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ON DYAKONOV TYPE THEOREMS FOR HARMONIC QUASIREGULAR MAPPINGS

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Abstract. We prove two Dyakonov type theorems which relate the modulus of continuity of a function on the unit disc with the modulus of continuity of its absolute value. The methods we use are quite elementary, they cover the case of functions which are quasiregular and harmonic, briefly hqr, in the unit disc.

Keywords: modulus of continuity; harmonic mapping; quasiregular mapping

MSC 2010: 30C62, 30C80

1. Introduction and notation

Let $\mathbb{D}$ denote the open unit disc in the complex plane and let $\mathbb{T}$ denote its boundary. For a function g defined on $\mathbb{D}$ (or $\overline{\mathbb{D}}$) its modulus of continuity is defined by

$$\omega(g,\delta) = \sup_{|z_1-z_2|\leqslant\delta} |g(z_1)-g(z_2)|, \quad 0<\delta\leqslant 2.$$

Let ω be a positive function on $(0,2]$ such that $\omega(0+)=0$. The space $\Lambda_\omega = \Lambda_\omega(\mathbb{D})$ consists of all functions g defined on $\mathbb{D}$ such that

$$\omega(g,\delta)\leqslant C\omega(\delta), \quad 0<\delta\leqslant 2 \tag{1.1}$$

for some constant C. Let, for $g\in\Lambda_\omega$, $C_{g,\omega}$ denote the smallest constant C such that (1.1) holds. Then $\|g\|_\omega = C_{g,\omega}+\|g\|_\infty$ defines a norm on the vector space Λ_ω, which turns it into a Banach space. Similarly we have a Banach space $\Lambda_\omega(\mathbb{T})$.

If $g\in\Lambda_\omega$, then g is uniformly continuous on $\mathbb{D}$ and therefore has a continuous extension g_1 to $\overline{\mathbb{D}}$; moreover, we have $\omega(g,\delta)=\omega(g_1,\delta)$, $0<\delta\leqslant 2$.

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In the following we will assume that ω is a majorant, i.e., an increasing function defined on $[0,2]$ such that $\omega(0)=\omega(0+)=0$ and that $\omega(t)/t$ is decreasing on $(0,2]$. The majorant ω is called *fast* if $\omega(t)/t^\alpha$ is almost increasing in t for some $\alpha>0$, it is called *slow* if $\omega(t)/t^\beta$ is almost decreasing in t for some $\beta<1$. It is called *regular* if it is both slow and fast.

Let $A(\mathbb{D})$ denote the algebra of functions continuous on $\overline{\mathbb{D}}$ and holomorphic in $\mathbb{D}$, i.e. the disc algebra. If f is in $C(\overline{\mathbb{D}})$, f_* denotes the restriction of f to $\mathbb{T}$. For a continuous function g on $\mathbb{T}$, $\mathcal{P}[g]$ denotes its continuous extension to $\overline{\mathbb{D}}$, harmonic in $\mathbb{D}$.

We follow the usual convention: letter C denotes a constant that can change its value from one occurrence to the next.

The following theorem is due to Dyakonov, see [2].

Theorem 1.1. *Let ω be fast, and $f\in A(\mathbb{D})$. Then the following conditions are equivalent:*

(a) $f\in\Lambda_\omega$.

(b) $|f|\in\Lambda_\omega$.

(c) *There is a constant C such that for all $z\in\mathbb{D}$ and all $\zeta\in\mathbb{T}$ we have*

$$\big||f(z)|-|f_*(\zeta)|\big|\leqslant C\omega(|z-\zeta|).$$

As the author of this theorem remarked in [2], this "looks tantalizingly (and perhaps deceptively) simple". Certainly it is a beautiful and fundamental result and thus attracted much attention. As "All great theorems have simple proofs",[1] a simple proof was given by the second author, see [5]. It should be noted that Dyakonov proved this theorem under the hypothesis that the majorant ω is regular. The same hypothesis was used in [5], but the proof given there relies only on the assumption that ω is fast.

Theorem 1.2. *Let $\omega(t)=t^\alpha$, $0<\alpha<1$, and let f be a quasiregular function defined in the unit disc. Then f belongs to Λ_ω if and only if so does $|f|$.*

This partial generalization of Theorem 1.1 was proved in [3] for quasiregular local homeomorphisms, and in [4], Theorems 24 and 40, for quasiregular mappings.

In this note we present an elementary proof of Theorem 1.2 for a fast majorant ω under the additional hypothesis that the mapping is hqr. Namely, we prove the following theorem.

Theorem 1.3. *Let ω be fast and let f be a quasiregular harmonic function defined in the unit disc. Then f belongs to Λ_ω if and only if so does $|f|$.*

[1] We invite readers to find a reference for this statement.

