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Abstract. Let $\mathfrak{M}$ and $\mathfrak{R}$ be algebras of subsets of a set Ω with $\mathfrak{M} \subset \mathfrak{R}$, and denote by $E(\mu)$ the set of all quasi-measure extensions of a given quasi-measure μ on $\mathfrak{M}$ to $\mathfrak{R}$. We show that $E(\mu)$ is order bounded if and only if it is contained in a principal ideal in $ba(\mathfrak{R})$ if and only if it is weakly compact and $\operatorname{extr} E(\mu)$ is contained in a principal ideal in $ba(\mathfrak{R})$. We also establish some criteria for the coincidence of the ideals, in $ba(\mathfrak{R})$, generated by $E(\mu)$ and $\operatorname{extr} E(\mu)$.

Keywords: linear lattice; ideal; order bounded; ideal dominated; order unit; Banach lattice; *AM*-space; convex set; extreme point; weakly compact; additive set function; quasi-measure; atomic; extension

Classification: 06F20, 28A12, 28A33, 46A55, 46B42

1. Introduction

By a *quasi-measure* we mean a positive additive function on an algebra of sets. Let $\mathfrak{M}$ and $\mathfrak{R}$ be algebras of subsets of a set Ω with $\mathfrak{M} \subset \mathfrak{R}$ and let μ be a quasi-measure on $\mathfrak{M}$. The 'sets' appearing in the title of the paper[1] are the convex set $E(\mu)$ of all quasi-measure extensions of μ to $\mathfrak{R}$ and the set $\operatorname{extr} E(\mu)$ of its extreme points. These sets have been studied in many earlier papers by the author, including [4]–[7]. So far, their topological and linear-topological properties as subsets of the dual Banach lattice $ba(\mathfrak{R})$ have been of main concern. A systematic presentation of most of the results obtained is given in the memoir [8].

This paper is a continuation of [9]. Its starting point is the following consequence of classical results: if $E(\mu)$ is order bounded, then it is weakly compact ([9, Proposition 2(c)]). To fill the gap between order boundedness and weak compactness, we introduce, in Section 2, a property of subsets of a general linear lattice X, which we call ideal domination. (By definition, $V \subset X$ is *ideal dominated* if it is contained in a principal ideal in X.) This property is weaker than order boundedness, in general, but coincides with it for compact convex subsets of X, the topology involved being compatible with the linear structure and order of X (Theorem 1 in Section 2). Compactness alone does not suffice here; see the passage introducing Proposition 1 in Section 2. It follows from Theorem 1

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[1]Some results of Section 2 of the paper were presented at the 43rd Winter School in Abstract Analysis (Svratka, Czech Republic, 2015).

that $E(\mu)$ is order bounded if and only if it is ideal dominated (Theorem 4 in Section 6). A further equivalent condition is the following one: $E(\mu)$ is weakly compact and $\operatorname{extr} E(\mu)$ is ideal dominated (Theorem 6 in Section 6).

Order boundedness of $E(\mu)$ is equivalent to that of $\operatorname{extr} E(\mu)$, according to [9, Theorem 2, (i) $\Leftrightarrow$ (ii)]. This is still true for ideal domination provided μ is nonatomic (see Theorem 5 in Section 6), but not in general. Therefore, we establish some criteria for ideal domination of $\operatorname{extr} E(\mu)$. One of them is concerned with the case where μ has finite range or, more generally, is atomic (Proposition 3 and Theorem 3 in Section 5). Another one applies in the situation where $\mathfrak{M}$, $\mathfrak{R}$ and μ are related in a special way (Theorem 7 in Section 6).

Finally, Section 7 is concerned with the question when the ideals, in $ba(\mathfrak{R})$, generated by $E(\mu)$ and $\operatorname{extr} E(\mu)$ coincide. A necessary and sufficient condition is provided in the case where μ has finite range or is atomic (Proposition 4 and Theorem 8). The answer is, moreover, affirmative when $\mathfrak{R}$ is generated, as an algebra, by $\mathfrak{M}$ and a finite family of subsets of Ω, and μ is arbitrary (Proposition 6(b)).

Many results of [8] and [9] are applied extensively in the paper. We also frequently appeal to the Baire category theorem, both for compact topological spaces and complete metric spaces (see the proofs of Theorem 1 in Section 2, and of Lemma 2 in Section 4 and Theorem 5 in Section 6, respectively).

The measure-theoretic notation and terminology we use are explained in Section 3. They are mostly standard and coincide with those of [8]. Section 3 also contains a few auxiliary results on $E(\mu)$ and $\operatorname{extr} E(\mu)$. More auxiliary results on these sets are presented in Section 4.

2. Ideal domination in linear lattices and in Banach lattices

Let X be a real linear lattice (= Riesz space in the terminology of [2]), with the order and lattice operations denoted by $\leq$ and $\wedge$, $\vee$, respectively. As usual, $|x|$ stands for the modulus or absolute value of $x \in X$ and X_+ for the positive cone of X.

The *order interval* $[x, y]$, where $x, y \in X$ and $x \leq y$, is the set

$$\{z \in X : x \leq z \leq y\}.$$

Let V be a subset of X. Recall that V is *order bounded* if $V \subset [x, y]$ for some x and y as above. We denote by A_V the ideal in X generated by V. The notation $A_{\{x\}}$, where $x \in X$, is abbreviated to A_x. Such ideals are called *principal*. We have

$$A_x = \bigcup_{n \in \mathbb{N}} [-n|x|, n|x|].$$

We shall tacitly make use of this simple formula, combined with the Baire category theorem, in some proofs.

We call V *ideal dominated* if $V \subset A_x$ for some $x \in X$. Clearly, every principal ideal is ideal dominated. Also, every order interval $[x, y]$ is ideal dominated since

$x, y \in A_{|x| \vee |y|}$. Note that X is itself ideal dominated if and only if it has a (*strong*) *order unit* e, i.e., $e \in X$ and for every $x \in X$ there exists $n \in \mathbb{N}$ with $|x| \le ne$.

We start by a result which will be used in establishing Proposition 2 in this section and Theorem 4 in Section 6. For a special case see [2, Chapter 6, Exercise 5].

