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CERTAIN DECOMPOSITIONS OF MATRICES OVER ABELIAN RINGS

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Abstract. A ring R is (weakly) nil clean provided that every element in R is the sum of a (weak) idempotent and a nilpotent. We characterize nil and weakly nil matrix rings over abelian rings. Let R be abelian, and let $n \in \mathbb{N}$. We prove that $M_n(R)$ is nil clean if and only if $R/J(R)$ is Boolean and $M_n(J(R))$ is nil. Furthermore, we prove that R is weakly nil clean if and only if R is periodic; $R/J(R)$ is $\mathbb{Z}_3$, B or $\mathbb{Z}_3 \oplus B$ where B is a Boolean ring, and that $M_n(R)$ is weakly nil clean if and only if $M_n(R)$ is nil clean for all $n \geqslant 2$.

Keywords: idempotent element; nilpotent element; nil clean ring; weakly nil clean ring

MSC 2010: 16S34, 16U10, 16E50

Let R be a ring with an identity. An element a in a ring is called weak idempotent if a or $-a$ is an idempotent. An element in R is (weakly) nil clean provided that it is the sum of a (weak) idempotent and a nilpotent element [3], [5], [9], [10], and [12]. A ring R is (weakly) nil clean if every element in R is (weakly) nil clean. Many fundamental properties about commutative (weakly) nil clean rings were obtained in [1] and [2], and weakly nil clean rings were studied by Breaz et al. in [5].

In [10], Question 3, Diesl asked: Let R be a nil clean ring, and let n be a positive integer. Is $M_n(R)$ nil clean? In [4], Theorem 3, Breaz et al. proved their main theorem: for a field K, $M_n(K)$ is nil clean if and only if $K \cong \mathbb{Z}_2$. They also asked if this result could be extended to division rings. As a main result in [11], Koşan et al. gave an affirmative answer to this problem. They showed the preceding equivalence holds for any division ring. We shall extend [4], Theorem 3, and [11], Theorem 3, to an arbitrary abelian ring.

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A ring R is abelian if every idempotent in R is central. In this note, we are concerned with nil and weakly nil clean matrix rings over abelian rings, and investigate when a matrix over an abelian ring can be written as the sum of a (weak) idempotent matrix and a nilpotent matrix. We prove that if R is abelian then $M_n(R)$ is nil clean if and only if $R/J(R)$ is Boolean and $M_n(J(R))$ is nil. This extends the main results of Breaz et al. [4] and that of Koşan et al. [11]. A ring R is periodic if for any $a \in R$ there exist distinct $m, n \in \mathbb{N}$ such that $a^m = a^n$. Furthermore, we prove that if R is abelian then R is weakly nil clean if and only if R is periodic; $R/J(R)$ is $\mathbb{Z}_3$, B or $\mathbb{Z}_3 \oplus B$ where B is a Boolean ring, and that $M_n(R)$ is weakly nil clean if and only if $M_n(R)$ is nil clean for all $n \geqslant 2$.

Throughout, all rings are associative with an identity. We use $M_n(R)$ and $T_n(R)$ to stand for the rings of all $n \times n$ full matrices and triangular matrices over R, respectively. The Jacobson radical of R is denoted by $J(R)$, $\mathrm{Id}(R) = \{e \in R\colon e^2 = e \in R\}$, $-\mathrm{Id}(R) = \{e \in R\colon e^2 = -e \in R\}$, $U(R)$ is the set of all units in R, and $N(R)$ is the set of all nilpotent elements in R.

Recall that a ring R is an exchange ring if for every $a \in R$ there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. Clearly, every nil clean ring is an exchange ring. Let $BM(R)$ denote the Brown-McCoy radical of the ring R. Then $BM(R)$ is just the intersection of all maximal two-sided ideals of R. Obviously, $J(R) \subseteq BM(R)$. In general, they are not the same, e.g., $\mathrm{End}_F(V)$ and V is an infinite-dimensional vector space over a field F. A ring R is right (left) quasi-duo if every right (left) maximal ideal of R is two-sided. By Burgess and Stephenson [6], Theorem 3.1, (ii) (b), every abelian exchange ring R is a left and right quasi-duo ring. This would imply immediately the equality of the Brown-McCoy radical and the Jacobson radical of R. That is,

Lemma 1. *Let R be an abelian exchange ring. Then $BM(R) = J(R)$.*

Lemma 2. *Let R be a ring, and let $n \in \mathbb{N}$. Then the following assertions are equivalent:*

(1) *$M_n(R)$ is nil clean and R has no nontrivial idempotents.*

(2) *$R/J(R) \cong \mathbb{Z}_2$ and $M_n(J(R))$ is nil.*

Proof. (1) $\Rightarrow$ (2) In view of [10], Proposition 3.16, $J(M_n(R))$ is nil, and then so is $M_n(J(R))$.

