in\mathbb{N}$ and A a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfying condition* (1.1), *we get the following relation:*

$$(4.11)\qquad \sum_{s=0}^{[n/2]}\frac{(-(I+(\sqrt{2A})^2))^s k^{n-s}((x_1+x_2+\ldots+x_k)(\sqrt{2A})^2)^{n-2s}}{s!(n-2s)!} = \sum_{n_1+n_2+\ldots+n_k=n}\frac{{}_HH_{n_1}(x_1,A)\,{}_HH_{n_2}(x_2,A)\ldots{}_HH_{n_k}(x_k,A)}{n_1!n_2!\ldots n_k!}.$$

Proof. Let

$$W(x_1, x_2, \dots, x_k, t, A) = \exp\big((x_1 + x_2 + \dots + x_k)t(\sqrt{2A})^2 - t^2(I + (\sqrt{2A})^2)\big).$$

Using (1.2), we can write

$$\begin{aligned}(4.12)\quad & W(x_1\sqrt{k}, x_2\sqrt{k}, \dots, x_k\sqrt{k}, t\sqrt{k}, A)\\ &= \exp\big(k(x_1 + x_2 + \dots + x_k)t(\sqrt{2A})^2 - kt^2(I + (\sqrt{2A})^2)\big)\\ &= \sum_{n=0}^{\infty} \frac{(k(x_1 + x_2 + \dots + x_k)(\sqrt{2A})^2 t)^n}{n!} \sum_{s=0}^{\infty} \frac{(-kt^2(I + (\sqrt{2A})^2))^s}{s!}\\ &= \sum_{n=0}^{\infty} \sum_{s=0}^{[n/2]} \frac{(-k(I + (\sqrt{2A})^2))^s (k(x_1 + x_2 + \dots + x_k)(\sqrt{2A})^2)^{n-2s}}{s!(n-2s)!} t^n.\end{aligned}$$

On the other hand, we get

$$\begin{aligned}(4.13)\quad & \exp\big(k(x_1 + x_2 + \dots + x_k)t(\sqrt{2A})^2 - kt^2(I + (\sqrt{2A})^2)\big)\\ &= \big[\exp\big((x_1 + x_2 + \dots + x_k)t(\sqrt{2A})^2 - t^2(I + (\sqrt{2A})^2)\big)\big]^k\\ &= \sum_{n=0}^{\infty} \Bigg[\sum_{n_1+n_2+\dots+n_k=n} \frac{{}_HH_{n_1}(x_1, A)\,{}_HH_{n_2}(x_2, A) \dots {}_HH_{n_k}(x_k, A)}{n_1!n_2!\dots n_k!} \Bigg] t^n.\end{aligned}$$

Comparing (4.12) and (4.13), we obtain (4.11), the proof is completed. □

It is obvious that we can define the generalized Hermite-Hermite matrix polynomials in the forms

$$(4.14)\qquad {}_HH_n(x_1 + \dots + x_k, A) = \sum_{s=0}^{[n/2]} \frac{(-1)^s (\sqrt{2A})^{n-2s} H_{n-2s}(x_1 + \dots + x_k)}{s!(n-2s)!}$$

and

$$(4.15)\qquad {}_HH_n(x_1, \dots, x_k, A) = \sum_{s=0}^{[n/2]} \frac{(-1)^s (\sqrt{2A})^{n-2s} H_{n-2s}(x_1, \dots, x_k)}{s!(n-2s)!}; \quad k \in \mathbb{N}.$$

5. Generalized Hermite-Hermite matrix polynomials

It is the purpose of this section to introduce a new matrix polynomial which represents a generalization of the Hermite-Hermite matrix polynomials as given by relation (1.1). For $n = 0, 1, 2, \ldots$, $\lambda \in \mathbb{R}$ and m a positive integer, we define the generalized Hermite-Hermite matrix polynomials by

$$ {}_{H}H^{\lambda}_{n,m}(x, A) = n! \sum_{k=0}^{[n/m]} \frac{(-1)^k \lambda^k (\sqrt{2A})^{n-mk} H^{\lambda}_{n-mk,m}(x, A)}{k!(n-mk)!}. \tag{5.1} $$

The generalized Hermite matrix polynomials are defined by (see [14])

$$ H^{\lambda}_{n,m}(x, A) = n! \lambda^n \sum_{k=0}^{[n/m]} \frac{(-1)^k}{\lambda^{(m-1)k} k!(n-mk)!} (x\sqrt{2A})^{n-mk} \tag{5.2} $$

and

$$ \sum_{n=0}^{\infty} \frac{t^n}{n!} H^{\lambda}_{n,m}(x, A) = \exp\big(\lambda(xt\sqrt{2A} - t^m I)\big). \tag{5.3} $$

Using (1.3), (5.1), (5.2) and (5.3), we arrange the series

$$ \begin{aligned} \sum_{n=0}^{\infty} \frac{{}_{H}H^{\lambda}_{n,m}(x, A) t^n}{n!} &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{(-1)^k \lambda^k (\sqrt{2A})^{n-mk} H^{\lambda}_{n-mk,m}(x, A)}{k!(n-mk)!} t^n \\ &= \sum_{n=0}^{\infty} \frac{(\sqrt{2A})^n H^{\lambda}_{n,m}(x, A)}{n!} t^n \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{k!} t^{mk} \\ &= \exp\big(\lambda(xt(\sqrt{2A})^2 - (t\sqrt{2A})^m)\big) \exp(-\lambda t^m I). \end{aligned} $$

Thus, we obtain an explicit representation for the matrix generating function of generalized Hermite-Hermite matrix polynomials in the form

$$ \sum_{n=0}^{\infty} \frac{{}_{H}H^{\lambda}_{n,m}(x, A) t^n}{n!} = \exp\big(\lambda(xt(\sqrt{2A})^2 - (I + (\sqrt{2A})^m) t^m)\big). \tag{5.4} $$

The above examples prove the usefulness of the method adopted in this paper. Here, we have obtained the matrix generating functions for associated Hermite-Hermite and generalized Hermite-Hermite polynomials, from a known result for Hermite-Hermite polynomials.

In a forthcoming paper, we will consider the problem of a unified approach to the theory of new orthogonal matrix polynomials following the technique discussed in this paper. The used notations are implied by the following matrix generating function for the generalized Hermite-Hermite matrix polynomials definitions: The matrix generating functions of the generalized Hermite-Hermite matrix polynomials of index two, three and p in terms of series are represented as follows:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{{}_HH_{n,m}(x,A)t^n u^m}{n!m!} = \exp\big(x(t+u)(\sqrt{2A})^2-(t+u)^2(I+(\sqrt{2A})^2)\big),$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{{}_HH_{n,m,p}(x,A)t^n u^m v^p}{n!m!p!} = \exp\big(x(t+u+v)(\sqrt{2A})^2-(t+u+v)^2(I+(\sqrt{2A})^2)\big)$$

and

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\dots\sum_{n_p=0}^{\infty}\frac{{}_HH_{n_1,n_2,\dots,n_p}(x,A)t_1^{n_1}t_2^{n_2}\dots t_p^{n_p}}{n_1!n_2!\dots n_p!} = \exp\big(x(t_1+t_2+\dots+t_p)(\sqrt{2A})^2-(t_1+t_2+\dots+t_p)^2(I+(\sqrt{2A})^2)\big).$$

6. Open problem

One can use the same class of new integral representation and operational methods for some other matrix polynomials of several variables. Hence, new results and further applications can be obtained.

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