Theorem 1. *Let τ be a linear topology on X with X_+ closed. For a compact convex subset W of X the following two conditions are equivalent:*

(i) *W is order bounded;*
(ii) *W is ideal dominated.*

Proof: Suppose (ii) holds. The order intervals in X being τ-closed, the Baire category theorem for compact spaces applied in W yields a nonempty relatively τ-open and order bounded subset U of W. A translation argument allows us to assume that $0 \in U$. Using the continuity of the mapping

$$[0,1] \ni t \longmapsto tx \in W, \quad x \in W,$$

we can find for each $x \in W$ some $0 < \varepsilon_x < 1$ with $\varepsilon_x x \in U$. We then have

$$W = \bigcup_{x \in W} \left(\frac{1}{\varepsilon_x} U \right) \cap W.$$

The sets $(tU) \cap W$, where $t > 1$, being relatively τ-open in W, there exist $x_1, \ldots, x_n \in W$ with

$$W \subset \bigcup_{i=1}^{n} \frac{1}{\varepsilon_{x_i}} U.$$

This yields (i). □

Clearly, the compactness assumption in Theorem 1 cannot be dispensed with. That this is also the case for the convexity assumption is seen from Theorem 2, (ii) ⇒ (i), and Proposition 1 below. The latter will be also used in establishing Proposition 2 in this section and Lemma 1 in Section 3. Needless to say, Proposition 1 is surely known.

Proposition 1. *Every countable subset V of a Banach lattice X is ideal dominated.*

Proof: Let $V = \{x_1, x_2, \ldots\}$. We may assume that $0 \notin V$. Set

$$x = \sum_{i=1}^{\infty} 2^{-i} \frac{|x_i|}{\|x_i\|}.$$

Clearly, we then have $V \subset A_x$. □

The next result is also known. The equivalence of conditions (i) and (ii) thereof is due to V. Schlotterbeck (see [14, Theorem IV.2.8]). The original proof of the implication (ii) $\Rightarrow$ (i) is somewhat involved. Therefore, we shall present below a simple and elementary proof based on a known idea (see [1, proof of Theorem 1]; cf. also [11, proof of Theorem 5]). The implication (i) $\Rightarrow$ (iii) is a consequence of standard results (see [15, Lemma 9.23]). The implication (ii) $\Rightarrow$ (i) will be used in the proof of Proposition 2 in this section. For the definition of an *AM*-space we refer the reader to [14, Definition II.7.1] or [15, Definition 9.1(i)].

Theorem 2. *For a Banach lattice X the following three conditions are equivalent:*

(i) *X is isomorphic to an AM-space;*
(ii) *every sequence (x_n) in X such that $\|x_n\| \to 0$ is order bounded;*
(iii) *every relatively compact subset of X is order bounded.*

Proof of the implication (ii) $\Rightarrow$ (i): Set

$$M = \sup\Big\{\||x_1| \vee \ldots \vee |x_k|\| : x_1, \ldots, x_k \in X,\ \max_{1\le i\le k} \|x_i\| \le 1,\ \text{and } k \in \mathbb{N}\Big\}.$$

We claim that $M < \infty$. Otherwise, for each $s \in \mathbb{N}$, we could find $x_1^s, \ldots, x_{k_s}^s \in X$ with $\||x_1^s| \vee \ldots \vee |x_{k_s}^s|\| > s^2$ and $\|x_i^s\| \le 1$ for all $1 \le i \le k_s$. Considering the sequence

$$x_1^1,\ \ldots,\ x_{k_1}^1,\ \frac{1}{2}x_1^2,\ \ldots,\ \frac{1}{2}x_{k_2}^2,\ \ldots,$$

we then see that (ii) fails, and so the claim is established. Set, for $x \in X$,

$$\|x\|' = \inf\Big\{\max_{1\le i\le k} \|x_i\| : x_1,\ \ldots,\ x_k \in X_+,\ |x| \le x_1 \vee \ldots \vee x_k,\ \text{and } k \in \mathbb{N}\Big\}.$$

As easily seen, $\|\cdot\|'$ is an M-norm in X and

$$\|x\|' \le \|x\| \le M\|x\|' \quad \text{for all } x \in X,$$

and so we are done. □

We note that there are straightforward examples showing that condition (ii) of Theorem 2 fails for $X = l_p$, where $1 \le p < \infty$. Indeed, set $x_n = t_n e_n$ for $n \in \mathbb{N}$, where (e_n) is the standard basis of l_p and t_n are real numbers with $t_n \to 0$ and $(t_n) \notin l_p$.

For a result related to Theorem 2, (i) $\Leftrightarrow$ (iii), see [16, Exercise 122.8].

Part (a) of the next result is in contrast with [9, Proposition 1], which implies that a compact convex set W of a linear lattice X equipped with a locally convex topology τ such that X_+ is closed is order bounded if and only if so is $\operatorname{extr} W$. Similarly, part (b) thereof shows that an analogue of [9, Lemma 1(b)] for ideal domination does not hold.

Proposition 2. *Let X be a Banach lattice nonisomorphic to an AM-space.*

(a) *There exists a compact convex subset W of X which is not ideal dominated but $\overline{\operatorname{extr} W}$ is ideal dominated. In particular, $A_W \ne A_{\overline{\operatorname{extr} W}}$.*

(b) *There exists an ideal dominated convex subset V of X such that $\overline{V}$ is compact but not ideal dominated.*

Proof: By Theorem 2, (ii) $\Rightarrow$ (i), there exists a sequence (x_n) in X such that $\|x_n\| \to 0$ but $\{x_n : n \in \mathbb{N}\}$ is not order bounded. Set

$$V = \operatorname{conv}\{x_n : n \in \mathbb{N}\} \quad \text{and} \quad W = \overline{V}.$$

According to Mazur's theorem [13, II.4.3], W is compact. It follows from Milman's theorem [13, II.10.5] that

$$\operatorname{extr} W \subset \{x_n : n \in \mathbb{N}\} \cup \{0\}.$$

Therefore, $\overline{\operatorname{extr} W}$ and V are both ideal dominated, by Proposition 1. Since W is not order bounded, Theorem 1 implies that it is not ideal dominated either. □

Proposition 2(a) is in contrast with Theorem 6, (ii) $\Rightarrow$ (i), in Section 6.

