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ESSENTIAL NORM AND A NEW CHARACTERIZATION OF WEIGHTED COMPOSITION OPERATORS FROM WEIGHTED BERGMAN SPACES AND HARDY SPACES INTO THE BLOCH SPACE

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Abstract. In this paper, we give some estimates for the essential norm and a new characterization for the boundedness and compactness of weighted composition operators from weighted Bergman spaces and Hardy spaces to the Bloch space.

Keywords: Bloch space; weighted Bergman space; Hardy space; essential norm; weighted composition operator

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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space, denoted by A^p_α, is the set of all functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|^p_{A^p_\alpha} = (\alpha+1)\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \,\mathrm{d}A(z) < \infty,$$

where A is the normalized Lebesgue area measure in $\mathbb{D}$ such that $A(\mathbb{D}) = 1$. The Hardy space H^p is the space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{H^p} = \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^p \,\mathrm{d}\theta < \infty.$$

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The Bloch space, denoted by $\mathcal{B} = \mathcal{B}(\mathbb{D})$, is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_\beta = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

Under the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_\beta$, the Bloch space is a Banach space. See [26] for more information on the Bloch space.

Let $v\colon \mathbb{D} \to \mathbb{R}_+$ be a continuous, strictly positive and bounded function. An $f \in H(\mathbb{D})$ is said to belong to the weighted space, denoted by H_v^∞, if

$$\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty.$$

H_v^∞ is a Banach space with the norm $\|\cdot\|_v$. The weight v is called radial, if $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. For a weight v, the associated weight $\tilde{v}$ is defined as

$$\tilde{v} = (\sup\{|f(z)|\colon\ f \in H_v^\infty,\ \|f\|_v \leqslant 1\})^{-1}, \quad z \in \mathbb{D}.$$

When $v = v_\alpha(z) = (1-|z|^2)^\alpha$, $0 < \alpha < \infty$, it is easy to check that $\tilde{v}_\alpha(z) = v_\alpha(z)$. In this case, we denote H_v^∞ by $H_{v_\alpha}^\infty$ and $\|f\|_{v_\alpha} = \sup_{z\in\mathbb{D}} |f(z)|(1-|z|^2)^\alpha$.

Let $S(\mathbb{D})$ denote the set of all analytic self-maps of $\mathbb{D}$. Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. For $f \in H(\mathbb{D})$, the composition operator C_φ and the multiplication operator M_u are defined by

$$(C_\varphi f)(z) = f(\varphi(z)) \quad \text{and} \quad (M_u f)(z) = u(z)f(z),$$

respectively. The weighted composition operator uC_φ is defined by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

It is clear that the weighted composition operator uC_φ is the generalization of C_φ and M_u. A basic and interesting problem concerning concrete operators (such as composition operator, multiplication operator, Volterra operator, Toeplitz operator, Hankel operator and other integral-type operators) is to relate operator-theoretic properties to the function-theoretic properties of their symbols, which attracted a lot of attention recently, we refer the reader to [3] and [26].

It is well known that C_φ is bounded on $\mathcal{B}$ by the Schwarz-Pick lemma for any $\varphi \in S(\mathbb{D})$. The compactness of C_φ on $\mathcal{B}$ was studied for example in [13], [19], [21]. In [21], Wulan, Zheng and Zhu proved that for any $\varphi \in S(\mathbb{D})$, $C_\varphi\colon \mathcal{B} \to \mathcal{B}$ is compact if and only if $\lim_{j\to\infty} \|\varphi^j\|_{\mathcal{B}} = 0$. This result has been generalized to Bloch-type spaces by Zhao in [25] and shows that $C_\varphi\colon \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if

$\lim\limits_{j\to\infty} j^{\alpha-1}\|\varphi^j\|_{\mathcal{B}^\beta} = 0$. For some results on composition operator and related operators mapping into the Bloch space see, for example, [1], [2], [7]–[14], [16]–[18], [22]–[25], [27] and the related references therein.

In [7], Li and Stević obtained a characterization of the boundedness and compactness of the weighted composition operator $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$. Among others, we proved the following result.

Theorem A. *Let $1 \leqslant p < \infty$, $\alpha > -1$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ is bounded. Then $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|u'(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha)/p}} = 0 \quad \text{and} \quad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha+p)/p}} = 0.$$

In [2], Colonna obtained a new characterization by using two families of functions, among others, she obtained the following result.

Theorem B. *Let $1 \leqslant p < \infty$, $\alpha > -1$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ is bounded. Then $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ is compact if and only if*

$$\lim_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{B}} = 0 \quad \text{and} \quad \lim_{|a|\to 1} \|uC_\varphi g_a\|_{\mathcal{B}} = 0,$$

where

$$f_a(z) = \frac{(1-|a|^2)^{1+(2+\alpha)(1-1/p)}}{(1-\overline{a}z)^{3+\alpha}}, \quad g_a(z) = \frac{(1-|a|^2)^{1+(2+\alpha)(1-1/p)+1/p}}{(1-\overline{a}z)^{3+\alpha+1/p}}.$$

In [2], Colonna also obtained two characterizations for the compactness of weighted composition operator $uC_\varphi\colon H^p \to \mathcal{B}$.