Let us note that a result of this type appeared in [1], with a different proof.

The proof we give does not use any results from the theory of quasiconformal or quasiregular mappings. We use only an elegant characterization of harmonic quasiregular mappings, which can be taken as a definition, see Definition 1.1. Note that any function f, harmonic in $\mathbb{D}$, can be represented in a unique way as a sum $f(z) = g(z) + \overline{h(z)}$, where g and h are analytic in $\mathbb{D}$ and $h(0) = 0$. Let us call this representation $f = g + \overline{h}$ the canonical representation of a harmonic function f.

Definition 1.1. A harmonic function f, defined on $\mathbb{D}$, is k-quasiregular ($0 \leqslant k < 1$) if its canonical representation $f = g + \overline{h}$ satisfies

$$|h'(z)| \leqslant k|g'(z)|, \quad z \in \mathbb{D}.$$

Before stating another theorem of Dyakonov note that for $f \in A(\mathbb{D})$ we have $\mathcal{P}[|f_*|](z) - |f(z)| \geqslant 0$ for all $z \in \mathbb{D}$, due to the subharmonicity of $|f|$. We also have

$$\mathcal{P}[|f_*|](z) - |f(z)| \to 0 \quad \text{as } |z| \to 1 \tag{1.2}$$

so it is a natural question how fast the convergence in (1.2) can be. It turns out that this is closely related to the modulus of continuity of f, i.e., to the growth of the first derivative of f. The next theorem, also due to Dyakonov (see [2]), gives a precise answer to the above question.

Theorem 1.4. *Let ω be a regular majorant and $f \in A(\mathbb{D})$. Then the following conditions are equivalent:*

(a) $|f_*| \in \Lambda_\omega(\mathbb{T})$ *and there is a constant C such that* $|f_*(\zeta)| - |f(r\zeta)| \leqslant C\omega(1 - r)$ *for all $\zeta \in \mathbb{T}$ and all $0 < r < 1$.*

(b) *There is a constant C such that $|f_*(\zeta)| - |f(z)| \leqslant C\omega(|\zeta - z|)$ for all $\zeta \in \mathbb{T}$ and all $z \in \mathbb{D}$.*

(c) $|f_*| \in \Lambda_\omega(\mathbb{T})$ *and there is a constant C such that*

$$\mathcal{P}[|f_*|](z) - |f(z)| \leqslant C\omega(1 - |z|), \quad z \in \mathbb{D}. \tag{1.3}$$

We extend this theorem to the case of harmonic quasiregular mappings, see Theorem 2.3.

2. Proofs

We commence with three quite elementary lemmata.

Lemma 2.1. *If a, b and c are complex numbers such that*

$$|a + b\mathrm{e}^{\mathrm{i}t} + c\mathrm{e}^{-\mathrm{i}t}| \leqslant 1, \quad t \in \mathbb{R},$$

then

$$\big||b| - |c|\big| \leqslant 1 - |a|.$$

Proof. As t runs from 0 to 2π, $b\mathrm{e}^{\mathrm{i}t} + c\mathrm{e}^{-\mathrm{i}t}$ traverses an ellipse whose larger semi-axis is $|b| + |c|$ and the smaller one is $\big||b| - |c|\big|$. Clearly this suffices. □

Lemma 2.2. *If the series*

$$f(r\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} \mathrm{e}^{\mathrm{i}n\theta} \tag{2.1}$$

converges for all $z = r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{D}$ and $|f(z)| \leqslant 1$ for all $z \in \mathbb{D}$, then

$$\Big|c_0 + \frac{1}{2}(c_1\mathrm{e}^{\mathrm{i}t} + c_{-1}\mathrm{e}^{-\mathrm{i}t})\Big| \leqslant 1, \quad t \in \mathbb{R}. \tag{2.2}$$

Proof. We use the following result: if $f \in L^\infty(-\pi, \pi)$, $s_n(f, t)$ is the nth partial sum of its Fourier series and $\sigma_n(f, t) = (n+1)^{-1} \sum_{k=0}^{n} s_k(f, t)$ is the corresponding $(C, 1)$ mean, then $\|\sigma_n(f)\|_\infty \leqslant \|f\|_\infty$. In fact, we need this for $n = 1$ only: taking $f_r(\theta) = f(r\mathrm{e}^{\mathrm{i}\theta})$ we get $|\sigma_1(f_r, t)| \leqslant \|f_r\|_\infty \leqslant 1$ for all real t, which reads as

$$\Big|c_0 + \frac{r}{2}(c_1\mathrm{e}^{\mathrm{i}t} + c_{-1}\mathrm{e}^{-\mathrm{i}t})\Big| \leqslant 1, \quad t \in \mathbb{R},$$

and letting $r \to 1$ we obtain (2.2). □

Combining the above lemmata we deduce the following:

Lemma 2.3. *Under the hypothesis of the previous lemma, we have*

$$\frac{1}{2}\big||c_1| - |c_{-1}|\big| \leqslant 1 - |c_0| = 1 - |f(0)|. \tag{2.3}$$

Proposition 2.1. *Let f be a harmonic function defined on the unit disc and let*

$$f(z) = g(z) + \overline{h(z)}, \quad z \in \mathbb{D},$$

be the canonical representation of f. Assume $|f(z)| \leqslant 1$ for all $z \in \mathbb{D}$. Then

$$\frac{1}{2}\big||g'(0)| - |h'(0)|\big| \leqslant 1 - |f(0)|. \tag{2.4}$$

P r o o f. Since $g'(0) = c_1$ and $h'(0) = \overline{c_{-1}}$, this is an immediate consequence of Lemma 2.3. □

The first key ingredient in our proof of Theorem 1.3 is the following variant of Schwarz lemma for harmonic quasiregular functions, or, to be precise, its Corollary 2.1.

Theorem 2.1. *Assume f is harmonic and k-quasiregular in $\mathbb{D}$, where $0 \leqslant k < 1$, and $|f(z)| \leqslant 1$ for all $z \in \mathbb{D}$. Then*

$$\|f'(z)\| \leqslant \frac{4}{1-k}\frac{1-|f(z)|}{1-|z|}, \quad z \in \mathbb{D}, \tag{2.5}$$

where $f'(z)$ is the derivative of f at z treated as an $\mathbb{R}$-linear operator from $\mathbb{C}$ into $\mathbb{C}$.

P r o o f. It suffices to consider the case $z = 0$ and then apply the inequality to the function $g(w) = f(z + (1 - |z|)w)$. Let $f(z) = g(z) + \overline{h(z)}$ be the canonical representation of f. Since $\|f'(z)\| = |h'(0)| + |h'(0)|$ we have to prove the inequality

$$|g'(0)| + |h'(0)| \leqslant \frac{4}{1-k}(1 - |f(0)|). \tag{2.6}$$

Using (2.4) and Definition 1.1 we have

$$1 - |f(0)| \geqslant \frac{1}{2}(|g'(0)| - |h'(0)|) \geqslant \frac{1-k}{2}\,|g'(0)|$$

and therefore we obtained the estimate

$$|g'(0)| \leqslant \frac{2}{1-k}(1 - |f(0)|).$$

Since $|h'(0)| \leqslant |g'(0)|$, (2.6) follows at once. □

Corollary 2.1. *Assume f is harmonic and k-quasiregular in $\mathbb{D}$, where $0 \leqslant k < 1$, and assume $|f| \in \Lambda_\omega$. Then*

$$\|f'(z)\| \leqslant C_{|f|,\omega} \frac{4}{1-k} \frac{\omega(1-|z|)}{1-|z|}, \quad z \in \mathbb{D}. \tag{2.7}$$

Proof. For a given $z \in \mathbb{D}$ let us consider an auxiliary function

$$g(w) = \frac{1}{M_z} f(z + (1-|z|)w), \quad w \in \mathbb{D},$$

where

$$M_z = \sup_{|z-w| \leqslant 1-|z|} |f(w)|.$$

Clearly $g(w)$ satisfies the conditions of Theorem 2.1, which gives us an estimate of its derivative at $w = 0$:

$$\frac{1}{M_z}(1-|z|)\|f'(z)\| = \|g'(0)\| \leqslant \frac{4}{1-k}(1-|g(0)|) = \frac{4}{1-k}\Big(1 - \frac{|f(z)|}{M_z}\Big).$$

Now (2.7) follows from the above estimate and the obvious inequality

$$M_z - |f(z)| \leqslant \sup_{|w-z| \leqslant 1-|z|} |f(w)| - |f(z)| \leqslant C_{|f|,\omega}\omega(1-|z|).$$

□

The second key result we need is the following theorem, due to Hardy and Littlewood, see [6], Theorem 8.1. In fact, only the "if" part of that theorem is needed.

Theorem 2.2. *Let ω be a fast majorant. A harmonic complex valued function f defined on $\mathbb{D}$ belongs to Λ_ω if and only if there is a constant C such that*

$$\|f'(z)\| \leqslant C \frac{\omega(1-|z|)}{1-|z|}, \quad z \in \mathbb{D}.$$

Clearly, Theorem 1.3 follows immediately from Corollary 2.1 and Theorem 2.2. The next theorem is a generalization of Theorem 1.4.