3. Further notation and measure-theoretic preliminaries

The set of nonzero $\{0, 1\}$-valued additive functions on a Boolean algebra A is denoted by $ult(A)$.

For a set Ω we denote by 2^Ω the family of all subsets of Ω and by $|\Omega|$ the cardinality of Ω.

Throughout the rest of the paper, Ω stands for a nonempty set and $\mathfrak{M}$ for an algebra of subsets of Ω.

Given $\mathfrak{E} \subset 2^\Omega$, we denote by $\mathfrak{E}_b$ the algebra of subsets of Ω generated by $\mathfrak{E}$.

We denote by $ba(\mathfrak{M})$ the Banach lattice of all real-valued bounded additive functions on $\mathfrak{M}$ (see [3, Section 2.2]). By definition, $\|\varphi\| = |\varphi|(\Omega)$ for $\varphi \in ba(\mathfrak{M})$. In addition to the strong topology, $ba(\mathfrak{M})$ is equipped with its weak and weak* topologies; see [3, Section 4.7] for the canonical Banach-lattice predual of $ba(\mathfrak{M})$.

Let $\mu \in ba_+(\mathfrak{M})$. Adapting a general linear-lattice-theoretical terminology (see [2, p. 13]), we say that $\nu \in ba(\mathfrak{M})$ is a *component of* μ if

$$\nu \wedge (\mu - \nu) = 0.$$

We denote by $\mathcal{U}_\mu$ the set of all components of μ which take at most two values. As easily seen (cf. [3, Proposition 5.2.2]), for different $\nu_1, \nu_2 \in \mathcal{U}_\mu$ we have $\nu_1 \wedge \nu_2 = 0$. Therefore, $\mathcal{U}_\mu$ is countable.

We say that $\mu \in ba_+(\mathfrak{M})$ is *nonatomic* provided for every $\varepsilon > 0$ there exists an $\mathfrak{M}$-partition $\{M_1, \dots, M_n\}$ of Ω with $\mu(M_i) < \varepsilon$ for all i (see [3, Definition 5.1.4], where the term *strongly continuous* is used). We say that μ is (*purely*) *atomic* provided $\mu \wedge \nu = 0$ for every nonatomic $\nu \in ba_+(\mathfrak{M})$. According to the Sobczyk–Hammer decomposition theorem [3, Theorem 5.2.7], μ is atomic if and only if $\mu = \sum_{\nu \in \mathcal{U}_\mu} \nu$, while μ is nonatomic if and only if $\mathcal{U}_\mu = \{0\}$. Moreover, $\mu = \mu_1 + \mu_2$, where $\mu_1, \mu_2 \in ba_+(\mathfrak{M})$, μ_1 is atomic and μ_2 is nonatomic. We shall use this decomposition in the proofs of Theorems 6 and 7 in Section 6.

As usual, we associate with $\mu \in ba_+(\mathfrak{M})$ the outer quasi-measure μ^*, defined, for all $E \subset \Omega$, by the formula:

$$\mu^*(E) = \inf\{\mu(M) : E \subset M \in \mathfrak{M}\}.$$

Throughout the rest of the paper, $\mathfrak{R}$ *stands for an algebra of subsets of* Ω *with* $\mathfrak{M} \subset \mathfrak{R}$. Given $\mu \in ba_+(\mathfrak{M})$, we set

$$E(\mu) = \{\varrho \in ba_+(\mathfrak{R}) : \varrho|\mathfrak{M} = \mu\}.$$

It is a classical result that $E(\mu)$ is always nonempty (see [3, Chapter 3]). Moreover, it is, clearly, convex. In some other papers by the author, including [8], the more comprehensive notation $E(\mu, \mathfrak{R})$ instead of $E(\mu)$ is occasionally used.

We shall also need the following notation (see [8, p. 18]). Given $\mu \in ba_+(\mathfrak{M})$, we set

$$\mathfrak{J}_\mu = \{R \in \mathfrak{R} : \text{there exists } M \in \mathfrak{M} \text{ with } R \subset M \text{ and } \mu(M) = 0\}.$$

Clearly, $\mathfrak{J}_\mu$ is an ideal in $\mathfrak{R}$.

The following result will be often applied below.

(D)$'$ *For* $\mu \in ult(\mathfrak{M})$ *we have* $\operatorname{extr} E(\mu) = E(\mu) \cap ult(\mathfrak{R})$.

See [8, p. 19] or [5, p. 396].

We shall also make frequent use of the following two formulas:

$$E\Big(\sum_{j=1}^{n} \mu_j\Big) = \sum_{j=1}^{n} E(\mu_j) \quad \text{for } \mu_1, \ldots, \mu_n \in ba_+(\mathfrak{M}); \tag{1}$$

$$\operatorname{extr} E\Big(\sum_{j=1}^{n} \mu_j\Big) = \sum_{j=1}^{n} \operatorname{extr} E(\mu_j) \quad \text{for } \mu_1, \ldots, \mu_n \in ba_+(\mathfrak{M}) \text{ with } \mu_j \wedge \mu_{j'} = 0 \text{ whenever } j \neq j'. \tag{2}$$

They are immediate consequences of the corresponding parts of [8, Theorem 6.1] or [5, Theorem 1].

Formulas (1) and (2) imply, in view of [2, Theorem 1.2] applied in $ba(\mathfrak{R})$, the next two formulas, respectively:

$$A_{E\left(\sum_{j=1}^{n} \mu_j\right)} = \sum_{j=1}^{n} A_{E(\mu_j)} \quad \text{for } \mu_1, \ldots, \mu_n \in ba_+(\mathfrak{M}); \tag{3}$$

$$A_{\operatorname{extr} E\left(\sum_{j=1}^{n} \mu_j\right)} = \sum_{j=1}^{n} A_{\operatorname{extr} E(\mu_j)} \quad \text{for } \mu_1, \ldots, \mu_n \in ba_+(\mathfrak{M}) \text{ with } \mu_j \wedge \mu_{j'} = 0 \text{ whenever } j \neq j'. \tag{4}$$

They will be used in the proofs of Lemmas 2 and 4 in Section 4 and of Proposition 3 and Theorem 3 in Section 5.

The next assertion will be used in the proofs of Lemma 4 in Section 4 and Theorem 8 in Section 7.

