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SOME INEQUALITIES FOR RADIAL BLASCHKE-MINKOWSKI HOMOMORPHISMS

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Abstract. We establish some Brunn-Minkowski type inequalities for radial Blaschke-Minkowski homomorphisms with respect to Orlicz radial sums and differences of dual quermassintegrals.

Keywords: radial Blaschke-Minkowski homomorphism; Orlicz radial sum

MSC 2010: 52A20, 52A40

1. Introduction

During the last three decades, convex geometric analysis has achieved important developments. The classical Brunn-Minkowski theory has been extended to the L_p Brunn-Minkowski theory (see e.g. [6], [21]), and more recently to the more general Orlicz-Brunn-Minkowski theory, see [7], [8], [28], [29], [34].

Projection bodies and intersection bodies play a critical role in the solution of Shephard's problem, respectively the Busemann-Petty problem. We refer the reader to [16], [17], [33], [5], [4], [9], [10], [12], [13], [14], [18], [19], [20], [23], [22]. The projection body operator and intersection body operator are continuous and $\mathrm{GL}(n)$ contravariant valuations, see [16], [17]. Schuster [25], [26] introduced the notion of Blaschke-Minkowski homomorphisms, respectively radial Blaschke-Minkowski homomorphisms, which are more general than the well known projection body operator, respectively the intersection body operator.

For $n \geqslant 3$, let $\mathcal{K}^n$ be the space of compact convex sets with nonempty interior in $\mathbb{R}^n$ endowed with the Hausdorff topology. A map $\Phi\colon \mathcal{K}^n \to \mathcal{K}^n$ is called a Blaschke-Minkowski homomorphism, see [25], if it satisfies the following conditions:

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(a) Φ is continuous.

(b) For $K_1, K_2 \in \mathcal{K}^n$,

$$\Phi(K_1 \# K_2) = \Phi K_1 + \Phi K_2,$$

where $K_1 \# K_2$ denotes the Blaschke sum of K_1 and K_2, and $\Phi K_1 + \Phi K_2$ is the Minkowski sum of ΦK_1 and ΦK_2.

(c) For all $K \in \mathcal{K}^n$ and every $\upsilon \in SO(n)$,

$$\Phi(\upsilon K) = \upsilon \Phi K,$$

where $SO(n)$ is the group of rotations of $\mathbb{R}^n$.

A star body is a compact subset of $\mathbb{R}^n$ that is star-shaped with respect to the origin and has a positive continuous radial function (see Section 2). For $n \geqslant 3$, we denote by $\mathcal{S}^n$ the set of all star bodies of $\mathbb{R}^n$ endowed with the Hausdorff topology. A map $\Psi\colon \mathcal{S}^n \to \mathcal{S}^n$ is called a radial Blaschke-Minkowski homomorphism, see [25], if it satisfies the following conditions:

(a) Ψ is continuous.

(b) For $K_1, K_2 \in \mathcal{S}^n$,

$$\Psi(K_1 \mathbin{\widetilde{\sharp}} K_2) = \Psi K_1 \mathbin{\widetilde{+}} \Psi K_2,$$

where $K_1 \mathbin{\widetilde{\sharp}} K_2$ denotes the radial Blaschke sum of K_1 and K_2, and $\Psi K_1 \mathbin{\widetilde{+}} \Psi K_2$ is the radial Minkowski sum of ΨK_1 and ΨK_2.

(c) For all $K \in \mathcal{S}^n$ and every $\upsilon \in SO(n)$,

$$\Psi(\upsilon K) = \upsilon \Psi K,$$

where $SO(n)$ is the group of rotations of $\mathbb{R}^n$.

Volume inequalities for convex body and star body valued valuations are an active field of research (see [1], [2], [25], [26], [27], [30], [32], [11]). In particular, Schuster in [25] established the following Brunn-Minkowski type inequalities.

Theorem A. *Let $\Psi\colon \mathcal{S}^n \to \mathcal{S}^n$ be a radial Blaschke-Minkowski homomorphism. If $K_1, K_2 \in \mathcal{S}^n$, then*

$$V(\Psi(K_1 \mathbin{\widetilde{+}} K_2))^{1/n(n-1)} \leqslant V(\Psi K_1)^{1/n(n-1)} + V(\Psi K_2)^{1/n(n-1)},$$

with equality if and only if K_1 and K_2 are dilates.

In fact, a more general form of the Brunn-Minkowski inequality for radial Blaschke-Minkowski homomorphisms Ψ and dual quermassintegrals $\widetilde{W}_j$ holds (see [25], Theorem 7.6): If $K_1, L_1 \in \mathcal{S}^n$, $0 \leqslant i \leqslant n-1$ and $0 \leqslant j < n-2$, then

$$\begin{aligned} (1.1) \qquad & \widetilde{W}_i(\Psi_j(K_1 \,\widetilde{+}\, K_2))^{1/(n-i)(n-j-1)} \\ & \leqslant \widetilde{W}_i(\Psi_j K_1)^{1/(n-i)(n-j-1)} + \widetilde{W}_i(\Psi_j K_2)^{1/(n-i)(n-j-1)}, \end{aligned}$$

with equality if and only if K_1 and K_2 are dilates.

Wang in [27] established the following L_p Brunn-Minkowski type inequalities for mixed radial Blaschke-Minkowski homomorphisms.

Theorem B. *Let* $\Psi\colon \underbrace{\mathcal{S}^n \times \ldots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ *be a mixed radial Blaschke-Minkowski homomorphism. If* $K_1, K_2 \in \mathcal{S}^n$, *then for* $0 \leqslant i \leqslant n-2$, $0 \leqslant j \leqslant n-2$, $0 < p < n-j-1$,

$$\begin{aligned} (1.2) \qquad & \widetilde{W}_i(\Psi_j(K_1 \,\widetilde{+}_p\, K_2))^{p/(n-i)(n-j-1)} \\ & \leqslant \widetilde{W}_i(\Psi_j K_1)^{p/(n-i)(n-j-1)} + \widetilde{W}_i(\Psi_j K_2)^{p/(n-i)(n-j-1)}, \end{aligned}$$

with equality if and only if K_1 *and* K_2 *are dilates.*

Leng in [15] established the following Brunn-Minkowski type inequality for the volume difference function.