Theorem C. *Let $1 \leqslant p < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi\colon H^p \to \mathcal{B}$ is bounded. Then the following statements are equivalent:*

(a) *$uC_\varphi\colon H^p \to \mathcal{B}$ is compact.*

(b)
$$\lim_{|a|\to 1} \|uC_\varphi p_a\|_{\mathcal{B}} = 0 \quad \text{and} \quad \lim_{|a|\to 1} \|uC_\varphi q_a\|_{\mathcal{B}} = 0,$$

where

$$p_a(z) = \frac{(1-|a|^2)^{2-1/p}}{(1-\overline{a}z)^2}, \quad q_a(z) = \frac{(1-|a|^2)^2}{(1-\overline{a}z)^{2+1/p}}.$$

(c)
$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|u'(z)|}{(1-|\varphi(z)|^2)^{1/p}} = 0 \quad \text{and} \quad \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{(1+p)/p}} = 0.$$

The purpose of this paper is to give some estimates for the essential norm of the operator $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ (as well as $uC_\varphi\colon H^p \to \mathcal{B}$), in particular, by using $\|uC_\varphi f_a\|_{\mathcal{B}}$ and $\|uC_\varphi g_a\|_{\mathcal{B}}$ (as well as $\|uC_\varphi p_a\|_{\mathcal{B}}$ and $\|uC_\varphi q_a\|_{\mathcal{B}}$). Moreover, we give a new characterization for the boundedness, compactness and essential norm of the operator $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ (as well as $uC_\varphi\colon H^p \to \mathcal{B}$) by using φ^j.

Recall that the essential norm of a bounded linear operator $T\colon X \to Y$ is its distance to the set of compact operators K mapping X into Y, that is,

$$\|T\|_{\mathrm{es},X\to Y} = \inf\{\|T-K\|_{X\to Y}\colon K \text{ is compact}\},$$

where X, Y are Banach spaces and $\|\cdot\|_{X\to Y}$ is the operator norm.

Throughout this paper, we say that $A \lesssim B$ if there exists a constant C such that $A \leqslant CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Essential norm of uC_φ

In this section, we give two estimates for the essential norm of the operator $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ and the operator $uC_\varphi\colon H^p \to \mathcal{B}$, respectively.

Theorem 2.1. *Let $1 \leqslant p < \infty$, $\alpha > -1$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ is bounded. Then*

$$\|uC_\varphi\|_{\mathrm{es},A^p_\alpha\to\mathcal{B}} \approx \max\{A,B\} \approx \max\{P,Q\},$$

where

$$A := \limsup_{|a|\to 1} \|uC_\varphi(f_a)\|_{\mathcal{B}}, \quad B := \limsup_{|a|\to 1} \|uC_\varphi(g_a)\|_{\mathcal{B}},$$

$$P := \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|u'(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha)/p}}, \quad Q := \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha+p)/p}}.$$

Proof. First we prove that

$$\max\{A,B\} \lesssim \|uC_\varphi\|_{\mathrm{es},A^p_\alpha\to\mathcal{B}}.$$

Let $a \in \mathbb{D}$. It is easy to check that $f_a, g_a \in A^p_\alpha$ and $\|f_a\|_{A^p_\alpha} \lesssim 1$, $\|g_a\|_{A^p_\alpha} \lesssim 1$ for all $a \in \mathbb{D}$ and f_a, g_a converge to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Thus, for any compact operator $K\colon A^p_\alpha \to \mathcal{B}$, by Lemma 3.7 of [20] we have

$$\lim_{|a|\to 1} \|Kf_a\|_{\mathcal{B}} = 0, \quad \lim_{|a|\to 1} \|Kg_a\|_{\mathcal{B}} = 0.$$

Hence

$$\|uC_\varphi - K\|_{A^p_\alpha\to\mathcal{B}} \gtrsim \|(uC_\varphi - K)f_a\|_{\mathcal{B}} \geqslant \|uC_\varphi f_a\|_{\mathcal{B}} - \|Kf_a\|_{\mathcal{B}},$$

and

$$\|uC_\varphi - K\|_{A^p_\alpha\to\mathcal{B}} \gtrsim \|(uC_\varphi - K)g_a\|_{\mathcal{B}} \geqslant \|uC_\varphi g_a\|_{\mathcal{B}} - \|Kg_a\|_{\mathcal{B}}.$$

Taking $\limsup\limits_{|a|\to 1}$ to the last two inequalities on both sides, we obtain

$$\|uC_\varphi - K\|_{A^p_\alpha\to\mathcal{B}} \gtrsim A, \quad \|uC_\varphi - K\|_{A^p_\alpha\to\mathcal{B}} \gtrsim B.$$

Therefore, by the definition of the essential norm, we get

$$\|uC_\varphi\|_{\mathrm{es},A^p_\alpha\to\mathcal{B}} = \inf_K \|uC_\varphi - K\|_{A^p_\alpha\to\mathcal{B}} \gtrsim \max\{A, B\}.$$

Next, set

$$h_a(z) = f_a - g_a, \quad k_a(z) = f_a - \frac{3+\alpha}{3+\alpha+1/p}\, g_a.$$