(5) If $\mu_1, \mu_2 \in ba(\mathfrak{M})$ and $\mu_1 \wedge \mu_2 = 0$, then $\varrho_1 \wedge \varrho_2 = 0$ whenever $\varrho_1 \in E(\mu_1)$ and $\varrho_2 \in E(\mu_2)$.

This holds, since $\mu_1 \wedge \mu_2 = 0$ if and only if for every $\varepsilon > 0$ there exists $M \in \mathfrak{M}$ with $\mu_1(M) + \mu_2(M^c) < \varepsilon$ (see [3, Theorem 2.2.1(7)]).

4. Auxiliary results on $E(\mu)$ and $\operatorname{extr} E(\mu)$

The following lemma will be used in establishing Proposition 3 and Theorem 3 in Section 5.

Lemma 1. *Let $\mu \in ult(\mathfrak{M})$. Then the following two conditions are equivalent:*

(i) $\operatorname{extr} E(\mu)$ *is ideal dominated;*

(ii) $\operatorname{extr} E(\mu)$ *is countable.*

Proof: The implication (ii) $\Rightarrow$ (i) holds, by Proposition 1.

To get a contradiction, suppose that (ii) fails, but (i) holds. Then there exists an uncountable subset $\mathcal{E}$ of $\operatorname{extr} E(\mu)$ and $\tau \in ba_+(\mathfrak{R})$ such that $\tau \geq \pi$ for each $\pi \in \mathcal{E}$. In view of (D)$'$, this implies $\tau \geq \sum_{\pi \in \mathcal{F}} \pi$ whenever $\mathcal{F}$ is a finite subset of $\mathcal{E}$. Hence $\tau(\Omega) = \infty$, which is impossible. □

We continue with a lemma which will be used in the proof of Theorem 3 in Section 5.

Lemma 2. *Suppose $\mu, \mu_j \in ba_+(\mathfrak{M})$ are such that $\sum_{j=1}^{\infty} \mu_j = \mu$ and $\mu_j \wedge \mu_{j'} = 0$ whenever $j \neq j'$. Then the following two conditions are equivalent:*

(i) $\operatorname{extr} E(\mu)$ *is ideal dominated;*

(ii) $\operatorname{extr} E(\mu_j)$ *is ideal dominated for each $j \in \mathbb{N}$ and there exists $n \in \mathbb{N}$ such that $\operatorname{extr} E\big(\sum_{j=n+1}^{\infty} \mu_j\big)$ is order bounded.*

Proof: That (ii) implies (i) is clear, in view of formula (4). By the same formula, (i) implies the first part of condition (ii). According to [8, Proposition 4.4(b)] or [4, Proposition 1(b)], $\operatorname{extr} E(\mu)$ is closed in $ba(\mathfrak{R})$. Thus, by an application of the Baire category theorem combined with (i), there exist $\pi \in \operatorname{extr} E(\mu)$, $\varepsilon > 0$ and $\tau \in ba_+(\mathfrak{R})$ such that

$$\{\pi' \in \operatorname{extr} E(\mu) : \|\pi - \pi'\| < \varepsilon\} \subset [0, \tau].$$

Fix $n \in \mathbb{N}$ with $\sum_{j=n+1}^{\infty} \mu_j(\Omega) < \varepsilon/2$. To establish the second part of condition (ii), it is enough to prove the following claim:

$$\operatorname{extr} E\bigg(\sum_{j=n+1}^{\infty} \mu_j\bigg) \subset [0, \tau].$$

According to [8, Theorem 6.1(b)] or [5, Theorem 1(b)], there exist (unique) $\pi_j \in \operatorname{extr} E(\mu_j)$, $j \in \mathbb{N}$, such that $\sum_{j=1}^{\infty} \pi_j = \pi$. By the same result, given

$\pi'_j \in \operatorname{extr} E(\mu_j)$, $j = n+1, n+2, \ldots$, we have

$$\pi' := \sum_{j=1}^{n} \pi_j + \sum_{j=n+1}^{\infty} \pi'_j \ \in \ \operatorname{extr} E(\mu).$$

In addition, $\|\pi' - \pi\| < \varepsilon$. It follows that π' is in $[0, \tau]$, and the same is true for $\sum_{j=n+1}^{\infty} \pi'_j$. Thus, the claim holds, by one more application of [8, Theorem 6.1(b)] or [5, Theorem 1(b)]. □

The next two lemmas will be used in establishing Proposition 4 and Theorem 8 in Section 7.

Lemma 3. *Let $\mu \in ult(\mathfrak{M})$. Then the following two conditions are equivalent:*

(i) $A_{E(\mu)} = A_{\operatorname{extr} E(\mu)}$;
(ii) $\operatorname{extr} E(\mu)$ *is finite.*

Proof: Suppose (ii) holds. Since $E(\mu)$ is weak* compact (see [8, Proposition 4.4(a)] or [4, Proposition 1(a)]), the Krein–Milman theorem implies that

$$E(\mu) = \operatorname{conv} \operatorname{extr} E(\mu).$$

Hence (i) holds.

Suppose (ii) fails, and let $\pi_1, \pi_2, \ldots$ be different elements of $\operatorname{extr} E(\mu)$. Setting $\varrho = \sum_{n=1}^{\infty} 2^{-n} \pi_n$, we have $\varrho \in E(\mu)$. On the other hand, it follows from (D)$'$ that $\varrho \notin A_{\operatorname{extr} E(\mu)}$. Thus, (i) fails, too. □

Lemma 4. *Let $\mu_1, \ldots, \mu_n \in ba_+(\mathfrak{M})$ and $\mu_j \wedge \mu_{j'} = 0$ whenever $j \neq j'$. Then the following two conditions are equivalent:*

(i) $A_{E(\sum_{j=1}^{n} \mu_j)} = A_{\operatorname{extr} E(\sum_{j=1}^{n} \mu_j)}$;
(ii) $A_{E(\mu_j)} = A_{\operatorname{extr} E(\mu_j)}$ *for each $j = 1, \ldots, n$.*

Proof: In view of formulas (3) and (4), (ii) implies (i).