Let $a \in R$. By hypothesis, $M_n(R)$ is nil clean. If $n = 1$, then R is nil clean. Then $a \in N(R)$ or $a - 1 \in N(R)$. This shows that $a \in U(R)$ or $1 - a \in U(R)$, and so R is local. That is, $R/J(R)$ is a division ring. As $R/J(R)$ is nil clean, we easily see that $R/J(R) \cong \mathbb{Z}_2$. We now assume that $n \geqslant 2$. Then there exist an idempotent

$E \in M_n(R)$ and a nilpotent $W \in M_n(R)$ such that

$$I_n + \begin{pmatrix} a & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} = E + W.$$

Set $U = -I_n + W$. Then $U \in GL_n(R)$. Hence,

$$U^{-1} \begin{pmatrix} a & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} = U^{-1}E + I_n = (U^{-1}EU)U^{-1} + I_n.$$

Set $F = U^{-1}EU$. Then $F = F^2 \in M_n(R)$, and that

$$(I_n - F)U^{-1} \begin{pmatrix} a & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} = I_n - F.$$

By computing the left side of this equality, we may write

$$I_n - F = \begin{pmatrix} e & 0 & \dots & 0 \\ * & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \dots & 0 \end{pmatrix}.$$

As R possesses no nontrivial idempotents, $e = 0$ or 1. If $e = 0$, then $I_n - F$ is both idempotent and nilpotent. This shows that $I_n - F = 0$, and so $E = I_n$. This shows that

$$\begin{pmatrix} a & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} = W$$

is nilpotent; hence $a \in R$ is nilpotent. Thus, $1 - a \in U(R)$.

If $e = 1$, then

$$F = \begin{pmatrix} 0 & 0 & \dots & 0 \\ * & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \dots & 1 \end{pmatrix}.$$

Write

$$U^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

$\alpha \in R$, $\beta \in M_{1\times(n-1)}(R)$, $\gamma \in M_{(n-1)\times 1}(R)$, $\delta \in M_{(n-1)\times(n-1)}(R)$. Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x & I_{n-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + I_n,$$

where $x \in M_{(n-1)\times 1}(R)$. Thus, we get

$$\alpha a = 1, \quad \gamma a = x\alpha + \gamma, \quad 0 = x\beta + \delta + I_{n-1}.$$

One easily checks that

$$\begin{pmatrix} 1 & \beta \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & I_{n-1} \end{pmatrix} U^{-1} \begin{pmatrix} 1 & 0 \\ \gamma a & I_{n-1} \end{pmatrix} = \begin{pmatrix} \alpha + \beta\gamma a & 0 \\ 0 & -I_{n-1} \end{pmatrix}.$$

This implies that $u := \alpha + \beta\gamma a \in U(R)$. Hence, $\alpha = u - \beta\gamma a$. It follows from $\alpha a = 1$ that $(u - \beta\gamma a)a = 1$. As R has no nontrivial idempotents, we see that $a(u - \beta\gamma a) = 1$, and so $a \in U(R)$. This shows that $a \in U(R)$ or $1 - a \in U(R)$. Therefore R is local, and then $R/J(R)$ is a division ring. Since $M_n(R)$ is nil clean, we see hence so is $M_n(R/J(R))$. Therefore, $R/J(R) \cong \mathbb{Z}_2$, as desired.

(2) $\Rightarrow$ (1) In light of [4], Theorem 3, $M_n(R/J(R))$ is nil clean.