It is also easy to check that $h_a, k_a \in A^p_\alpha$ and $\|h_a\|_{A^p_\alpha} \lesssim 1$, $\|k_a\|_{A^p_\alpha} \lesssim 1$ for all $a \in \mathbb{D}$ and h_a, k_a converge to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Hence, for any $b_j \in \mathbb{D}$ such that $|\varphi(b_j)| \to 1$ and any compact operator $K\colon A^p_\alpha \to \mathcal{B}$, we have

$$\|uC_\varphi - K\|_{A^p_\alpha\to\mathcal{B}} \gtrsim \|(uC_\varphi - K)h_{\varphi(b_j)}\|_{\mathcal{B}} \geqslant \|uC_\varphi h_{\varphi(b_j)}\|_{\mathcal{B}} - \|Kh_{\varphi(b_j)}\|_{\mathcal{B}},$$

and

$$\|uC_\varphi - K\|_{A^p_\alpha\to\mathcal{B}} \gtrsim \|(uC_\varphi - K)k_{\varphi(b_j)}\|_{\mathcal{B}} \geqslant \|uC_\varphi k_{\varphi(b_j)}\|_{\mathcal{B}} - \|Kk_{\varphi(b_j)}\|_{\mathcal{B}}.$$

Taking $\limsup\limits_{|\varphi(b_j)|\to 1}$ to the last two inequalities on both sides we get

$$\|uC_\varphi - K\|_{A^p_\alpha\to\mathcal{B}} \gtrsim \limsup_{|\varphi(b_j)|\to 1} \|uC_\varphi h_{\varphi(b_j)}\|_{\mathcal{B}} \gtrsim \limsup_{|\varphi(b_j)|\to 1} \frac{(1-|b_j|^2)|u'(b_j)|}{(1-|\varphi(b_j)|^2)^{(2+\alpha)/p}} = P,$$

and

$$\|uC_\varphi - K\|_{A^p_\alpha\to\mathcal{B}} \gtrsim \limsup_{|\varphi(b_j)|\to 1} \|uC_\varphi k_{\varphi(b_j)}\|_{\mathcal{B}} \gtrsim \limsup_{|\varphi(b_j)|\to 1} \frac{(1-|b_j|^2)|u(b_j)\varphi'(b_j)|}{(1-|\varphi(b_j)|^2)^{(2+\alpha+p)/p}} = Q.$$

By the definition of the essential norm, we obtain

$$\|uC_\varphi\|_{\mathrm{es},A^p_\alpha\to\mathcal{B}} = \inf_K \|uC_\varphi - K\|_{A^p_\alpha\to\mathcal{B}} \gtrsim \max\{P, Q\}.$$

Finally, we prove that

$$\|uC_\varphi\|_{\mathrm{es},A^p_\alpha\to\mathcal{B}} \lesssim \max\{A,B\} \quad \text{and} \quad \|uC_\varphi\|_{\mathrm{es},A^p_\alpha\to\mathcal{B}} \lesssim \max\{P,Q\}.$$

For $r\in[0,1)$, set $K_r\colon H(\mathbb{D})\to H(\mathbb{D})$ by

$$(K_rf)(z)=f_r(z)=f(rz),\quad f\in H(\mathbb{D}).$$

It is clear that K_r is compact on A^p_α and $\|K_r\|_{A^p_\alpha\to A^p_\alpha}\leqslant 1$. Let $\{r_j\}\subset(0,1)$ be a sequence such that $r_j\to 1$ as $j\to\infty$. Then for all positive integers j, the operator $uC_\varphi K_{r_j}\colon A^p_\alpha\to\mathcal{B}$ is compact. By the definition of the essential norm we have

$$\|uC_\varphi\|_{\mathrm{es},A^p_\alpha\to\mathcal{B}} \leqslant \limsup_{j\to\infty}\|uC_\varphi-uC_\varphi K_{r_j}\|_{A^p_\alpha\to\mathcal{B}}. \tag{2.1}$$

Thus, we only need to show that

$$\limsup_{j\to\infty}\|uC_\varphi-uC_\varphi K_{r_j}\|_{A^p_\alpha\to\mathcal{B}}\lesssim\max\{A,B\}, \tag{2.2}$$

and

$$\limsup_{j\to\infty}\|uC_\varphi-uC_\varphi K_{r_j}\|_{A^p_\alpha\to\mathcal{B}}\lesssim\max\{P,Q\}. \tag{2.3}$$

For any $f\in A^p_\alpha$ such that $\|f\|_{A^p_\alpha}\leqslant 1$, we consider

$$\|(uC_\varphi-uC_\varphi K_{r_j})f\|_{\mathcal{B}}=|u(0)f(\varphi(0))-u(0)f(r_j\varphi(0))|+\|u(f-f_{r_j})\circ\varphi\|_\beta.$$