Suppose (i) holds. Using formula (4), we then get

$$E(\mu_{j'}) \subset \sum_{j=1}^{n} A_{\operatorname{extr} E(\mu_j)} \quad \text{for } j' = 1, \ldots, n.$$

By (5), it follows that $E(\mu_{j'}) \subset A_{\operatorname{extr} E(\mu_{j'})}$, and so (ii) holds. □

The next lemma is an essential tool in establishing Theorem 7 in Section 6 and Proposition 5 in Section 7. Both results assume condition $(*)$, which is intermediate between the condition of independence and that of almost independence of algebras of sets considered by E. Marczewski (see [10, p. 220]). For other uses of $(*)$ see [8, Proposition 12.4] or [6, Proposition 2] as well as [7, Theorem 7] and [9, Corollaries 2 and 3].

Lemma 5. *Let $\mathfrak{N}$ be an algebra of subsets of Ω with $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{N})_b$ and let $\mu \in ult(\mathfrak{M})$. Then $\mathfrak{R}/\mathfrak{J}_\mu$ is homomorphic image of $\mathfrak{N}$. If, in addition,*

$$(*) \qquad M \cap N \neq \varnothing \text{ for all } M \in \mathfrak{M} \text{ with } \mu(M) > 0 \text{ and nonempty } N \in \mathfrak{N}$$

holds, then $\mathfrak{R}/\mathfrak{J}_\mu$ and $\mathfrak{N}$ are isomorphic. In particular,

$$|ult(\mathfrak{R}/\mathfrak{J}_\mu)| = |ult(\mathfrak{N})|.$$

Proof: Denote by h the canonical mapping from $\mathfrak{R}$ onto $\mathfrak{R}/\mathfrak{J}_\mu$. For $M \in \mathfrak{M}$ and $N \in \mathfrak{N}$ we have

$$h(M \cap N) = \begin{cases} 0 & \text{if } \mu(M) = 0, \\ h(N) & \text{if } \mu(M) = 1. \end{cases}$$

It follows that $h(\mathfrak{N}) = \mathfrak{R}/\mathfrak{J}_\mu$. Condition $(*)$ implies that $\mathfrak{N} \cap \mathfrak{J}_\mu = \{\varnothing\}$, and so the injectivity of $h|\mathfrak{N}$. □

5. extr $E(\mu)$ for atomic μ

We start by an extension of Lemma 1. It is worth-while to compare it with [9, Proposition 5], which, under the same assumption, asserts that extr $E(\mu)$ is order bounded if and only if it is finite.

Proposition 3. *Let $\mu \in ba_+(\mathfrak{M})$ have finite range. Then the following three conditions are equivalent:*

(i) extr $E(\mu)$ *is ideal dominated;*
(ii) extr $E(\mu)$ *is countable;*
(iii) $ult(\mathfrak{R}/\mathfrak{J}_\nu)$ *is countable for each* $\nu \in \mathcal{U}_\mu$.

Proof: The assumption implies that μ is atomic and $\mathcal{U}_\mu$ is finite (see [8, Lemma 3.2] and [3, Lemma 11.1.3]). Therefore, it follows from formula (2) that (ii) holds if and only if extr $E(\nu)$ is countable for each $\nu \in \mathcal{U}_\mu$. Thus, (ii) and (iii) are equivalent, by [8, Proposition 7.1, 4°] or [6, Proposition 1]. The equivalence of (i) and (ii) follows from Lemma 1 and formula (4). □

The next result is a partial generalization of Proposition 3.

Theorem 3. *Let $\mu \in ba_+(\mathfrak{M})$ be atomic, and set*

$$\mathcal{D} = \{\nu \in \mathcal{U}_\mu : ult(\mathfrak{R}/\mathfrak{J}_\nu) \text{ is infinite}\}.$$

Then the following three conditions are equivalent:

(i) extr $E(\mu)$ *is ideal dominated;*
(ii) $ult(\mathfrak{R}/\mathfrak{J}_\nu)$ *is countable for each* $\nu \in \mathcal{U}_\mu$, $\mathcal{D}$ *is finite, and*

$$\operatorname{extr} E\Big(\sum_{\nu \in \mathcal{U}_\mu \setminus \mathcal{D}} \nu\Big) \text{ is order bounded};$$

(iii) $ult(\mathfrak{R}/\mathfrak{J}_\nu)$ *is countable for each* $\nu \in \mathcal{U}_\mu$, $\mathcal{D}$ *is finite, and*

$$\sum_{\nu \in \mathcal{U}_\mu \setminus \mathcal{D}} \nu(\Omega)\, |ult(\mathfrak{R}/\mathfrak{J}_\nu)| < \infty.$$

Proof: Using [8, Proposition 7.1, 4°] or [6, Proposition 1], and Lemma 1, we can reword the first part of condition (ii) as follows: $\operatorname{extr} E(\nu)$ is ideal dominated for each $\nu \in \mathcal{U}_\mu$. Moreover, in view of [9, Proposition 5, (i) $\Rightarrow$ (iii)], $\operatorname{extr} E(\nu)$ is not order bounded for each $\nu \in \mathcal{D}$. Thus, (i) and (ii) are equivalent by formula (4) and Lemma 2.

The equivalence of (ii) and (iii) is a direct consequence of [9, Theorem 3]. □

Remark 1. Condition (i) of Theorem 3 neither implies nor is implied by the condition that $E(\mu)$ be weakly compact (equivalently, $\operatorname{extr} E(\mu)$ be relatively weakly compact; see [8, Theorem 5.1]), even for atomic $\mu \in ba_+(\mathfrak{M})$. Indeed, in Example 1 of [4] μ is two-valued, $\operatorname{extr} E(\mu)$ has cardinality $\aleph_0$, and so is ideal dominated, by Proposition 1, but $E(\mu)$ is not weakly compact (cf. [9, Proposition 5, (ii) $\Rightarrow$ (iii)]). On the other hand, in Example 1 of [9] $E(\mu)$ is weakly compact, but not order bounded. Therefore, $\operatorname{extr} E(\mu)$ is not ideal dominated, by Theorem 6 in the next section.

6. $E(\mu)$ and $\operatorname{extr} E(\mu)$ for arbitrary μ and $\operatorname{extr} E(\mu)$ for nonatomic μ

The functionals

$$ba(\mathfrak{R}) \ni \varphi \longmapsto \varphi(R) \in \mathbb{R}, \quad \text{where } R \in \mathfrak{R},$$

can be identified with elements of the predual of $ba(\mathfrak{R})$. Consequently, the positive cone of $ba(\mathfrak{R})$ is weak* closed. In fact, the positive cone of an arbitrary dual Banach lattice is weak* closed, in view of a classical result (see [14, Proposition II.5.5]). Therefore, the following result is a direct consequence of Theorem 1 above, and [8, Proposition 4.4(a)] or [4, Proposition 1(a)].