Since $M_n(R)/J(M_n(R)) \cong M_n(R/J(R))$ and $J(M_n(R)) = M_n(J(R))$ is nil, it follows from [10], Corollary 3.17, that $M_n(R)$ is nil clean, as asserted. □

Example 3. Let K be a field, and let $R = K[x,y]/(x,y)^2$. Then $M_n(R)$ is nil clean if and only if $K \cong \mathbb{Z}_2$. As $J(R) = (x,y)/(x,y)^2$, $R/J(R) \cong K$. Thus, R is a local ring with a nilpotent Jacobson radical. Hence, R has no nontrivial idempotents. Thus, we are done by Lemma 2.

We are now ready to prove

Theorem 4. *Let R be abelian, and let $n \in \mathbb{N}$. Then the following assertions are equivalent:*

(1) *$M_n(R)$ is nil clean.*

(2) *$R/J(R)$ is Boolean and $M_n(J(R))$ is nil.*

P r o o f. (1) $\Rightarrow$ (2) Clearly, $M_n(J(R))$ is nil. Let M be a maximal ideal of R, and let $\varphi_M\colon R \to R/M$. Since $M_n(R)$ is nil clean, it follows by [10], Proposition 3.4, that

$M_n(R)$ is clean. Thus, $M_n(R)$ is an exchange ring in terms of [13], Proposition 1.8. By [13], Proposition 1.10, R is an exchange ring; hence, so is R/M. In light of [13], Corollary 1.3, we see that every idempotent lifts modulo M, and hence R/M is abelian. Therefore R/M is an exchange ring with all idempotents central. In view of [8], Lemma 17.2.5, R/M is local, and so R/M has only trivial idempotents. It follows from Lemma 2 that $R/M/J(R/M) \cong \mathbb{Z}_2$. Write $J(R/M) = K/M$. Then K is a maximal ideal of R, and $M \subseteq K$. This implies that $M = K$; hence, $R/M \cong \mathbb{Z}_2$. Construct a map $\varphi_M\colon\ R/BM(R) \to R/M$, $r + BM(R) \mapsto r + M$. Here, $BM(R)$ is the Brown-McCoy radical of R. Then

$$\bigcap_M \operatorname{Ker} \varphi_M = \bigcap_M \{r + BM(R)\colon\ r \in M\} = 0,$$

and so $R/BM(R)$ is isomorphic to a subdirect product of some $\mathbb{Z}_2$. Thus, $R/BM(R)$ is Boolean. In light of Lemma 1, $R/J(R)$ is Boolean, as desired.

(2) $\Rightarrow$ (1) Since $R/J(R)$ is Boolean, it follows by [4], Corollary 6, that $M_n(R/J(R))$ is nil clean. That is, $M_n(R)/J(M_n(R))$ is nil clean. But $J(M_n(R)) = M_n(J(R))$ is nil. Therefore we complete the proof by virtue of [10], Corollary 3.17. □

We note that the "(2) $\Rightarrow$ (1)" in Theorem 4 always holds, but "abelian" condition is necessary in "(1) $\Rightarrow$ (2)". Let $R = M_n(\mathbb{Z}_2)$, $n \geqslant 2$. Then R is nil clean. But $R/J(R)$ is not Boolean. Here, R is not abelian.

Corollary 5. *Let R be (left) right quasi-duo, and let $n \in \mathbb{N}$. Then the following conditions are equivalent:*

(1) *$M_n(R)$ is nil clean.*

(2) *$R/J(R)$ is Boolean and $M_n(J(R))$ is nil.*

P r o o f. (1) $\Rightarrow$ (2) By hypothesis, $M_n(R)$ is nil clean, and then $M_n(R)$ is exchange. This implies that R is exchange. Set $S = R/J(R)$. Let $\overline{e} \in S$ be an idempotent and let $\overline{x} \in S$. Then we may assume that $e \in R$ is an idempotent. In view of [14], Lemma 2.3, $ex(1-e), (1-e)xe \in J(R)$. Hence, $\overline{ex} = \overline{exe} = \overline{xe}$. That is, S is abelian. As $M_n(R)$ is nil clean, so is $M_n(S)$. In light of Theorem 4, $S/J(S)$ is Boolean. But $J(S) = 0$, so we proved that $R/J(R)$ is Boolean. Furthermore, $M_n(J(R))$ is nil, by virtue of [10], Corollary 3.17.