Theorem C. *If* K, L, K_1 *and* L_1 *are compact domains,* $K_1 \subseteq K, L_1 \subseteq L$ *and* K_1 *is a homothetic copy of* L_1, *then*

$$(V(K+L) - V(K_1+L_1))^{1/n} \geqslant (V(K) - V(K_1))^{1/n} + (V(L) - V(L_1))^{1/n}.$$

Equality holds if and only if K *and* L *are homothetic and* $(V(K), V(K_1)) = R(V(L), V(L_1))$, *where* R *is a constant.*

The aim of this paper is to establish Brunn-Minkowski type inequalities for radial Blaschke-Minkowski homomorphisms. Let $\mathcal{C}^+$ be the set of all Orlicz functions, that is, strictly decreasing convex functions $\phi\colon (0,\infty) \to (0,\infty)$ such that $\lim\limits_{t\to 0^+} \phi(t) = \infty$ and $\lim\limits_{t\to\infty} \phi(t) = 0$. For $\phi \in \mathcal{C}^+$ the associated radial Orlicz addition is denoted by $\widetilde{+}_\phi$, see (2.1). First we show the following Orlicz-Brunn-Minkowski type inequality for radial Blaschke-Minkowski homomorphisms:

Theorem 1.1. *Let* $\Psi\colon \underbrace{\mathcal{S}^n \times \ldots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ *be a mixed radial Blaschke-Minkowski homomorphism. If* $K_1, K_2 \in \mathcal{S}^n$, $0 \leqslant i \leqslant n-1$, *and* $0 \leqslant j < n-2$, *while* $\phi \in \mathcal{C}^+$, *then*

$$\phi(1) \geqslant \phi\biggl(\biggl(\frac{\widetilde{W}_i(\Psi_j K_1)}{\widetilde{W}_i(\Psi_j(K_1 \mathbin{\widetilde{+}_\phi} K_2))}\biggr)^{1/(n-i)(n-j-1)}\biggr) + \phi\biggl(\biggl(\frac{\widetilde{W}_i(\Psi_j K_2)}{\widetilde{W}_i(\Psi_j(K_1 \mathbin{\widetilde{+}_\phi} K_2))}\biggr)^{1/(n-i)(n-j-1)}\biggr),$$

with equality if and only if K_1 *and* K_2 *are dilates.*

In particular, for $\phi(t) = t^p$, $p < 0$, Theorem 1.1 implies

$$\widetilde{W}_i(\Psi_j(K_1 \mathbin{\widetilde{+}_p} K_2))^{p/(n-i)(n-j-1)} \geqslant \widetilde{W}_i(\Psi_j K_1)^{p/(n-i)(n-j-1)} + \widetilde{W}_i(\Psi_j K_2)^{p/(n-i)(n-j-1)},$$

which complements Theorem B obtained by Wang.

Theorem 1.2. *Let* $K, L, K_1, L_1 \in \mathcal{S}^n$, $K \subseteq K_1$, $L \subseteq L_1$ *and* K_1 *is a dilate of* L_1. *Then for* $0 \leqslant i < n$, $0 \leqslant j < n-2$, $0 < p < n-j-1$,

$$[\widetilde{W}_i(\Psi_j(K_1 \mathbin{\widetilde{+}_p} L_1)) - \widetilde{W}_i(\Psi_j(K \mathbin{\widetilde{+}_p} L))]^{p/(n-i)(n-j-1)} \geqslant (\widetilde{W}_i(\Psi_j K_1) - \widetilde{W}_i(\Psi_j K))^{p/(n-i)(n-j-1)} + (\widetilde{W}_i(\Psi_j L_1) - \widetilde{W}_i(\Psi_j L))^{p/(n-i)(n-j-1)},$$

with equality if and only if K *and* L *are dilates and*

$$\widetilde{W}_i(\Psi_j K) \,/\, \widetilde{W}_i(\Psi_j K_1) = \widetilde{W}_i(\Psi_j L) \,/\, \widetilde{W}_i(\Psi_j L_1).$$

2. Notation and background material

Let $\mathcal{K}^n$ denote the class of convex bodies (compact, convex subsets with nonempty interiors) in the Euclidean space $\mathbb{R}^n$. For the class of convex bodies containing the origin in their interior, we write $\mathcal{K}_0^n$. $V(K)$ and $\widetilde{W}_i(K)$, $0 \leqslant i < n$, denote the n-dimensional volume and the quermassintegrals of a convex body K. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$.

For $K \in \mathcal{K}^n$, its support function $h_K(\cdot) := h(K, \cdot)\colon \mathbb{R}^n \to \mathbb{R}$ is defined for $u \in \mathbb{R}^n$ by $h(K, u) = \max\{x \cdot u\colon x \in K\}$, where $x \cdot u$ denotes the standard inner product of u and x in $\mathbb{R}^n$. The support function determines a convex body uniquely. The Minkowski sum $\alpha K + \beta L$ of $K, L \in \mathcal{K}^n$ with $\alpha, \beta > 0$ is determined by $h(\alpha K + \beta L, u) = \alpha h(K, u) + \beta h(L, u)$.

For a set $K \subset \mathbb{R}^n$ that is star-shaped with respect to the origin, the radial function $\varrho_K(\cdot) := \varrho(K, \cdot)\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is defined for $u \in \mathbb{R}^n \setminus \{0\}$ by $\varrho(K, u) = \max\{\lambda \geqslant 0\colon x \in \lambda K\}$. If ϱ_K is positive and continuous, then K is called a star body about the origin. We write $\mathcal{S}^n$ for the space of all star bodies in $\mathbb{R}^n$. A star body is uniquely determined by its radial function, and the radial sum $\alpha K \mathbin{\widetilde{+}} \beta L$ of $K, L \in \mathcal{S}^n$ with $\alpha, \beta > 0$ is determined by $\varrho(\alpha K \mathbin{\widetilde{+}} \beta L, u) = \alpha\varrho(K, u) + \beta\varrho(L, u)$. If $\varrho(K, u)/\varrho(L, u)$ is independent of $u \in S^{n-1}$, then K and L are dilates.

The Orlicz radial sum is defined by Zhu, Zhou and Xu in [34]: Let $\alpha, \beta > 0$ and $\phi \in \mathcal{C}^+$. The Orlicz radial sum $\alpha \cdot K \mathbin{\widetilde{+}_\phi} \beta \cdot L$ is given for all $u \in \mathbb{R}^n \setminus \{0\}$ by

$$\varrho(\alpha \cdot K \mathbin{\widetilde{+}_\phi} \beta \cdot L, u) = \sup\Big\{t > 0\colon\ \alpha\phi\Big(\frac{\varrho(K, u)}{t}\Big) + \beta\phi\Big(\frac{\varrho(L, u)}{t}\Big) \leqslant \phi(1)\Big\}. \tag{2.1}$$

If $\phi(t) = t^p$, $p < 0$, then the Orlicz radial sum reduces to the L_p radial sum $\widetilde{+}_p$, which is given by

$$\varrho(\alpha \cdot K \mathbin{\widetilde{+}_\phi} \beta \cdot L, u)^p = \alpha\varrho(K, u)^p + \beta\varrho(L, u)^p$$

for all $u \in \mathbb{R}^n \setminus \{0\}$.

