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# RESTRICTED BOOLEAN GROUP RINGS

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Abstract. In this paper we study restricted Boolean rings and group rings. A ring $R$ is *restricted Boolean* if every proper homomorphic image of $R$ is boolean. Our main aim is to characterize restricted Boolean group rings. A complete characterization of non-prime restricted Boolean group rings has been obtained. Also in case of prime group rings necessary conditions have been obtained for a group ring to be restricted Boolean. A counterexample is given to show that these conditions are not sufficient.

## 1. Introduction

Throughout this paper $R$ will denote an associative ring with identity $1 \neq 0$ and $G$ a non-trivial group. A ring $R$ is called *Boolean* if $r^2 = r$ for all $r \in R$. The Jacobson radical of a ring $R$ is denoted by $J(R)$, and $l(J(R))$, $r(J(R))$ respectively will denote the left, right annihilator of $J(R)$ in $R$. We define that a ring $R$ is *restricted Boolean* if every proper homomorphic image of $R$ is Boolean. Clearly every Boolean ring is restricted Boolean, but the converse is not true. For example, let $R = \mathbb{Z}_4$, the ring of integers modulo 4. Then $J(R) = \langle 2 \rangle$ is the only proper ideal of $R$ and $R/J(R) \cong \mathbb{Z}_2$. So $R$ is a restricted Boolean ring, but it is not a Boolean ring. A ring $R$ is called *clean* if every element of it can be written as a sum of an idempotent and a unit. A ring $R$ is *neat* if every proper homomorphic image of $R$ is clean. Since every Boolean ring is clean, the class of restricted Boolean rings is a proper subclass of neat rings. Commutative neat rings were studied by Warren Wm. McGovern [5].

The group ring of a group $G$ and a ring $R$ is denoted by $RG$. If $H$ is a subgroup of $G$, then $\omega H$ will denote the right ideal of $RG$ generated by $\{1 - h \mid h \in H\}$. In particular, if $H$ is a normal subgroup of $G$ then $\omega H$ is a two sided ideal of $RG$ and $RG/\omega H \cong R(G/H)$. If $H = G$, then $\omega G$ is the *augmentation ideal* of $RG$. It is easy to see that $\omega G$ is the kernel of the *augmentation map*, $\omega\colon RG \to R$, where $\omega(\Sigma_{g \in G} r_g g) = \Sigma_{g \in G} r_g$ and $RG/\omega G \cong R$. If $I$ is an ideal of $R$, then $IG$ is an ideal of $RG$ and $RG/IG \cong (R/I)G$. For group ring related results we refer to Connell [2] and Passman [6]; and for ring theory we refer to Lam [4].

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It is easy to see that a group ring $RG$ is Boolean if and only if $R$ is Boolean and $G$ is trivial. In this paper our main focus is on the study of restricted Boolean group rings which are not Boolean. Various properties of restricted Boolean group rings have been investigated. We obtain a complete characterization of non-prime restricted Boolean group rings. It is proved that a non-prime group ring $RG$ is restricted Boolean, but not Boolean if and only if $R \cong \mathbb{Z}_2$ and $G \cong C_2$, the finite cyclic group of order 2. Certain necessary conditions have been obtained in case of prime restricted Boolean group rings. A counterexample has been given to show that these conditions are not sufficient.

## 2. Main results

**Lemma 2.1.** *If $R$ is a restricted Boolean ring, then any non-trivial prime ideal of $R$ is maximal.*

**Proof.** Let $P$ be a non-trivial prime ideal of $R$. So $R/P$ is prime and Boolean. As a Boolean ring is commutative, $R/P$ is a commutative prime ring, and hence it is a domain. A Boolean domain is $\mathbb{Z}_2$. So $P$ is a maximal ideal. □

The following lemma is easy to verify.

**Lemma 2.2.** *A restricted Boolean ring which is semiprime, but not prime is Boolean.*

The next theorem characterizes commutative restricted Boolean rings.