(2) $\Rightarrow$ (1) As $R/J(R)$ is abelian, it follows from Theorem 4 that $M_n(R/J(R))$ is nil clean. By hypothesis, $J(M_n(R))$ is nil, thus yielding the result, by virtue of [10], Corollary 3.17. □

We note that the class of (left) right quasi-duo rings is much larger. Evidently, commutative rings, duo rings, uniquely clean rings, uniquely π-clean rings and strongly nil clean rings are all (left) right quasi-duo. If $R/J(R)$ is commutative, then $M_n(R)$ is nil clean if and only if $M_n(J(R))$ is nil. Since R is (left) right quasi-duo, we are through by Corollary 5.

Corollary 6. *Let R be a commutative ring, and let $n \in \mathbb{N}$. Then the following conditions are equivalent:*
(1) *$M_n(R)$ is nil clean.*
(2) *$R/J(R)$ is Boolean and $J(R)$ is nil.*
(3) *For any $a \in R$, $a - a^2 \in R$ is nilpotent.*

Proof. (1) $\Rightarrow$ (3) Let $a \in R$. In view of Theorem 4, $a - a^2 \in J(R)$. Since R is commutative, $J(R)$ is nil if and only if $J(M_n(R))$ is nil. Therefore $a - a^2 \in R$ is nilpotent.

(3) $\Rightarrow$ (2) Clearly, $R/J(R)$ is Boolean. For any $a \in J(R)$, we have $(a - a^2)^n = 0$ for some $n \geqslant 1$. Hence, $a^n(1-a)^n = 0$, and so $a^n = 0$. This implies that $J(R)$ is nil.

(2) $\Rightarrow$ (1) As R is commutative, we see that $M_n(J(R))$ is nil. This completes the proof, by Theorem 4. □

In [4], Corollary 7, Breaz et al. proved that if R is any commutative nil clean ring then $M_n(R)$ is nil clean. We indeed have

Corollary 7. *A commutative ring R is nil clean if and only if $M_n(R)$ is nil clean.*

Proof. One direction is obvious by [10], Corollary 7. Suppose that $M_n(R)$ is nil clean. In view of Corollary 5, $R/J(R)$ is Boolean, and $J(R)$ is nil. Therefore R is nil clean, by [10], Corollary 3.17. □

Example 8. Let $m, n \in \mathbb{N}$. Then $M_n(\mathbb{Z}_m)$ is nil clean if and only if $m = 2^r$ for some $r \in \mathbb{N}$. Write $m = p_1^{r_1} \dots p_s^{r_s}$, where $p_1, \dots, p_s$ are distinct primes, $r_1, \dots, r_s \in \mathbb{N}$. Then $Z_m \cong Z_{p_1^{r_1}} \oplus \dots \oplus Z_{p_m^{r_s}}$. In light of Corollary 7, $M_n(\mathbb{Z}_m)$ is nil clean if and only if $s = 1$ and $Z_{p_1^{r_1}}$ is nil clean. Therefore we are done by Lemma 2.

We now pass to consideration of the weakly nil clean rings. For the reader's convenience, we include the main theorem of [5].

Lemma 9 ([5], Theorem 20). *Let D be a division ring, and let $n \in \mathbb{N}$. Then $M_n(D)$ is weakly nil clean if and only if*
(1) *$D \cong \mathbb{Z}_2$; or*
(2) *$D \cong \mathbb{Z}_3$ and $n = 1$.*

Lemma 10. *Let R be a ring, and let $n \in \mathbb{N}$. Then $M_n(R)$ is weakly nil clean and R has no nontrivial idempotents if and only if*
(1) *$M_n(J(R))$ is nil;*
(2) *$R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$, $n = 1$.*

Proof. $\Rightarrow$: In view of [10], Proposition 3.16, $M_n(J(R)))$ is nil.

Since $M_n(R)$ is weakly nil clean, it is clean by [5], Corollary 8, and then R is exchange. As in the proof of Lemma 2, R is local. Clearly, $M_n(R/J(R))$ is weakly nil clean. It follows by Lemma 9 that $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$ and $n = 1$.

$\Leftarrow$: In view of Lemma 9, $M_n(R/J(R))$ is weakly nil clean. Therefore we complete the proof by [5], Proposition 3.15. $\square$

We have at our disposal all the information necessary to prove the following result.