2.1. Mixed volumes and mixed quermassintegrals. Zhao in [31] defines dual Orlicz mixed quermassintegrals $\overline{W}_{\psi,i}(K, L)$ of K, L with respect to a strictly increasing convex function $\psi\colon (0, \infty) \to (0, \infty)$ with $\lim\limits_{t\to 0^+} = 0$ and $\lim\limits_{t\to\infty} = \infty$, by

$$\overline{W}_{\psi,i}(K, L) := \frac{1}{n}\int_{S^{n-1}} \psi\Big(\frac{\varrho(K, u)}{\varrho(L, u)}\Big)\varrho(K, u)^{n-i}\,\mathrm{d}S(u).$$

We use a slightly different definition: For $K, L \in \mathcal{S}^n$ and $\phi \in \mathcal{C}^+$ we define $\widetilde{W}_{\phi,i}(K, L)$ by

$$\widetilde{W}_{\phi,i}(K, L) := \frac{1}{n}\int_{S^{n-1}} \phi\Big(\frac{\varrho(L, u)}{\varrho(K, u)}\Big)\varrho(K, u)^{n-i}\,\mathrm{d}S(u).$$

If $\phi \in \mathcal{C}^+$ is such that $\psi(t) = \phi(1/t)$ is convex, then $\overline{W}_{\psi,i}(K, L) = \widetilde{W}_{\phi,i}(K, L)$.

For $K, M, N \in \mathcal{S}^n$, $0 \leqslant i \leqslant n-2$ and $\phi \in \mathcal{C}^+$, the dual mixed Orlicz quermassintegral $\widetilde{W}_{\phi,i,1}(K, M, N)$ is defined by

$$\frac{n-i-1}{\phi'_r(1)}\widetilde{W}_{\phi,i,1}(K, M, N) = \lim_{\varepsilon\to 0^+} \frac{\widetilde{W}_i(K \mathbin{\widetilde{+}_\phi} \varepsilon \cdot M, N) - \widetilde{W}_i(K, N)}{\varepsilon}, \tag{2.2}$$

where $\phi_r'(1)$ is the right-derivative of ϕ at 1, which exists, since ϕ is convex (see, e.g., [24], Theorem 1.5.4), and $\phi_r'(1) < 0$, since ϕ is strictly decreasing. For the unit ball B in $\mathbb{R}^n$, by Lemma 3.3 (see Section 3) we have $\widetilde{W}_{\phi,i,1}(K,M,B) = \widetilde{W}_{\phi,i+1}(K,M)$.

Lutwak in [19] showed that the volume of a radial Minkowski combination $\lambda_1 L_1 \widetilde{+} \ldots \widetilde{+} \lambda_m L_m$ of star bodies $L_1,\ldots,L_m$ can be expressed as a homogeneous polynomial of degree n:

$$V(\lambda_1 L_1 \widetilde{+} \ldots \widetilde{+} \lambda_m L_m) = \sum_{i_1,\ldots,i_n=1}^{m} \widetilde{V}(L_{i_1},\ldots,L_{i_n})\lambda_{i_1}\ldots\lambda_{i_n}.$$

The coefficients $\widetilde{V}(L_{i_1},\ldots,L_{i_n})$ are called the dual mixed volumes of $L_{i_1},\ldots,L_{i_n}$. The definition of dual mixed volumes $\widetilde{W}_i(L,N)$ and $\widetilde{W}_i(L)$ are analogous to those for the mixed volume. The dual mixed volume $\widetilde{W}_i(L,N)$ has the integral representation

$$\widetilde{W}_i(L,N) = \frac{1}{n}\int_{S^{n-1}} \varrho(L,u)^{n-i-1}\varrho(N,u)\,\mathrm{d}S(u), \tag{2.3}$$

where $\mathrm{d}S(u)$ is the spherical Lebesgue measure of S^{n-1}.

The Minkowski inequality for dual mixed quermassintegrals is stated by Wang in [27] as follows: For $L_1, L_2 \in \mathcal{S}^n$ and $0 \leqslant i < n-1$,

$$\widetilde{W}_i(L_1,L_2)^{n-i} \leqslant \widetilde{W}_i(L_1)^{n-i-1}\widetilde{W}_i(L_2), \tag{2.4}$$

with equality if and only if L_1 and L_2 are dilates. It is a direct consequence of Hölder's inequality for integrals (see, e.g., [24], Lemma 9.3.1). A variant, which also follows from Hölder's inequality, is the following: For $L_1, L_2, Q \in \mathcal{S}^n$ and $0 \leqslant i < n-2$, set

$$\widetilde{W}_{i,1}(L_1,L_2,Q) = \frac{1}{n}\int_{S^{n-1}} \varrho(L_1,u)^{n-i-2}\varrho(L_2,u)\varrho(Q,u)\,\mathrm{d}S(u).$$

Then

$$\widetilde{W}_{i,1}(L_1,L_2,Q)^{n-i-1} \leqslant \widetilde{W}_i(L_1,Q)^{n-i-2}\widetilde{W}_i(L_2,Q). \tag{2.5}$$

2.2. Mixed Blaschke-Minkowski homomorphisms and mixed radial Blaschke-Minkowski homomorphisms. Let $\Phi\colon \mathcal{K}^n \to \mathcal{K}^n$ be a Blaschke-Minkowski homomorphism. Schuster in [25], Theorem 1.2, showed that if $K_1,\ldots,K_m \in \mathcal{K}^n$ and $\lambda_1,\ldots,\lambda_m \geqslant 0$, then there is a continuous operator

$$\Phi\colon \underbrace{\mathcal{K}^n \times \ldots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n,$$

symmetric in its arguments, such that

$$\Phi(\lambda_1 K_1 + \ldots + \lambda_m K_m) = \sum_{i_1,\ldots,i_{n-1}=1}^{m} \Phi(K_{i_1},\ldots,K_{i_{n-1}})\lambda_{i_1}\ldots\lambda_{i_{n-1}},$$

where the $\Phi(K_{i_1},\ldots,K_{i_{n-1}})$ are called mixed Blaschke-Minkowski homomorphisms, and the sum is with respect to Minkowski addition.