It is clear that $\lim\limits_{j\to\infty}|u(0)f(\varphi(0))-u(0)f(r_j\varphi(0))|=0$. Now we estimate

$$\begin{aligned}
&\limsup_{j\to\infty}\|u(f-f_{r_j})\circ\varphi\|_\beta\\
&\qquad\leqslant \limsup_{j\to\infty}\sup_{|\varphi(z)|\leqslant r_N}(1-|z|^2)|(f-f_{r_j})'(\varphi(z))||\varphi'(z)||u(z)|\\
&\qquad\quad+\limsup_{j\to\infty}\sup_{|\varphi(z)|>r_N}(1-|z|^2)|(f-f_{r_j})'(\varphi(z))||\varphi'(z)||u(z)|\\
&\qquad\quad+\limsup_{j\to\infty}\sup_{|\varphi(z)|\leqslant r_N}(1-|z|^2)|(f-f_{r_j})(\varphi(z))||u'(z)|\\
&\qquad\quad+\limsup_{j\to\infty}\sup_{|\varphi(z)|>r_N}(1-|z|^2)|(f-f_{r_j})(\varphi(z))||u'(z)|\\
&\qquad=Q_1+Q_2+Q_3+Q_4,
\end{aligned} \tag{2.4}$$

where $N \in \mathbb{N}$ is large enough such that $r_j \geqslant 1/2$ for all $j \geqslant N$,

$$Q_1 := \limsup_{j\to\infty} \sup_{|\varphi(z)|\leqslant r_N} (1-|z|^2)|(f-f_{r_j})'(\varphi(z))||\varphi'(z)||u(z)|,$$

$$Q_2 := \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)|(f-f_{r_j})'(\varphi(z))||\varphi'(z)||u(z)|,$$

$$Q_3 := \limsup_{j\to\infty} \sup_{|\varphi(z)|\leqslant r_N} (1-|z|^2)|(f-f_{r_j})(\varphi(z))||u'(z)|,$$

and

$$Q_4 := \limsup_{j\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)|(f-f_{r_j})(\varphi(z))||u'(z)|.$$

Since $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ is bounded, applying the operator uC_φ to 1 and z, we easily get that $u \in \mathcal{B}$ and

$$\widetilde{K} := \sup_{z\in\mathbb{D}}(1-|z|^2)|\varphi'(z)||u(z)| < \infty.$$

Since $r_j f'_{r_j} \to f'$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$, we have

$$Q_1 \leqslant \widetilde{K} \limsup_{j\to\infty} \sup_{|w|\leqslant r_N} |f'(w) - r_j f'(r_j w)| = 0. \tag{2.5}$$

Also, from the fact that $u \in \mathcal{B}$ and $f_{r_j} \to f$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$, we have

$$Q_3 \leqslant \|u\|_{\mathcal{B}} \limsup_{j\to\infty} \sup_{|w|\leqslant r_N} |f(w) - f(r_j w)| = 0. \tag{2.6}$$

Next we consider Q_2. We have $Q_2 \leqslant \limsup_{j\to\infty}(S_1^j + S_2^j)$, where

$$S_1^j := \sup_{|\varphi(z)|> r_N} (1-|z|^2)|f'(\varphi(z))||\varphi'(z)||u(z)|$$

and

$$S_2^j := \sup_{|\varphi(z)|> r_N} (1-|z|^2) r_j |f'(r_j\varphi(z))||\varphi'(z)||u(z)|.$$

First we estimate S_1^j. Using the fact that $\|f\|_{A^p_\alpha} \leqslant 1$, we have

$$\begin{aligned}
S_1^j &= \sup_{|\varphi(z)|> r_N} (1-|z|^2)|f'(\varphi(z))||\varphi'(z)||u(z)| \\
&\lesssim \frac{1}{r_N}\|f\|_{A^p_\alpha} \sup_{|\varphi(z)|> r_N} (1-|z|^2)|\varphi'(z)||u(z)|\frac{|\varphi(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha+p)/p}} \\
&\lesssim \frac{1}{p} \sup_{|\varphi(z)|> r_N} \sup_{|a|> r_N} (1-|z|^2)|\varphi'(z)||u(z)|\frac{|\varphi(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha+p)/p}} \\
&\lesssim \sup_{|a|>r_N} \|uC_\varphi(f_a - g_a)\|_{\mathcal{B}} \\
&\lesssim \sup_{|a|>r_N} \|uC_\varphi f_a\|_{\mathcal{B}} + \sup_{|a|>r_N} \|uC_\varphi g_a\|_{\mathcal{B}}.
\end{aligned}$$

Taking limit as $N \to \infty$ we obtain

$$\begin{aligned}\limsup_{j\to\infty} S_1^j &\lesssim \limsup_{|a|\to 1} \frac{(1-|z|^2)|\varphi'(z)||u(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha+p)/p}} = Q \\ &\lesssim \limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{B}} + \limsup_{|a|\to 1} \|uC_\varphi g_a\|_{\mathcal{B}}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\limsup_{j\to\infty} S_2^j &\lesssim \limsup_{|a|\to 1} \frac{(1-|z|^2)|\varphi'(z)||u(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha+p)/p}} = Q \\ &\lesssim \limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{B}} + \limsup_{|a|\to 1} \|uC_\varphi g_a\|_{\mathcal{B}},\end{aligned}$$

i.e., we get that

$$Q_2 \lesssim Q \lesssim A + B \lesssim \max\{A, B\}. \tag{2.7}$$

Next we consider Q_4. We have $Q_4 \leqslant \limsup_{j\to\infty}(S_3^j + S_4^j)$, where

$$S_3^j := \sup_{|\varphi(z)|>r_N} (1-|z|^2)|f(\varphi(z))||u'(z)|, \quad S_4^j := \sup_{|\varphi(z)|>r_N} (1-|z|^2)|f(r_j\varphi(z))||u'(z)|.$$