Theorem 11. *Let R be abelian. Then*
(1) *R is weakly nil clean if and only if*
  (a) *R is periodic;*
  (b) *$R/J(R)$ is $\mathbb{Z}_3$, B or $\mathbb{Z}_3 \oplus B$ where B is a Boolean ring.*
(2) *$M_n(R)$ is weakly nil clean if and only if $M_n(R)$ is nil clean for all $n \geqslant 2$.*

Proof. (1) $\Rightarrow$: (a) Let $a \in R$. Then there exists an idempotent $e \in R$ such that $a - e$ or $a + e \in N(R)$. Hence, $a - a^2$ or $a + a^2 \in N(R)$. This shows that $a^n = a^{n+1}f(a)$ for some $n \in \mathbb{N}$, where $f(t) \in \mathbb{Z}[t]$. By virtue of Chacron's theorem, R is periodic (see [7]). (b) This could be proved by [5], Theorem 12. We include an alternative proof. Let M be a maximal ideal of R, and let $\varphi_M\colon R \to R/M$. Since R is weakly nil clean, it is clean, by [5], Corollary 7, and then R/M is an exchange ring with all idempotents central. As in the proof of Theorem 4, R/M has only trivial idempotents. According to Lemma 10, $R/M/J(R/M) \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. Write $J(R/M) = K/M$. Then K is a maximal ideal of R, and so $M = K$. This shows that $R/M \cong \mathbb{Z}_2, \mathbb{Z}_3$.

Let P and Q be distinct maximal ideals of R such that $R/P, R/Q \cong \mathbb{Z}_3$. As $P + Q = R$, by the Chinese remainder theorem we get

$$R/(P \cap Q) \cong R/P \oplus R/Q \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3.$$

Since R is weakly nil clean, so is $R/(P \cap Q)$. This shows that $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ is weakly nil clean. Hence, $(1, -1)$ or $(-1, 1)$ is nil clean in $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, a contradiction. Thus, there is at most one maximal ideal M such that $R/M \cong \mathbb{Z}_3$. Similarly to the discussion in Theorem 4, $R/J(R)$ is isomorphic to the subdirect product of finitely many $\mathbb{Z}_2$ and/or one $\mathbb{Z}_3$. Accordingly, for any $\overline{a} \in R/J(R), \overline{a} = \overline{a}^2$ or $-\overline{a^2}$. In light of [1], Theorem 1.12, $R/J(R)$ is $\mathbb{Z}_3$, B or $\mathbb{Z}_3 \oplus B$ where B is a Boolean ring.

$\Leftarrow$: In view of Lemma 10, $R/J(R)$ is weakly nil clean. Since R is periodic, $J(R)$ is nil, and therefore R is weakly nil clean, by [5], Theorem 2.

(2) $\Rightarrow$: Let M be a maximal ideal of R and $n \geqslant 2$. Then $M_n(R/M)$ is weakly nil clean. As in the previous discussion, it follows by Lemma 10 that $R/M \cong \mathbb{Z}_2$. Thus, $R/J(R)$ is isomorphic to the subdirect product of some $\mathbb{Z}_2$'s. Hence, $R/J(R)$ is Boolean. Clearly, $J(M_n(R))$ is nil. Accordingly, $M_n(R)$ is nil clean, by Theorem 4.

$\Leftarrow$: This is obvious. □

As in the proof of Corollary 5, applying Theorem 11 to $R/J(R)$, we now derive

Corollary 12. *Let R be (left) right quasi-duo. Then*

(1) *R is weakly nil clean if and only if*
   (a) *R is periodic;*
   (b) *$R/J(R)$ is $\mathbb{Z}_3$, B or $\mathbb{Z}_3 \oplus B$ where B is a Boolean ring;*

(2) *$M_n(R)$ is weakly nil clean if and only if $M_n(R)$ is nil clean for all $n \geqslant 2$.*

Corollary 13. *Let R be a commutative ring. Then*

(1) *R is weakly nil clean if and only if for any $a \in R$, $a - a^2$ or $a + a^2$ is nilpotent.*

(2) *$M_n(R)$, $n \geqslant 2$, is weakly nil clean if and only if for any $a \in R$, $a - a^2$ is nilpotent.*

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