If $K_1 = \ldots = K_{n-i-1} = K$ and $K_{n-i} = \ldots = K_{n-1} = M$, then $\Phi(K_1,\ldots,K_{n-1})$ will be written as $\Phi_i(K,M)$. In particular, $\Phi_i K = \Phi_i(K,B)$ and $\Phi K = \Phi_0 K$.

Let $\Psi\colon \mathcal{S}^n \to \mathcal{S}^n$ be a radial Blaschke-Minkowski homomorphism. Schuster in [25], Theorem 1.2_d, showed that if $K_1,\ldots,K_m \in \mathcal{S}^n$ and $\lambda_1,\ldots,\lambda_m \geqslant 0$, then there is a continuous operator

$$\Psi\colon \underbrace{\mathcal{S}^n \times \ldots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n,$$

symmetric in its arguments, such that

$$\Psi(\lambda_1 K_1 \mathbin{\tilde{+}} \ldots \mathbin{\tilde{+}} \lambda_m K_m) = \sum_{i_1,\ldots,i_{n-1}=1}^{m} \Psi(K_{i_1},\ldots,K_{i_{n-1}})\lambda_{i_1}\ldots\lambda_{i_{n-1}},$$

where the $\Psi(K_{i_1},\ldots,K_{i_{n-1}})$ are called mixed radial Blaschke-Minkowski homomorphisms, and the sum is with respect to the radial sum.

If $K_1 = \ldots = K_{n-i-1} = K$ and $K_{n-i} = \ldots = K_{n-1} = M$, then $\Psi(K_1,\ldots,K_{n-1})$ will be written as $\Psi_i(K,M)$. In particular, $\Psi_i K = \Psi_i(K,B)$ and $\Psi K = \Psi_0 K$.

3. Lemmas and main results

Lemma 3.1 (Jensen's inequality [8]). *Let μ be a probability measure in a space X, let U be an open convex set in $\mathbb{R}^n$, and let φ be a convex real-valued function on U. Assume that $g\colon X \to U$ is measurable and component-wise μ-integrable, and that $\varphi \circ g$ is μ-integrable. Let $z_0 = \int_X g(x)\,\mathrm{d}\mu(x)$. Then $z_0 \in U$ and*

$$\int_X \varphi(g(x))\,\mathrm{d}\mu(x) \geqslant \varphi\Big(\int_X g(x)\,\mathrm{d}\mu(x)\Big).$$

Lemma 3.2 ([34]). *If $\phi \in \mathcal{C}^+$ and $K, L \in \mathcal{S}^n$, then*

$$\lim_{\varepsilon\to 0^+} \frac{\varrho(K \mathbin{\tilde{+}}_\phi \varepsilon\cdot L, u) - \varrho(K,u)}{\varepsilon} = \frac{\varrho(K,u)}{\phi_r'(1)}\phi\Big(\frac{\varrho(L,u)}{\varrho(K,u)}\Big), \quad \textit{uniformly on } S^{n-1}.$$

Lemma 3.3. *If $\phi \in \mathcal{C}^+$ and $K, L, Q \in \mathcal{S}^n$, then for each $i = 0, 1, \ldots, n-2$,*

$$\lim_{\varepsilon\to0^+} \frac{\widetilde{W}_i(K \mathbin{\widetilde{+}_\phi} \varepsilon\cdot L, Q) - \widetilde{W}_i(K,Q)}{\varepsilon} = \frac{n-i-1}{\phi_r'(1)n}\int_{S^{n-1}} \phi\Big(\frac{\varrho(L,u)}{\varrho(K,u)}\Big)\varrho(K,u)^{n-i-1}\varrho(Q,u)\,\mathrm{d}S(u).$$

P r o o f. From Lemma 3.2 we derive

$$\begin{aligned}
&\lim_{\varepsilon\to0^+} \frac{\widetilde{W}_i(K \mathbin{\widetilde{+}_\phi} \varepsilon\cdot L, Q) - \widetilde{W}_i(K,Q)}{\varepsilon}\\
&= \lim_{\varepsilon\to0^+}\Big[\frac{1}{n\varepsilon}\int_{S^{n-1}} \varrho(K \mathbin{\widetilde{+}_\phi} \varepsilon\cdot L, u)^{n-i-1}\varrho(Q,u)\\
&\qquad - \frac{1}{n\varepsilon}\int_{S^{n-1}}\varrho(K,u)^{n-i-1}\varrho(Q,u)\,\mathrm{d}S(u)\Big]\\
&= \lim_{\varepsilon\to0^+}\frac{1}{n}\int_{S^{n-1}} \frac{\varrho(K \mathbin{\widetilde{+}_\phi} \varepsilon\cdot L, u)^{n-i-1} - \varrho(K,u)^{n-i-1}}{\varepsilon}\varrho(Q,u)\,\mathrm{d}S(u)\\
&= \frac{1}{n}\int_{S^{n-1}} \lim_{\varepsilon\to0^+}\frac{\varrho(K \mathbin{\widetilde{+}_\phi} \varepsilon\cdot L, u)^{n-i-1} - \varrho(K,u)^{n-i-1}}{\varepsilon}\varrho(Q,u)\,\mathrm{d}S(u)\\
&= \frac{n-i-1}{\phi_r'(1)n}\int_{S^{n-1}} \phi\Big(\frac{\varrho(L,u)}{\varrho(K,u)}\Big)\varrho(K,u)^{n-i-1}\varrho(Q,u)\,\mathrm{d}S(u).
\end{aligned}$$

□

Lemma 3.4. *If $\phi \in \mathcal{C}^+$ and $K, L, Q \in \mathcal{S}^n$, while $0 \leqslant i \leqslant n-2$, then*

$$\frac{\widetilde{W}_{\phi,i,1}(K,L,Q)}{\widetilde{W}_i(K,Q)} \geqslant \phi\bigg(\Big(\frac{\widetilde{W}_i(L,Q)}{\widetilde{W}_i(K,Q)}\Big)^{1/(n-i-1)}\bigg). \tag{3.1}$$