Theorem 2.3. *Let ω be a regular majorant and assume $f \in C(\overline{\mathbb{D}})$ is harmonic and k-quasiregular in $\mathbb{D}$, where $0 \leqslant k < 1$. Then the following conditions are equivalent:*

(a) $|f_*| \in \Lambda_\omega(\mathbb{T})$ *and there is a constant C such that* $|f_*(\zeta)| - |f(r\zeta)| \leqslant C\omega(1-r)$ *for all $\zeta \in \mathbb{T}$ and all $0 < r < 1$.*

(b) *There is a constant C such that* $|f_*(\zeta)| - |f(z)| \leqslant C\omega(|\zeta - z|)$ *for all $\zeta \in \mathbb{T}$ and all $z \in \mathbb{D}$.*

(c) $|f_*| \in \Lambda_\omega(\mathbb{T})$ *and there is a constant C such that*

$$\mathcal{P}[|f_*|](z) - |f(z)| \leqslant C\omega(1-|z|), \quad z \in \mathbb{D}. \tag{2.8}$$

(d) $f \in \Lambda_\omega(\mathbb{D})$.

Proof. The implication (a) $\Rightarrow$ (b) is easy and valid without the assumption that f is a hqr mapping. The implication (b) $\Rightarrow$ (c) rests on subharmonicity of $|f|$ and the assumption that ω is slow. For details on both of these implications see [6], Chapter 8.

Now we assume (c) holds and set $h = \mathcal{P}[|f_*|]$. Since $|f|$ is subharmonic in $\mathbb{D}$ we have $|f| \leqslant h$ on $\mathbb{D}$. Also, $|f_*| \in \Lambda_\omega(\mathbb{T})$ implies $h \in \Lambda_\omega(\mathbb{D})$ by Theorem 8.3 from [6]. Therefore, using (2.8), we see that for any two points z and w in $\mathbb{D}$ we have

$$\begin{aligned}|f(w)| - |f(z)| &\leqslant h(w) - |f(z)| = h(w) - h(z) + [h(z) - f(z)] \\ &\leqslant C\omega(|w-z|) + C\omega(|1-|z|).\end{aligned}$$

Now assume $|w-z| < 1-|z|$. Then (2.9) gives $|f(w)|-|f(z)| \leqslant C\omega(1-|z|)$. However, interchanging z and w in (2.9) we also obtain

$$|f(z)| - |f(w)| \leqslant C\omega(1-|z|) + C\omega(1-|w|) \leqslant C\omega(1-|z|)$$

since $1-|w| < 2(1-|z|)$ for $|w-z| < 1-|z|$. Hence we proved

$$\big||f(w)| - |f(z)|\big| \leqslant C\omega(1-|z|), \quad |w-z| < 1-|z|. \tag{2.9}$$

Next, we fix $z \in \mathbb{D}$ and apply Theorem 2.1 to an auxiliary function

$$g(w) = \frac{f(z+(1-|z|)w) - f(z)}{C\omega(1-|z|)}, \quad w \in \mathbb{D},$$

estimate (2.10) ensures $|g(w)| \leqslant 1$ for all $w \in \mathbb{D}$. The estimate of $\|g'(0)\|$ gives

$$\|f'(z)\| \leqslant \frac{4C}{1-k}\frac{\omega(1-|z|)}{1-|z|}. \tag{2.10}$$

This estimate, combined with Theorem 2.2, gives (d). Note that this is the only implication where we have used quasiregularity. Finally, implication (d) $\Rightarrow$ (a) is trivial. $\square$

For extensions of Dyakonov's theorems in other directions, see [6].

References

[1] *S. Chen, S. Ponnusamy, X. Wang*: On planar harmonic Lipschitz and planar harmonic Hardy classes. Ann. Acad. Sci. Fenn., Math. *36* (2011), 567–576.

[2] *K. M. Dyakonov*: Equivalent norms on Lipschitz-type spaces of holomorphic functions. Acta Math. *178* (1997), 143–167.

[3] *K. M. Dyakonov*: Holomorphic functions and quasiconformal mappings with smooth moduli. Adv. Math. *187* (2004), 146–172.

[4] *M. Mateljević*: Distortions of quasiregular mappings and equivalent norms on Lipschitz-type spaces. Abstr. Appl. Anal. *2014* (2014), Article ID 895074, 20 pages.

[5] *M. Pavlović*: On K. M. Dyakonov's paper "Equivalent norms on Lipschitz-type spaces of holomorphic functions". Acta Math. *183* (1999), 141–143.

[6] *M. Pavlović*: Function Classes on the Unit Disc. An Introduction. De Gruyter Studies in Mathematics 52, De Gruyter, Berlin, 2014.

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