Theorem 4. *For* $\mu \in ba_+(\mathfrak{M})$ *the following two conditions are equivalent:*

(i) $E(\mu)$ *is order bounded;*
(ii) $E(\mu)$ *is ideal dominated.*

The next result is a partial strengthening of Theorem 4.

Theorem 5. *Let* $\mu \in ba_+(\mathfrak{M})$ *be nonatomic. Then the following three conditions are equivalent:*

(i) $E(\mu)$ *is order bounded;*
(ii) $E(\mu)$ *is ideal dominated;*
(iii) $\operatorname{extr} E(\mu)$ *is ideal dominated.*

Proof: Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Suppose (iii) holds. To derive (i), note that there exist $\pi_0 \in \operatorname{extr} E(\mu)$, $\varepsilon > 0$ and $\tau \in ba_+(\mathfrak{R})$ such that

$$\{\pi \in \operatorname{extr} E(\mu) : \|\pi_0 - \pi\| < \varepsilon\} \subset [0, \tau].$$

Indeed, $\operatorname{extr} E(\mu)$ being closed in $ba(\mathfrak{R})$ (see [8, Proposition 4.4(b)] or [4, Proposition 1(b)]), this is a consequence of (iii) and the Baire category theorem. We shall show that $\mu^*|\mathfrak{R} \leq \tau$, which is equivalent to $E(\mu) \subset [0,\tau]$ (see [8, p. 19, (C)*]). To this end, fix an $\mathfrak{M}$-partition $\{M_1, \dots, M_n\}$ of Ω with $\mu(M_i) < \varepsilon/2$ for each i and $R_0 \in \mathfrak{R}$. Appealing to [8, p. 19, (C)*] again, we find, for each $i = 1, \dots, n$,

$$\pi_i \in \operatorname{extr} E(\mu) \quad \text{with} \quad \pi_i(R_0 \cap M_i) = \mu^*(R_0 \cap M_i).$$

Set

$$\tilde{\pi}_i(R) = \pi_i(R \cap M_i) + \pi_0(R \cap M_i^c) \quad \text{for } R \in \mathfrak{R} \text{ and } i = 1, \dots, n.$$

By [8, Lemma 4.5(d)] or [5, Lemma 4(d)], $\tilde{\pi}_i \in \operatorname{extr} E(\mu)$. Moreover, we have

$$\|\pi_0 - \tilde{\pi}_i\| < \varepsilon, \quad \text{and so} \quad \tilde{\pi}_i \leq \tau,\ i = 1, \dots, n.$$

It follows that

$$\mu^*(R_0) = \sum_{i=1}^{n} \mu^*(R_0 \cap M_i) = \sum_{i=1}^{n} \tilde{\pi}_i(R_0 \cap M_i) \leq \sum_{i=1}^{n} \tau(R_0 \cap M_i) = \tau(R_0).$$

□

Remark 2. The nonatomicity assumption is essential for the validity of the implications (iii) ⇒ (i), (ii) of Theorem 5. Indeed, in Example 1 of [4] $\operatorname{extr} E(\mu)$ is countable, and so ideal dominated, by Proposition 1, but $E(\mu)$ is seen not to be order bounded (cf. Remark 1). In view of Theorem 4, nor is $E(\mu)$ ideal dominated.

According to Remark 1, neither part of condition (ii) of Theorem 6 below implies the other part thereof, in general.

Theorem 6. *For $\mu \in ba_+(\mathfrak{M})$ the following two conditions are equivalent:*

(i) *$E(\mu)$ is order bounded;*

(ii) *$E(\mu)$ is weakly compact and $\operatorname{extr} E(\mu)$ is ideal dominated.*

Proof: The nontrivial part of the implication (i) ⇒ (ii) coincides with [9, Proposition 2(c)].

Suppose (ii) holds. Let μ_1 and μ_2 stand for the atomic and nonatomic components of μ, respectively. Then $E(\mu_i)$ is weakly compact and $\operatorname{extr} E(\mu_i)$ is ideal dominated for $i = 1, 2$, by [8, Corollary 6.3] and formula (2), respectively. Thus, $E(\mu_2)$ is order bounded, according to Theorem 5, (iii) ⇒ (i). From [8, Theorem 7.7, (ii) ⇒ (iii)] we infer that $\mathfrak{R}/\mathfrak{J}_\nu$ is finite for each $\nu \in \mathcal{U}_{\mu_1}$, and so Theorem 3 yields that $\operatorname{extr} E(\mu_1)$ is order bounded. By [9, Theorem 2, (ii) ⇒ (i)], $E(\mu_1)$ is also order bounded. An application of formula (1) completes the proof of (i). □

Theorem 7. *Let $\mathfrak{N}$ be an algebra of subsets of Ω with $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{N})_b$, let $\mu \in ba_+(\mathfrak{M})$ and let μ_1 and μ_2 stand for the atomic and nonatomic components of μ, respectively. Suppose*

(∗) $\quad M \cap N \neq \varnothing$ *for all $M \in \mathfrak{M}$ with $\mu(M) > 0$ and nonempty $N \in \mathfrak{N}$.*

Then the following two conditions are equivalent:

(i) $\operatorname{extr} E(\mu)$ *is ideal dominated;*

(ii) μ_1 *has finite range,* $\mu_2 = 0$ *and* $ult(\mathfrak{N})$ *is countable, or* $\mathfrak{N}$ *is finite or* $\mu = 0$.