P r o o f. Since

$$\widetilde{W}_i(K,Q) = \frac{1}{n}\int_{S^{n-1}}\varrho(K,u)^{n-i-1}\varrho(Q,u)\,\mathrm{d}S(u),$$

and since $\varrho(K,\cdot)$ and $\varrho(Q,\cdot)$ are positive functions, we have $\widetilde{W}_i(K,Q) > 0$. Hence, $\varrho(K,u)^{n-i-1}\varrho(Q,u)\,\mathrm{d}S(u)/n\widetilde{W}_i(K,Q)$ is a probability measure on S^{n-1}. It follows from Lemma 3.1, Lemma 3.3 and (2.5) and the fact that ϕ is decreasing, that

$$\begin{aligned}\frac{\widetilde{W}_{\phi,i,1}(K,L,Q)}{\widetilde{W}_i(K,Q)} &= \frac{1}{\widetilde{W}_i(K,Q)n}\int_{S^{n-1}}\phi\Big(\frac{\varrho(L,u)}{\varrho(K,u)}\Big)\varrho(K,u)^{n-i-1}\varrho(Q,u)\,\mathrm{d}S(u)\\ &\geqslant \phi\Big(\frac{1}{n\widetilde{W}_i(K,Q)}\int_{S^{n-1}}\varrho(L,u)\varrho(K,u)^{n-i-2}\varrho(Q,u)\,\mathrm{d}S(u)\Big)\\ &= \phi\Big(\frac{\widetilde{W}_{i,1}(K,L,Q)}{\widetilde{W}_i(K,Q)}\Big)\\ &\geqslant \phi\Big(\frac{\widetilde{W}_i(K,Q)^{(n-i-2)/(n-i-1)}\widetilde{W}_i(L,Q)^{1/(n-i-1)}}{\widetilde{W}_i(K,Q)}\Big)\\ &= \phi\Big(\Big(\frac{\widetilde{W}_i(L,Q)}{\widetilde{W}_i(K,Q)}\Big)^{1/(n-i-1)}\Big).\end{aligned}$$

□

Lemma 3.5. *If $\phi\in\mathcal{C}^+$ and $K_1,K_2,Q\in\mathcal{S}^n$, while $0\leqslant i\leqslant n-2$, then*

$$\phi(1)\geqslant\phi\Big(\Big(\frac{\widetilde{W}_i(K_1,Q)}{\widetilde{W}_i(K_1\,\widetilde{+}_\phi\,K_2,Q)}\Big)^{1/(n-i-1)}\Big)+\phi\Big(\Big(\frac{\widetilde{W}_i(K_2,Q)}{\widetilde{W}_i(K_1\,\widetilde{+}_\phi\,K_2,Q)}\Big)^{1/(n-i-1)}\Big). \tag{3.2}$$

Proof. For brevity, let $\widetilde{K}=K_1\,\widetilde{+}_\phi\,K_2$. The definition of $\widetilde{K}$ necessarily implies for all $u\in S^{n-1}$ that

$$\phi(1)=\phi\Big(\frac{\varrho(K_1,u)}{\varrho(\widetilde{K},u)}\Big)+\phi\Big(\frac{\varrho(K_2,u)}{\varrho(\widetilde{K},u)}\Big).$$

From

$$\widetilde{W}_i(\widetilde{K},Q)=\frac{1}{n}\int_{S^{n-1}}\varrho(\widetilde{K},u)^{n-i-1}\varrho(Q,u)\,\mathrm{d}S(u),$$

and by Lemma 3.4, we derive

$$\begin{aligned}\phi(1)\widetilde{W}_i(\widetilde{K},Q) &= \frac{1}{n}\int_{S^{n-1}}\Big(\phi\Big(\frac{\varrho(K_1,u)}{\varrho(\widetilde{K},u)}\Big)+\phi\Big(\frac{\varrho(K_2,u)}{\varrho(\widetilde{K},u)}\Big)\Big)\varrho(\widetilde{K},u)^{n-i-1}\varrho(Q,u)\,\mathrm{d}S(u)\\ &= \widetilde{W}_{\phi,i,1}(\widetilde{K},K_1,Q)+\widetilde{W}_{\phi,i,1}(\widetilde{K},K_2,Q)\\ &\geqslant \widetilde{W}_i(\widetilde{K},Q)\Big[\phi\Big(\Big(\frac{\widetilde{W}_i(K_1,Q)}{\widetilde{W}_i(\widetilde{K},Q)}\Big)^{1/(n-i-1)}\Big)+\phi\Big(\Big(\frac{\widetilde{W}_i(K_2,Q)}{\widetilde{W}_i(\widetilde{K},Q)}\Big)^{1/(n-i-1)}\Big)\Big].\end{aligned}$$

Thus, (3.2) is established. □

Lemma 3.6 ([25]). *Let* $\Psi\colon \underbrace{\mathcal{S}^n \times \ldots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ *be a mixed radial Blaschke-Minkowski homomorphism. If* $K, M \in \mathcal{S}^n$, *while* $0 \leqslant i \leqslant n-2$ *and* $0 \leqslant j \leqslant n-2$, *then*

$$\widetilde{W}_i(K, \Psi_j M) = \widetilde{W}_j(M, \Psi_i K). \tag{3.3}$$

Furthermore, the image of a ball B *under a radial Blaschke-Minkowski homomorphism is again a ball. Let* r_Ψ *be such that* $\Psi B = r_\Psi B$. *Then*

$$\widetilde{W}_{n-1}(\Psi_i K) = r_\Psi \widetilde{W}_{i+1}(K). \tag{3.4}$$

We are now ready to prove Theorem 1.1:

Theorem 3.7. *Let* $\Psi\colon \underbrace{\mathcal{S}^n \times \ldots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ *be a mixed radial Blaschke-Minkowski homomorphism. If* $K_1, K_2 \in \mathcal{S}^n$, $0 \leqslant i \leqslant n-2$ *and* $0 \leqslant j < n-2$, *while* $\phi \in \mathcal{C}^+$, *then*

$$\begin{aligned}\phi(1) \geqslant \phi\biggl(\biggl(\frac{\widetilde{W}_i(\Psi_j K_1)}{\widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_\phi K_2))}\biggr)^{1/(n-i)(n-j-1)}\biggr) \\ + \phi\biggl(\biggl(\frac{\widetilde{W}_i(\Psi_j K_2)}{\widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_\phi K_2))}\biggr)^{1/(n-i)(n-j-1)}\biggr),\end{aligned} \tag{3.5}$$

with equality if and only if K_1 *and* K_2 *are dilates.*

Proof. By (2.4), (3.2), (3.3) and the fact that ϕ is decreasing, we have for $M \in \mathcal{S}^n$ that