Similarly, we have

$$\begin{aligned}S_3^j &\lesssim \sup_{|\varphi(z)|>r_N} \sup_{|a|>r_N} (1-|z|^2)|u'(z)| \frac{1}{(1-|\varphi(z)|^2)^{(2+\alpha)/p}} \\ &\lesssim \sup_{|a|>r_N} \Big\| uC_\varphi f_a - \frac{3+\alpha}{3+\alpha+1/p} uC_\varphi g_a \Big\|_{\mathcal{B}} \\ &\leqslant \sup_{|a|>r_N} \|uC_\varphi f_a\|_{\mathcal{B}} + \frac{3+\alpha}{3+\alpha+1/p} \sup_{|a|>r_N} \|uC_\varphi g_a\|_{\mathcal{B}} \\ &\leqslant \sup_{|a|>r_N} \|uC_\varphi f_a\|_{\mathcal{B}} + \sup_{|a|>r_N} \|uC_\varphi g_a\|_{\mathcal{B}}.\end{aligned}$$

Taking limit as $N \to \infty$ we obtain

$$\begin{aligned}\limsup_{j\to\infty} S_3^j &\lesssim \limsup_{|a|\to 1} \frac{(1-|z|^2)|u'(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha)/p}} = P \\ &\lesssim \limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{B}} + \limsup_{|a|\to 1} \|uC_\varphi g_a\|_{\mathcal{B}} = A + B.\end{aligned}$$

Similarly, we have $\limsup_{j\to\infty} S_4^j \lesssim P \lesssim A + B$, i.e., we get that

$$Q_4 \lesssim P \lesssim A + B. \tag{2.8}$$

Hence, by (2.4), (2.5), (2.6), (2.7) and (2.8) we get

$$(2.9)\quad \begin{aligned}\limsup_{j\to\infty}\|uC_\varphi-uC_\varphi K_{r_j}\|_{A^p_\alpha\to\mathcal{B}} &= \limsup_{j\to\infty}\sup_{\|f\|_{A^p_\alpha}\leqslant 1}\|(uC_\varphi-uC_\varphi K_{r_j})f\|_{\mathcal{B}}\\ &= \limsup_{j\to\infty}\sup_{\|f\|_{A^p_\alpha}\leqslant 1}\|u(f-f_{r_j})\circ\varphi\|_{\mathcal{B}}\\ &\lesssim P+Q\lesssim A+B.\end{aligned}$$

Therefore, by (2.1) and (2.9), we obtain

$$\|uC_\varphi\|_{\mathrm{es},A^p_\alpha\to\mathcal{B}}\lesssim P+Q\lesssim\max\{P,Q\}$$

and

$$\|uC_\varphi\|_{\mathrm{es},A^p_\alpha\to\mathcal{B}}\lesssim A+B\lesssim\max\{A,B\}.$$

This completes the proof of the theorem. □

The Hardy space H^p can be viewed as the limiting space of A^p_α as α decreases to -1. In fact, carefully check the proof of Theorem 2.1 and replacing A^p_α and α by H^p and -1, respectively, we get the following result.

Theorem 2.2. *Let $1\leqslant p<\infty$, $u\in H(\mathbb{D})$ and $\varphi\in S(\mathbb{D})$ such that $uC_\varphi\colon\ H^p\to\mathcal{B}$ is bounded. Then*

$$\begin{aligned}\|uC_\varphi\|_{\mathrm{es},H^p\to\mathcal{B}} &\approx \max\Big\{\limsup_{|a|\to1}\|uC_\varphi(p_a)\|_{\mathcal{B}},\ \limsup_{|a|\to1}\|uC_\varphi(q_a)\|_{\mathcal{B}}\Big\}\\ &\approx \max\Big\{\limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)|u'(z)|}{(1-|\varphi(z)|^2)^{1/p}},\ \limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{(1+p)/p}}\Big\}.\end{aligned}$$

From Theorems 2.1 and 2.2, we immediately get the following two corollaries.

Corollary 2.1. *Let $1\leqslant p<\infty$, $\alpha>-1$ and $\varphi\in S(\mathbb{D})$ such that $C_\varphi\colon\ A^p_\alpha\to\mathcal{B}$ is bounded. Then*

$$\begin{aligned}\|C_\varphi\|_{\mathrm{es},A^p_\alpha\to\mathcal{B}} &\approx \limsup_{|a|\to1}\|C_\varphi(f_a)\|_{\mathcal{B}}\approx\limsup_{|a|\to1}\|C_\varphi(g_a)\|_{\mathcal{B}}\\ &\approx \limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)|\varphi'(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha+p)/p}}.\end{aligned}$$

Corollary 2.2. *Let $1 \leqslant p < \infty$ and $\varphi \in S(\mathbb{D})$ such that $C_\varphi\colon H^p \to \mathcal{B}$ is bounded. Then*

$$\|C_\varphi\|_{\mathrm{es}, H^p \to \mathcal{B}} \approx \limsup_{|a|\to 1} \|C_\varphi(p_a)\|_{\mathcal{B}} \approx \limsup_{|a|\to 1} \|C_\varphi(q_a)\|_{\mathcal{B}}$$
$$\approx \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|\varphi'(z)|}{(1-|\varphi(z)|^2)^{(1+p)/p}}.$$

3. New characterization of uC_φ

In this section, motivated by [4], we give a new characterization for the boundedness, compactness and essential norm for the weighted composition operators $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ and $uC_\varphi\colon H^p \to \mathcal{B}$. For this purpose, we state some lemmas which will be used.