Proof: Suppose (i) holds. By formula (2), $\operatorname{extr} E(\mu_i)$ is then ideal dominated for $i = 1, 2$. Hence $E(\mu_2)$ is order bounded, by Theorem 5, (iii) $\Rightarrow$ (i). If $\mu_2 \neq 0$, it follows by [9, Corollary 2, (i) $\Rightarrow$ (iii)], that $\mathfrak{N}$ is finite, and so (ii) holds. Suppose $\mu_2 = 0$ and, moreover, $\mu_1 \neq 0$ and $\mathfrak{N}$ is infinite. According to Lemma 5, we have $ult(\mathfrak{N}) = ult(\mathfrak{R}/\mathfrak{J}_\nu)$ for $\nu \in \mathcal{U}_{\mu_1}$ with $\nu \neq 0$. It follows from Theorem 3 that $ult(\mathfrak{N})$ is countable and $\mathcal{U}_{\mu_1}$ is finite, and so $\mu_1(\mathfrak{M})$ is also finite. Thus, the implication (i) $\Rightarrow$ (ii) is established.

Plainly, (i) holds if $\mu = 0$. It also holds if $\mathfrak{N}$ is finite, by [4, Theorem 1(a)]; see also Proposition 6(a) in Section 7. Suppose the first part of condition (ii) holds and $\mu_1 \neq 0$. By Lemma 5 again, $ult(\mathfrak{R}/\mathfrak{J}_\nu)$ is countable for each $\nu \in \mathcal{U}_{\mu_1}$. Proposition 3, (iii) $\Rightarrow$ (i), now yields (i). Thus, the implication (ii) $\Rightarrow$ (i) is also established. □

7. Coincidence of $A_{E(\mu)}$ and $A_{\operatorname{extr} E(\mu)}$

The following result extends Lemma 3.

Proposition 4. *Let* $\mu \in ba_+(\mathfrak{M})$ *have finite range. Then the following three conditions are equivalent:*

(i) $A_{E(\mu)} = A_{\operatorname{extr} E(\mu)}$;

(ii) $\operatorname{extr} E(\mu)$ *is finite;*

(iii) $ult(\mathfrak{R}/\mathfrak{J}_\nu)$ *is finite for each* $\nu \in \mathcal{U}_\mu$.

Proof: As in the proof of Proposition 3, the assumption implies that μ is atomic and $\mathcal{U}_\mu$ is finite. Now, formula (2) shows that (ii) is equivalent to the condition that $\operatorname{extr} E(\nu)$ is finite for each $\nu \in \mathcal{U}_\mu$. Therefore, the equivalence of (i) and (ii) is a consequence of Lemmas 3 and 4, while the equivalence of (ii) and (iii) follows from [8, Proposition 7.1, 4°] or [6, Proposition 1]. Indeed, according to those results, $\operatorname{extr} E(\nu)$ and $ult(\mathfrak{R}/\mathfrak{J}_\nu)$ are equipotent for $\nu \in ult(\mathfrak{M})$. □

The next result is a partial extension of Proposition 4.

Theorem 8. *Let* $\mu \in ba_+(\mathfrak{M})$ *be atomic. Then the following two conditions are equivalent:*

(i) $A_{E(\mu)} = A_{\operatorname{extr} E(\mu)}$;

(ii) *there exists* $n \in \mathbb{N}$ *such that* $|ult(\mathfrak{R}/\mathfrak{J}_\nu)| \leq n$ *for each* $\nu \in \mathcal{U}_\mu$.

Under these conditions, $E(\mu)$ *is order bounded.*

Proof: We shall consider below an equivalent version of condition (ii) with "$ult(\mathfrak{R}/\mathfrak{J}_\nu)$" replaced by "$\operatorname{extr} E(\nu)$" (see [8, Proposition 7.1, 4°] or [6, Proposition 1]).

Suppose (i) holds. We first show that $\operatorname{extr} E(\nu)$ is then finite for each $\nu \in \mathcal{U}_\mu$. Indeed, fix $\nu \in \mathcal{U}_\mu$. Applying Lemma 4 to ν and $\mu - \nu$, we get $A_{E(\nu)} = A_{\operatorname{extr} E(\nu)}$. Lemma 3 now shows that $\operatorname{extr} E(\nu)$ is, in fact, finite.

Suppose, moreover, that (ii) fails. By what we have proved so far, there exist different $\nu_1, \nu_2, \ldots$ in $\mathcal{U}_\mu$ such that $\operatorname{extr} E(\nu_n)$ contains different elements $\pi_1^{\nu_n}, \ldots, \pi_n^{\nu_n}$, $n = 1, 2, \ldots$. Fix $\varrho^\nu \in E(\nu)$ for $\nu \in \mathcal{U}_\mu$ with $\nu \neq \nu_1, \nu_2, \ldots$, and set

$$\varrho = \sum_{n=1}^{\infty} \frac{1}{n} (\pi_1^{\nu_n} + \ldots + \pi_n^{\nu_n}) + \sum_{\substack{\nu \in \mathcal{U}_\mu \\ \nu \neq \nu_1, \nu_2, \ldots}} \varrho^\nu.$$

Clearly, $\varrho \in E(\mu)$. We claim that $\varrho \notin A_{\operatorname{extr} E(\mu)}$, which contradicts (i). To establish the claim, fix $\pi_1, \ldots, \pi_p \in \operatorname{extr} E(\mu)$. In view of [8, Theorem 6.1(b)] or [5, Theorem 1(b)], we have

$$\pi_j = \sum_{\nu \in \mathcal{U}_\mu} \sigma_j^\nu, \quad \text{where } j = 1, \ldots, p \text{ and } \sigma_j^\nu \in \operatorname{extr} E(\nu) \text{ for } \nu \in \mathcal{U}_\mu.$$

It follows that for $n > p$ and some $1 \leq j_n \leq n$ we have $\pi_{j_n}^{\nu_n} \wedge \pi_j = 0$, $j = 1, \ldots, p$ (see (D)$'$ and (5)). Thus, the claim is established.

Suppose (ii) holds. Let, for $\nu \in \mathcal{U}_\mu$,

$$\operatorname{extr} E(\nu) = \{\pi_1^\nu, \ldots, \pi_n^\nu\},$$

repetitions being allowed. Set

$$\pi_j = \sum_{\nu \in \mathcal{U}_\mu} \pi_j^\nu, \quad j = 1, \ldots, n.$$

In view of [8, Theorem 6.1(b)] or [5, Theorem 1(b)], we have $\pi_j \in \operatorname{extr} E(\mu)$. To establish (i), it is enough to show that

$$\varrho \leq \sum_{j=1}^{n} \pi_j \quad \text{for every } \varrho \in E(\mu).$$

Fix $\varrho \in E(\mu)$, and choose, for $\nu \in \mathcal{U}_\mu$,

$$\varrho^\nu \in E(\nu) \quad \text{with} \quad \sum_{\nu \in \mathcal{U}_\mu} \varrho^\nu = \varrho$$

(see [8, Theorem 6.1(a)] or [5, Theorem 1(a)]). As in the proof of Lemma 3, we have

$$E(\nu) = \operatorname{conv}\{\pi_1^\nu, \ldots, \pi_n^\nu\}.$$

Consequently, $\varrho^\nu \leq \sum_{j=1}^n \pi_j^\nu$. It follows that $\varrho \leq \sum_{j=1}^n \pi_j$, and so (i) is established.