$$\begin{aligned}&\phi(1)\\ &\geqslant \phi\biggl(\biggl(\frac{\widetilde{W}_j(K_1, \Psi_i M)}{\widetilde{W}_j(K_1 \widetilde{+}_\phi K_2, \Psi_i M)}\biggr)^{1/(n-j-1)}\biggr) + \phi\biggl(\biggl(\frac{\widetilde{W}_j(K_2, \Psi_i M)}{\widetilde{W}_j(K_1 \widetilde{+}_\phi K_2, \Psi_i M)}\biggr)^{1/(n-j-1)}\biggr)\\ &= \phi\biggl(\biggl(\frac{\widetilde{W}_i(M, \Psi_j K_1)}{\widetilde{W}_i(M, \Psi_j(K_1 \widetilde{+}_\phi K_2))}\biggr)^{1/(n-j-1)}\biggr) + \phi\biggl(\biggl(\frac{\widetilde{W}_i(M, \Psi_j K_2)}{\widetilde{W}_i(M, \Psi_j(K_2 \widetilde{+}_\phi K_2))}\biggr)^{1/(n-j-1)}\biggr)\\ &\geqslant \phi\biggl(\biggl(\frac{\widetilde{W}_i(M)^{(n-i-1)/(n-i)}\widetilde{W}_i(\Psi_j K_1)^{1/(n-i)}}{\widetilde{W}_i(M, \Psi_j(K_1 \widetilde{+}_\phi K_2))}\biggr)^{1/(n-j-1)}\biggr)\\ &\quad + \phi\biggl(\biggl(\frac{\widetilde{W}_i(M)^{(n-i-1)/(n-i)}\widetilde{W}_i(\Psi_j K_2)^{1/(n-i)}}{\widetilde{W}_i(M, \Psi_j(K_1 \widetilde{+}_\phi K_2))}\biggr)^{1/(n-j-1)}\biggr).\end{aligned}$$

By the equality conditions of (2.4), equality holds if and only if M, $\Psi_j K_1$, and $\Psi_j K_2$ are dilates.

Set $M = \Psi_j(K_1 \widetilde{+}_\phi K_2)$ and note that $\widetilde{W}_i(M,M) = \widetilde{W}_i(M)$ to obtain (3.5). If there is equality in (3.5), then there exist $\lambda_1, \lambda_2 > 0$ such that

$$\Psi_j K_1 = \lambda_1 \Psi_j(K_1 \widetilde{+}_\phi K_2) \quad \text{and} \quad \Psi_j K_2 = \lambda_2 \Psi_j(K_1 \widetilde{+}_\phi K_2). \tag{3.6}$$

From equality (3.5), it follows that

$$\phi(\lambda_1^{1/(n-j-1)}) + \phi(\lambda_2^{1/(n-j-1)}) = \phi(1).$$

Moreover, (3.4) and (3.6) imply

$$\lambda_1 = \frac{\widetilde{W}_{j+1}(K_1)}{\widetilde{W}_{j+1}(K_1 \widetilde{+}_\phi K_2)} \quad \text{and} \quad \lambda_2 = \frac{\widetilde{W}_{j+1}(K_2)}{\widetilde{W}_{j+1}(K_1 \widetilde{+}_\phi K_2)}. \tag{3.7}$$

Hence, we have

$$\phi(1) = \phi\Big(\Big(\frac{\widetilde{W}_{j+1}(K_1)}{\widetilde{W}_{j+1}(K_1 \widetilde{+}_\phi K_2)}\Big)^{1/(n-j-1)}\Big) + \phi\Big(\Big(\frac{\widetilde{W}_{j+1}(K_2)}{\widetilde{W}_{j+1}(K_1 \widetilde{+}_\phi K_2)}\Big)^{1/(n-j-1)}\Big),$$

and using the equality conditions of an Orlicz-Brunn-Minkowski inequality for the dual mixed quermassintegral, see [31], we obtain K_1 and K_2 are dilates. □

The intersection body operator $I\colon \mathcal{S}^n \to \mathcal{S}^n$ is a radial Blaschke-Minkowski homomorphism and we derive the following corollary from Theorem 3.7:

Corollary 3.8. *If $K_1, K_2 \in \mathcal{S}^n$ and $\phi \in \mathcal{C}^+$, then*

$$\phi(1) \geqslant \phi\Big(\Big(\frac{V(IK_1)}{V(I(K_1 \widetilde{+}_\phi K_2))}\Big)^{1/n(n-1)}\Big) + \phi\Big(\Big(\frac{V(IK_2)}{V(I(K_1 \widetilde{+}_\phi K_2))}\Big)^{1/n(n-1)}\Big), \tag{3.8}$$

with equality if and only if K_1 and K_2 are dilates.

Lemma 3.9 ([3]). *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two series of positive real numbers. If $p > 1$, $a_1^p - \sum\limits_{i=2}^n a_i^p > 0$ and $b_1^p - \sum\limits_{i=2}^n b_i^p > 0$, then*

$$\Big(a_1^p - \sum_{i=2}^n a_i^p\Big)^{1/p} + \Big(b_1^p - \sum_{i=2}^n b_i^p\Big)^{1/p} \leqslant \Big((a_1+b_1)^p - \sum_{i=2}^n (a_i+b_i)^p\Big)^{1/p},$$

with equality if and only if $a_1/b_1 = a_2/b_2 = \ldots = a_n/b_n$.

Lemma 3.10 ([27]). *Let* $\Phi\colon \underbrace{\mathcal{K}^n \times \ldots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n$ *be a mixed Blaschke-Minkowski homomorphism. If* $K_1, K_2 \in \mathcal{K}^n$ *and* $p > 1$*, then for* $0 \leqslant i \leqslant n-2$*,* $0 \leqslant j \leqslant n-2$*,*

$$\begin{aligned} W_i(\Phi_j(K_1 +_p K_2))^{p/(n-i)(n-j-1)} \\ \geqslant W_i(\Phi_j K_1)^{p/(n-i)(n-j-1)} + W_i(\Phi_j K_2)^{p/(n-i)(n-j-1)}, \end{aligned} \tag{3.9}$$

with equality if and only if K_1 *and* K_2 *are dilates.*

Theorem 3.11. *Let* $K, L \in \mathcal{K}^n$*,* $K_1, L_1 \in \mathcal{S}^n$*,* $K_1 \subseteq K$*,* $L_1 \subseteq L$ *and let* K_1, L_1 *be dilates of each other. Then for* $0 \leqslant i < n$*,* $0 \leqslant j < n-2$*,* $1 < p < n-j-1$*,*

$$\begin{aligned} &[W_i(\Phi_j(K +_p L)) - \widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p L_1))]^{p/(n-i)(n-j-1)} \\ &\quad \geqslant (W_i(\Phi_j K) - \widetilde{W}_i(\Psi_j K_1))^{p/(n-i)(n-j-1)} \\ &\quad\quad + (W_i(\Phi_j L) - \widetilde{W}_i(\Psi_j L_1))^{p/(n-i)(n-j-1)}, \end{aligned}$$

with equality if and only if K *and* L *are dilates and*

$$W_i(\Phi_j K) \,/\, \widetilde{W}_i(\Psi_j K_1) = W_i(\Phi_j L) \,/\, \widetilde{W}_i(\Psi_j L_1).$$