Lemma 3.1 ([15])**.** *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then the following statements hold.*

(a) *The weighted composition operator $uC_\varphi\colon H^\infty_v \to H^\infty_w$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{w(z)}{\tilde{v}(\varphi(z))}|\varphi(z)| < \infty.$$

Moreover,

$$\|uC_\varphi\|_{H^\infty_v \to H^\infty_w} = \sup_{z\in\mathbb{D}} \frac{w(z)}{\tilde{v}(\varphi(z))}|\varphi(z)|.$$

(b) *Suppose $uC_\varphi\colon H^\infty_v \to H^\infty_w$ is bounded. Then*

$$\|uC_\varphi\|_{\mathrm{es}, H^\infty_v \to H^\infty_w} = \lim_{s\to 1^-} \sup_{|\varphi(z)|>s} \frac{w(z)}{\tilde{v}(\varphi(z))}|\varphi(z)|.$$

Lemma 3.2 ([5])**.** *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then the following statements hold.*

(a) *$uC_\varphi\colon H^\infty_v \to H^\infty_w$ is bounded if and only if*

$$\sup_{k\geqslant 0} \frac{\|u\varphi^k\|_w}{\|z^k\|_v} < \infty,$$

with the norm comparable to the above supremum.

(b) *Suppose $uC_\varphi\colon H^\infty_v \to H^\infty_w$ is bounded. Then*

$$\|uC_\varphi\|_{\mathrm{es}, H^\infty_v \to H^\infty_w} = \limsup_{k\to\infty} \frac{\|u\varphi^k\|_w}{\|z^k\|_v}.$$

Lemma 3.3 ([6]). *For* $\alpha > 0$, *we have* $\lim\limits_{k\to\infty} k^\alpha \|z^{k-1}\|_{v_\alpha} = (2\alpha/\mathrm{e})^\alpha$.

Theorem 3.1. *Let* $1 \leqslant p < \infty$, $\alpha > -1$, $u \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$. *Then the operator* $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ *is bounded if and only if*

$$(3.1)\qquad \sup_{j\geqslant 1} j^{(2+\alpha)/p}\|I_u(\varphi^j)\|_{\mathcal{B}} < \infty \quad \textit{and} \quad \sup_{j\geqslant 1} j^{(2+\alpha)/p}\|J_u(\varphi^{j-1})\|_{\mathcal{B}} < \infty,$$

where

$$I_u g(z) = \int_0^z g'(\xi)u(\xi)d\xi, \quad J_u g(z) = \int_0^z g(\xi)u'(\xi)d\xi, \quad z \in \mathbb{D},\ g \in H(\mathbb{D}).$$

Proof. By Theorem A, $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ is bounded if and only if

$$(3.2)\qquad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|u'(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha)/p}} < \infty \quad \text{and} \quad \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha+p)/p}} < \infty,$$

which are equivalent to the conditions that the weighted composition operator $u'C_\varphi\colon H^\infty_{v_{(2+\alpha)/p}} \to H^\infty_{v_1}$ is bounded and $u\varphi'C_\varphi\colon H^\infty_{v_{(2+\alpha+p)/p}} \to H^\infty_{v_1}$ is bounded, respectively. By Lemma 3.2, we see that the two inequalities in (3.2) are equivalent to

$$\sup_{j\geqslant 1} \frac{\|u'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{(2+\alpha)/p}}} < \infty \quad \text{and} \quad \sup_{j\geqslant 1} \frac{\|u\varphi'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{(2+\alpha+p)/p}}} < \infty,$$

respectively. Since $I_u f(0) = 0$, $J_u f(0) = 0$,

$$(I_u(\varphi^j)(z))' = ju(z)\varphi'(z)\varphi^{j-1}(z), \quad (J_u(\varphi^{j-1})(z))' = u'(z)\varphi^{j-1}(z),$$

by Lemma 3.3, we see that $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ is bounded if and only if

$$(3.3)\qquad \begin{aligned} \sup_{j\geqslant 1} j^{(2+\alpha)/p}\|J_u(\varphi^{j-1})\|_{\mathcal{B}} &= \sup_{j\geqslant 1} j^{(2+\alpha)/p}\|u'\varphi^{j-1}\|_{v_1} \\ &\approx \sup_{j\geqslant 1} \frac{j^{(2+\alpha)/p}\|u'\varphi^{j-1}\|_{v_1}}{j^{(2+\alpha)/p}\|z^{j-1}\|_{v_{(2+\alpha)/p}}} < \infty \end{aligned}$$

and

$$(3.4)\qquad \begin{aligned} \sup_{j\geqslant 1} j^{(2+\alpha)/p}\|I_u(\varphi^j)\|_{\mathcal{B}} &= \sup_{j\geqslant 1} j^{(2+\alpha+p)/p}\|u\varphi'\varphi^{j-1}\|_{v_1} \\ &\approx \sup_{j\geqslant 1} \frac{j^{(2+\alpha+p)/p}\|u\varphi'\varphi^{j-1}\|_{v_1}}{j^{(2+\alpha+p)/p}\|z^{j-1}\|_{v_{(2+\alpha+p)/p}}} < \infty. \end{aligned}$$