The final assertion is now an immediate consequence of [9, Theorem 3]. □

The final assertion of Theorem 8 is not equivalent to its conditions (i) and (ii), as the following example shows.

Example 1 (cf. [9, Example 1]). Set $\Omega = \mathbb{N}$, and let $\{M_1, M_2, \ldots\}$ be a partition of Ω with $|M_i| = i$ for each i. Define

$$\mathfrak{M} = \{M_1, M_2, \ldots\}_b \quad \text{and} \quad \mathfrak{R} = \{\{n\} : n \in \Omega\}_b.$$

Set, for $i \in \Omega$ and $M \in \mathfrak{M}$,

$$\nu_i(M) = 1/i^3 \text{ if } M \cap M_i \neq \varnothing \quad \text{and} \quad \nu_i(M) = 0 \text{ otherwise.}$$

Define $\mu = \sum_{i=1}^{\infty} \nu_i$. Clearly, $\mu \in ba_+(\mathfrak{M})$. Moreover, μ is atomic and $\mathcal{U}_\mu = \{0, \nu_1, \nu_2, \ldots\}$. As easily seen, μ does not satisfy condition (ii) of Theorem 8. On the other hand, $\sum_{n \in M_i} \mu^*(\{n\}) = 1/i^2$ for each i, and so $\sum_{n \in \Omega} \mu^*(\{n\}) < \infty$. Hence $E(\mu)$ is order bounded, by [9, Corollary 4].

The author does not know whether condition (i) of Theorem 8 implies that $E(\mu)$ is order bounded for arbitrary $\mu \in ba_+(\mathfrak{M})$.

Proposition 5. *Let $\mathfrak{N}$ be an algebra of subsets of Ω with $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{N})_b$ and let $\mu \in ba_+(\mathfrak{M})$ be atomic. Suppose*

(*) $\quad M \cap N \neq \varnothing$ *for all $M \in \mathfrak{M}$ with $\mu(M) > 0$ and nonempty $N \in \mathfrak{N}$.*

Then the following two conditions are equivalent:

(i) $A_{E(\mu)} = A_{\operatorname{extr} E(\mu)}$;

(ii) $\mu = 0$ *or $\mathfrak{N}$ is finite.*

This is a direct consequence of Theorem 8 and Lemma 5. The implication (ii) ⇒ (i) of Proposition 5 holds, in fact, in general (see Proposition 6(b) below).

Part (a) of our next result is an improvement of [4, Theorem 1(a)]. It is established by a slight modification of the original argument.

Proposition 6. *Let $\mathfrak{R} = (\mathfrak{M} \cup \{E_1, \ldots, E_n\})_b$, where $\{E_1, \ldots, E_n\}$ is a partition of Ω, and let $\mu \in ba_+(\mathfrak{M})$. Then*

(a) *there exist $\pi_1, \ldots, \pi_n \in \operatorname{extr} E(\mu)$ with $\varrho \leq \sum_{i=1}^{n} \pi_i$ for each $\varrho \in E(\mu)$;*

(b) $A_{E(\mu)} = A_{\operatorname{extr} E(\mu)}$.

Proof: Let $\tilde{\pi}_i \in \operatorname{extr}\{\varrho \in ba_+((\mathfrak{M} \cup \{E_i\})_b) : \varrho|\mathfrak{M} = \mu\}$ be such that

$$\tilde{\pi}_i(M \cap E_i) = \mu^*(M \cap E_i) \quad \text{for all } M \in \mathfrak{M} \text{ and } i = 1, \ldots, n$$

(see [12, Example 1]). Continuing in the same way, we get, after $n-1$ more steps, $\pi_i \in \operatorname{extr} E(\mu)$ such that

$$\pi_i|(\mathfrak{M} \cup \{E_i\})_b = \tilde{\pi}_i, \quad i = 1, \ldots, n.$$

Fix $R \in \mathfrak{R}$ and $\varrho \in E(\mu)$. We then have

$$R = \bigcup_{i=1}^{n} M_i \cap E_i, \quad \text{where } M_1, \ldots, M_n \in \mathfrak{M}.$$

It follows that

$$\varrho(R) = \sum_{i=1}^{n} \varrho(M_i \cap E_i) \le \sum_{i=1}^{n} \mu^*(M_i \cap E_i) = \sum_{i=1}^{n} \pi_i(M_i \cap E_i) \le \sum_{i=1}^{n} \pi_i(R).$$

Thus, (a) holds.

Part (b) is a direct consequence of (a). □

In Proposition 6 we cannot replace a finite partition by a countable one, even if μ is atomic (see Example 1). In fact, part (a) of Proposition 6 may then fail in a stronger sense. Namely, in the example below $\operatorname{extr} E(\mu)$ is not even ideal dominated.

Example 2. Set $\Omega = \mathbb{N}$ and let $\{M_1, M_2, \ldots\}$ be a partition of Ω with M_i infinite for each i. Define $\mathfrak{M}$ and $\mathfrak{R}$ as in Example 1. Let $\mu \in ba_+(\mathfrak{M})$ satisfy $\mu(M_i) > 0$ for each i. Then, as easily seen, $ult(\mathfrak{R}/\mathfrak{I}_\nu)$ is infinite whenever $\nu \in \mathcal{U}_\mu$ and $\nu \neq 0$. Therefore, $\operatorname{extr} E(\mu)$ is not ideal dominated, by Theorem 3.

Postscript. Related results on the sets $E(\mu)$ and $\operatorname{extr} E(\mu)$ are presented in another paper by the author, *Order-theoretic properties and separability of some sets of quasi-measures* (preprint).

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