P r o o f. By Theorem B and since K_1 and L_1 are dilates, we have

$$\begin{aligned} &\widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p L_1))^{p/(n-i)(n-j-1)} \\ &\quad = \widetilde{W}_i(\Psi_j K_1)^{p/(n-i)(n-j-1)} + \widetilde{W}_i(\Psi_j L_1)^{p/(n-i)(n-j-1)}. \end{aligned}$$

It follows from Lemma 3.9 and Lemma 3.10 that

$$\begin{aligned} &[W_i(\Phi_j(K +_p L)) - \widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p L_1))]^{p/(n-i)(n-j-1)} \\ &\quad \geqslant [(W_i(\Phi_j K)^{p/(n-i)(n-j-1)} + W_i(\Phi_j L)^{p/(n-i)(n-j-1)})^{(n-i)(n-j-1)/p} \\ &\quad\quad - (\widetilde{W}_i(\Psi_j K_1)^{p/(n-i)(n-j-1)} \\ &\quad\quad\quad + \widetilde{W}_i(\Psi_j L_1)^{p/(n-i)(n-j-1)})^{(n-i)(n-j-1)/p}]^{p/(n-i)(n-j-1)} \\ &\quad \geqslant (W_i(\Phi_j K) - \widetilde{W}_i(\Psi_j L))^{p/(n-i)(n-j-1)} \\ &\quad\quad + (W_i(\Phi_j K_1) - \widetilde{W}_i(\Psi_j L_1))^{p/(n-i)(n-j-1)}, \end{aligned}$$

with equality if and only if K and L are dilates and

$$W_i(\Phi_j K) \,/\, \widetilde{W}_i(\Psi_j K_1) = W_i(\Phi_j L) \,/\, \widetilde{W}_i(\Psi_j L_1).$$

□

Taking the mixed projection body Π_i and the mixed intersection operator I_i as the mixed Blaschke-Minkowski and the mixed radial Blaschke-Minkowski homomorphisms, we derive:

Corollary 3.12. *Let $K, L \in \mathcal{K}^n$, $K_1, L_1 \in \mathcal{S}^n$, $K_1 \subseteq K$, $L_1 \subseteq L$ and let K_1, L_1 be dilates of each other. Then for $0 \leqslant i < n$, $0 \leqslant j < n-2$, $1 < p < n-j-1$,*

$$\begin{aligned}[W_i(\Pi_j(K +_p L)) - \widetilde{W}_i(I_j(K_1 \widetilde{+}_p L_1))]^{p/(n-i)(n-j-1)} \\ \geqslant (W_i(\Pi_j K) - \widetilde{W}_i(I_j K_1))^{p/(n-i)(n-j-1)} \\ + (W_i(\Pi_j L) - \widetilde{W}_i(I_j L_1))^{p/(n-i)(n-j-1)},\end{aligned}$$

with equality if and only if K and L are dilates and

$$W_i(\Pi_j K) \,/\, \widetilde{W}_i(I_j K_1) = W_i(\Pi_j L) \,/\, \widetilde{W}_i(I_j L_1).$$

Taking $i = 0$, $j = 0$ in Corollary 3.12 yields

Corollary 3.13. *Let $K, L \in \mathcal{K}^n$, $K_1, L_1 \in \mathcal{S}^n$, $K_1 \subseteq K$, $L_1 \subseteq L$ and let K_1, L_1 be dilates of each other. Then for $1 < p < n-1$,*

$$\begin{aligned}[V(\Pi(K +_p L)) - V(I(K_1 \widetilde{+}_p L_1))]^{p/n(n-1)} \\ \geqslant (V(\Pi K) - V(I K_1))^{p/n(n-1)} + (V(\Pi L) - V(I L_1))^{p/n(n-1)},\end{aligned}$$

with equality if and only if K and L are dilates and

$$V(\Pi K) \,/\, V(I K_1) = V(\Pi L) \,/\, V(I L_1).$$

Theorem 3.14. *Let $K, L, K_1, L_1 \in \mathcal{S}^n$, $K \subseteq K_1$, $L \subseteq L_1$ and let K_1 be a dilate of L_1. Then for $0 \leqslant i < n$, $0 \leqslant j < n-2$, $0 < p < n-j-1$,*

$$\begin{aligned}[\widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p L_1)) - \widetilde{W}_i(\Psi_j(K \widetilde{+}_p L))]^{p/(n-i)(n-j-1)} \\ \geqslant (\widetilde{W}_i(\Psi_j K_1) - \widetilde{W}_i(\Psi_j K))^{p/(n-i)(n-j-1)} \\ + (\widetilde{W}_i(\Psi_j L_1) - \widetilde{W}_i(\Psi_j L))^{p/(n-i)(n-j-1)},\end{aligned}$$

with equality if and only if K and L are dilates and

$$\widetilde{W}_i(\Psi_j K) \,/\, \widetilde{W}_i(\Psi_j K_1) = \widetilde{W}_i(\Psi_j L) \,/\, \widetilde{W}_i(\Psi_j L_1).$$

Proof. We set $q = (n-i)(n-j-1)$. Applying Theorem B, we obtain

$$\begin{aligned}\widetilde{W}_i(\Psi_j(K \widetilde{+}_p L))^{p/q} \leqslant \widetilde{W}_i(\Psi_j K)^{p/q} + \widetilde{W}_i(\Psi_j L)^{p/q}, \\ \widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p L_1))^{p/q} = \widetilde{W}_i(\Psi_j K_1)^{p/q} + \widetilde{W}_i(\Psi_j L_1)^{p/q}.\end{aligned}$$