The proof is complete. □

Theorem 3.2. *Let $1 \leqslant p < \infty$, $\alpha > -1$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that the operator $uC_\varphi : A^p_\alpha \to \mathcal{B}$ is bounded. Then*

$$\|uC_\varphi\|_{\mathrm{es}, A^p_\alpha \to \mathcal{B}} \approx \max\Big\{\limsup_{j\to\infty} j^{(2+\alpha)/p}\|I_u(\varphi^j)\|_{\mathcal{B}}, \limsup_{j\to\infty} j^{(2+\alpha)/p}\|J_u(\varphi^{j-1})\|_{\mathcal{B}}\Big\}.$$

P r o o f. By Theorem A and Lemma 3.1, $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ is bounded if and only if the weighted composition operator $u'C_\varphi\colon H^\infty_{v_{(2+\alpha)/p}} \to H^\infty_{v_1}$ is bounded and $u\varphi'C_\varphi\colon H^\infty_{v_{(2+\alpha+p)/p}} \to H^\infty_{v_1}$ is bounded. By Lemmas 3.2 and 3.3, we get

$$(3.5)\qquad \begin{aligned} \|u'C_\varphi\|_{\mathrm{es}, H^\infty_{v_{(2+\alpha)/p}} \to H^\infty_{v_1}} &= \limsup_{j\to\infty} \frac{\|u'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{(2+\alpha)/p}}} \\ &= \limsup_{j\to\infty} \frac{j^{(2+\alpha)/p}\|u'\varphi^{j-1}\|_{v_1}}{j^{(2+\alpha)/p}\|z^{j-1}\|_{v_{(2+\alpha)/p}}} \\ &\approx \limsup_{j\to\infty} j^{(2+\alpha)/p}\|u'\varphi^{j-1}\|_{v_1} \\ &= \limsup_{j\to\infty} j^{(2+\alpha)/p}\|J_u(\varphi^{j-1})\|_{\mathcal{B}} \end{aligned}$$

and

$$(3.6)\qquad \begin{aligned} \|u\varphi'C_\varphi\|_{\mathrm{es}, H^\infty_{v_{(2+\alpha+p)/p}} \to H^\infty_{v_1}} &= \limsup_{j\to\infty} \frac{\|u\varphi'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{(2+\alpha+p)/p}}} \\ &\approx \limsup_{j\to\infty} j^{(2+\alpha+p)/p}\|u\varphi'\varphi^{j-1}\|_{v_1} \\ &= \limsup_{j\to\infty} j^{(2+\alpha)/p}\|I_u(\varphi^j)\|_{\mathcal{B}}. \end{aligned}$$

T h e u p p e r e s t i m a t e. From the fact $(uC_\varphi f)'(z) = u'(z)f(\varphi(z)) + u(z) \times \varphi'(z)f'(\varphi(z))$, it is easy to see that

$$(3.7)\qquad \|uC_\varphi\|_{\mathrm{es}, A^p_\alpha \to \mathcal{B}} \leqslant \|u'C_\varphi\|_{\mathrm{es}, H^\infty_{v_{(2+\alpha)/p}} \to H^\infty_{v_1}} + \|u\varphi'C_\varphi\|_{\mathrm{es}, H^\infty_{v_{(2+\alpha+p)/p}} \to H^\infty_{v_1}}.$$

Then, by (3.5), (3.6) and (3.7) we get

$$\begin{aligned} \|uC_\varphi\|_{\mathrm{es}, A^p_\alpha \to \mathcal{B}} &\lesssim \limsup_{j\to\infty} j^{(2+\alpha)/p}\|I_u(\varphi^j)\|_{\mathcal{B}} + \limsup_{j\to\infty} j^{(2+\alpha)/p}\|J_u(\varphi^{j-1})\|_{\mathcal{B}} \\ &\lesssim \max\Big\{\limsup_{j\to\infty} j^{(2+\alpha)/p}\|I_u(\varphi^j)\|_{\mathcal{B}}, \limsup_{j\to\infty} j^{(2+\alpha)/p}\|J_u(\varphi^{j-1})\|_{\mathcal{B}}\Big\}. \end{aligned}$$

T h e l o w e r e s t i m a t e. From Theorem 2.1 and Lemma 3.1, we have

$$\|uC_\varphi\|_{\mathrm{es}, A^p_\alpha \to \mathcal{B}} \gtrsim P = \|u'C_\varphi\|_{\mathrm{es}, H^\infty_{(2+\alpha)/p} \to H^\infty_{v_1}} \approx \limsup_{j\to\infty} j^{(2+\alpha)/p}\|J_u(\varphi^{j-1})\|_{\mathcal{B}}$$

and

$$\|uC_\varphi\|_{\mathrm{es},A^p_\alpha\to\mathcal{B}} \gtrsim Q = \|u\varphi' C_\varphi\|_{\mathrm{es},H^\infty_{v_{(2+\alpha+p)/p}}\to H^\infty_{v_1}} \approx \limsup_{j\to\infty} j^{(2+\alpha)/p}\|I_u(\varphi^j)\|_{\mathcal{B}}.$$

Therefore,

$$\|uC_\varphi\|_{\mathrm{es},A^p_\alpha\to\mathcal{B}} \gtrsim \max\Big\{\limsup_{j\to\infty} j^{(2+\alpha)/p}\|I_u(\varphi^j)\|_{\mathcal{B}}, \limsup_{j\to\infty} j^{(2+\alpha)/p}\|J_u(\varphi^{j-1})\|_{\mathcal{B}}\Big\}.$$

This completes the proof. □

From Theorem 3.2, we immediately get the following result.

Theorem 3.3. *Let $1 \leqslant p < \infty$, $\alpha > -1$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ is bounded. Then the operator $uC_\varphi\colon A^p_\alpha \to \mathcal{B}$ is compact if and only if*

$$\limsup_{j\to\infty} j^{(2+\alpha)/p}\|I_u(\varphi^j)\|_{\mathcal{B}} = 0 \quad \textit{and} \quad \limsup_{j\to\infty} j^{(2+\alpha)/p}\|J_u(\varphi^{j-1})\|_{\mathcal{B}} = 0.$$

We end this section with a new characterization of boundedness, compactness and essential norm of the operator $uC_\varphi\colon H^p \to \mathcal{B}$. Carefully check the proofs of Theorems 3.1 and 3.2, by replacing A^p_α and α by H^p and -1, respectively, we get the following result.

Theorem 3.4. *Let $1 \leqslant p < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the following statements hold.*

(a) *The operator $uC_\varphi\colon H^p \to \mathcal{B}$ is bounded if and only if*

$$\sup_{j\geqslant 1} j^{1/p}\|I_u(\varphi^j)\|_{\mathcal{B}} < \infty \quad \textit{and} \quad \sup_{j\geqslant 1} j^{1/p}\|J_u(\varphi^{j-1})\|_{\mathcal{B}} < \infty.$$

(b) *If the operator $uC_\varphi\colon H^p \to \mathcal{B}$ is bounded, then $uC_\varphi\colon H^p \to \mathcal{B}$ is compact if and only if*

$$\limsup_{j\to\infty} j^{1/p}\|I_u(\varphi^j)\|_{\mathcal{B}} = 0 \quad \textit{and} \quad \limsup_{j\to\infty} j^{1/p}\|J_u(\varphi^{j-1})\|_{\mathcal{B}} = 0.$$

Moreover,

$$\|uC_\varphi\|_{\mathrm{es},H^p\to\mathcal{B}} \approx \max\Big\{\limsup_{j\to\infty} j^{1/p}\|I_u(\varphi^j)\|_{\mathcal{B}}, \limsup_{j\to\infty} j^{1/p}\|J_u(\varphi^{j-1})\|_{\mathcal{B}}\Big\}.$$

From the above results, we immediately get the following new characterization of the operator $C_\varphi\colon A^p_\alpha$ (or H^p) $\to \mathcal{B}$.

Corollary 3.1. *Let $1 \leqslant p < \infty$, $\alpha > -1$ and $\varphi \in S(\mathbb{D})$. Then the following statements hold.*

(a) *The operator $C_\varphi\colon A^p_\alpha \to \mathcal{B}$ is bounded if and only if $\sup\limits_{j\geqslant 1} j^{(\alpha+2)/p}\|\varphi^j\|_{\mathcal{B}} < \infty$.*

(b) *If the operator $C_\varphi\colon A^p_\alpha \to \mathcal{B}$ is bounded, then $C_\varphi\colon A^p_\alpha \to \mathcal{B}$ is compact if and only if $\limsup\limits_{j\to\infty} j^{(\alpha+2)/p}\|\varphi^j\|_{\mathcal{B}} = 0$. Moreover,*

$$\|C_\varphi\|_{\mathrm{es}, A^p_\alpha\to\mathcal{B}} \approx \limsup_{j\to\infty} j^{(\alpha+2)/p}\|\varphi^j\|_{\mathcal{B}}.$$

Corollary 3.2. *Let $1 \leqslant p < \infty$ and $\varphi \in S(\mathbb{D})$. Then the following statements hold.*

(a) *The operator $C_\varphi\colon H^p \to \mathcal{B}$ is bounded if and only if $\sup\limits_{j\geqslant 1} j^{1/p}\|\varphi^j\|_{\mathcal{B}} < \infty$.*

(b) *If the operator $C_\varphi\colon H^p \to \mathcal{B}$ is bounded, then $C_\varphi\colon H^p \to \mathcal{B}$ is compact if and only if $\limsup\limits_{j\to\infty} j^{1/p}\|\varphi^j\|_{\mathcal{B}} = 0$. Moreover,*

$$\|C_\varphi\|_{\mathrm{es}, H^p\to\mathcal{B}} \approx \limsup_{j\to\infty} j^{1/p}\|\varphi^j\|_{\mathcal{B}}.$$

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