It follows from Lemma 3.9, which we can use since $q/p > 1$, that

$$[\widetilde{W}_i(\Psi_j(K_1 \widetilde{+}_p L_1)) - \widetilde{W}_i(\Psi_j(K \widetilde{+}_p L))]^{p/q}$$
$$\geqslant [(\widetilde{W}_i(\Psi_j K_1)^{p/q} + \widetilde{W}_i(\Psi_j L_1)^{p/q})^{q/p} - (\widetilde{W}_i(\Psi_j K)^{p/q} + \widetilde{W}_i(\Psi_j L)^{p/q})^{q/p}]^{p/q}$$
$$\geqslant (\widetilde{W}_i(\Psi_j K_1) - \widetilde{W}_i(\Psi_j K))^{p/q} + (W_i(\Psi_j L_1) - \widetilde{W}_i(\Psi_j L))^{p/q}.$$

Equality holds if and only if equality holds in Theorem B and Lemma 3.9, hence equality holds if and only if K and L are dilates and

$$\widetilde{W}_i(\Psi_j K) \,/\, \widetilde{W}_i(\Psi_j K_1) = \widetilde{W}_i(\Psi_j L) \,/\, \widetilde{W}_i(\Psi_j L_1).$$

□

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References

[1] *J. Abardia, A. Bernig*: Projection bodies in complex vector spaces. Adv. Math. *227* (2011), 830–846. zbl MR doi

[2] *S. Alesker, A. Bernig, F. E. Schuster*: Harmonic analysis of translation invariant valuations. Geom. Funct. Anal. *21* (2011), 751–773. zbl MR doi

[3] *E. F. Beckenbach, R. Bellman*: Inequalities. Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Band 30, Springer, Berlin, 1965. zbl MR

[4] *R. J. Gardner*: A positive answer to the Busemann-Petty problem in three dimensions. Ann. Math. (2) *140* (1994), 435–447. zbl MR doi

[5] *R. J. Gardner*: Intersection bodies and the Busemann-Petty problem. Trans. Am. Math. Soc. *342* (1994), 435–445. zbl MR doi

[6] *R. J. Gardner*: The Brunn-Minkowski inequality. Bull. Am. Math. Soc., New Ser. *39* (2002), 355–405. zbl MR doi

[7] *R. J. Gardner, D. Hug, W. Weil*: The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities. J. Differ. Geom. *97* (2014), 427–476. zbl MR doi

[8] *R. J. Gardner, D. Hug, W. Weil, D. Ye*: The dual Orlicz-Brunn-Minkowski theory. J. Math. Anal. Appl. *430* (2015), 810–829. zbl MR doi

[9] *R. J. Gardner, A. Koldobsky, T. Schlumprecht*: An analytic solution to the Busemann-Petty problem on sections of convex bodies. Ann. Math. (2) *149* (1999), 691–703. zbl MR doi

[10] *R. J. Gardner, L. Parapatits, F. E. Schuster*: A characterization of Blaschke addition. Adv. Math. *254* (2014), 396–418. zbl MR doi

[11] *C. Haberl*: Star body valued valuations. Indiana Univ. Math. J. *58* (2009), 2253–2276. zbl MR doi

[12] *A. Koldobsky*: Intersection bodies, positive definite distributions, and the Busemann-Petty problem. Am. J. Math. *120* (1998), 827–840. zbl MR doi

[13] *A. Koldobsky*: Stability in the Busemann-Petty and Shephard problems. Adv. Math. *228* (2011), 2145–2161. zbl MR doi

[14] *A. Koldobsky, D. Ma*: Stability and slicing inequalities for intersection bodies. Geom. Dedicata *162* (2013), 325–335. zbl MR doi

[15] *G. Leng*: The Brunn-Minkowski inequality for volume differences. Adv. Appl. Math. *32* (2004), 615–624. zbl MR doi

[16] *M. Ludwig*: Projection bodies and valuations. Adv. Math. *172* (2002), 158–168. zbl MR doi
[17] *M. Ludwig*: Intersection bodies and valuations. Am. J. Math. *128* (2006), 1409–1428. zbl MR doi
[18] *E. Lutwak*: Dual mixed volumes. Pac. J. Math. *58* (1975), 531–538. zbl MR doi
[19] *E. Lutwak*: Intersection bodies and dual mixed volumes. Adv. Math. *71* (1988), 232–261. zbl MR doi
[20] *E. Lutwak*: Inequalities for mixed projection bodies. Trans. Am. Math. Soc. *339* (1993), 901–916. zbl MR doi
[21] *E. Lutwak*: The Brunn-Minkowski-Firey theory I. Mixed volumes and the Minkowski problem. J. Differ. Geom. *38* (1993), 131–150. zbl MR doi
[22] *E. Lutwak, D. Yang, G. Zhang*: Orlicz centroid bodies. J. Differ. Geom. *84* (2010), 365–387. zbl MR doi
[23] *E. Lutwak, D. Yang, G. Zhang*: Orlicz projection bodies. Adv. Math. *223* (2010), 220–242. zbl MR doi
[24] *R. Schneider*: Convex Bodies: The Brunn-Minkowski Theory. Encyclopedia of Mathematics and Its Applications 151, Cambridge University Press, Cambridge, 2014. zbl MR doi
[25] *F. E. Schuster*: Volume inequalities and additive maps of convex bodies. Mathematika *53* (2006), 211–234. zbl MR doi
[26] *F. E. Schuster*: Valuations and Busemann-Petty type problems. Adv. Math. *219* (2008), 344–368. zbl MR doi
[27] *W. Wang*: L_p Brunn-Minkowski type inequalities for Blaschke-Minkowski homomorphisms. Geom. Dedicata *164* (2013), 273–285. zbl MR doi
[28] *D. Xi, H. Jin, G. Leng*: The Orlicz Brunn-Minkowski inequality. Adv. Math. *260* (2014), 350–374. zbl MR doi
[29] *G. Xiong, D. Zou*: Orlicz mixed quermassintegrals. Sci. China, Math. *57* (2014), 2549–2562. zbl MR doi
[30] *C.-J. Zhao*: On radial Blaschke-Minkowski homomorphisms. Geom. Dedicata *167* (2013), 1–10. zbl MR doi
[31] *C.-J. Zhao*: Orlicz dual mixed volumes. Result. Math. *68* (2015), 93–104. zbl MR doi
[32] *C. Zhao, W.-S. Cheung*: Radial Blaschke-Minkowski homomorphisms and volume differences. Geom. Dedicata *154* (2011), 81–91. zbl MR doi
[33] *C.-J. Zhao, G. Leng*: Brunn-Minkowski inequality for mixed intersection bodies. J. Math. Anal. Appl. *301* (2005), 115–123. zbl MR doi
[34] *B. Zhu, J. Zhou, W. Xu*: Dual Orlicz-Brunn-Minkowski theory. Adv. Math. *264* (2014), 700–725. zbl MR doi

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