**Theorem 2.3.** *Let $R$ be a commutative ring. Then $R$ is restricted Boolean if and only if any one of the following is satisfied:*

(1) *the ring $R$ is a field, or*

(2) *the ring $R$ is Boolean, or*

(3) *$J(R)$ is the only proper ideal of $R$ and $R/J(R) \cong \mathbb{Z}_2$.*

**Proof.** First, we assume that $J(R) = 0$. Then $(xR)^2 \neq 0$, for any $x(\neq 0) \in R$. Thus, by assumption, $R/(xR)^2$ is Boolean. Since $xR/(xR)^2$ is a nilpotent ideal in the Boolean ring $R/(xR)^2$, we must have $xR = (xR)^2$. Therefore, $x \in x^2R$. Hence $R$ is von Neumann regular. Now, if $R$ is prime, then $R$ is a von Neummann regular domain. Thus, $R$ is a field. This proves (1). The converse of (1) holds by using the fact that each simple ring has the property that all its proper factors are Boolean. Now if $R$ is not prime, then by Lemma 2.2, $R$ is Boolean. This proves the (2).

Now suppose that $J(R) \neq 0$. Then $R/J(R)$ is Boolean. So we get that $J(R) \subseteq I$, for all non-trivial ideals $I$ of $R$. Thus $l(J(R)) = r(J(R))$ is a prime ideal. Since $R$ is not prime, there exists two non-trivial ideals $I_1$ and $I_2$ of $R$ such that $I_1 I_2 = 0$. Thus we have $J(R)^2 = 0$. Therefore, $l(J(R)) \neq 0$. By Lemma 2.1, $l(J(R))$ is maximal. Further by [1, Theorem 5 and Theorem 6], $l(J(R)) \subseteq R$. Thus $l(J(R)) = R$. Hence, $J(R)$ is the only proper ideal $R$ and $R/J(R) \cong \mathbb{Z}_2$. This proves the (3).

The converse of (2) and (3) is straightforward. □

We now consider the restricted Boolean group rings.

**Theorem 2.4.** *Let $G$ be a non-trivial group. If $RG$ is restricted Boolean but not Boolean, then $R \cong \mathbb{Z}_2$ and $G$ is a simple group.*

**Proof.** First we prove that $R$ is a field with $R \cong \mathbb{Z}_2$. Since $RG$ is restricted Boolean and $RG/\omega(G) \cong R$, we get that $R$ is Boolean. Now, let $I \neq 0$ be an ideal of $R$, then $(R/I)G$ is Boolean, which implies that $G = \{1\}$. But this is not possible, because $G$ is non-trivial. Thus, $R$ does not have any non-trivial ideal $I$, so $R$ is simple. And any simple Boolean ring is $\mathbb{Z}_2$. Thus, $R \cong \mathbb{Z}_2$.

Now Let $H$ be a non-trivial normal subgroup of $G$. So $R(G/H)$ is Boolean, and thus $G/H$ is trivial. So $G$ has no non-trivial normal subgroups. Hence $G$ is simple. □

**Remark 2.5.** If $G$ is trivial, then the above Theorem need not hold. For example, if we take $R = \mathbb{Z}_4$ and $G = \{1\}$, then $RG$ is restricted Boolean, but not Boolean and $R \ncong \mathbb{Z}_2$.

The converse of the Theorem 2.4 need not hold.

**Example 2.6.** Let $R = \mathbb{Z}_2$ and $G$ be an infinite alternating group, i.e., $G = Alt_\Omega$, where $\Omega$ is an infinite set and each element of $G$ moves only finitely many points. Clearly $G$ is a simple locally finite group and $\Delta(G) = \{1\}$. We form the permutation module $V = \{\sum_{i\in\Omega} a_i i | a_i \in R, i \in \Omega, a_i = 0$ except for finitely many $i\}$ for $RG$. Now $V$ has as a $R$-basis the elements of $\Omega$ and $G$ acts on $V$ by appropriately permuting this basis. If $\sigma$ and $\tau$ are two disjoint permutations in $G$, for example, take $\sigma = (i_1, i_2, i_3)$ and $\tau = (i_4, i_5, i_6)$, where $i_1, i_2, i_3, i_4, i_5$ and $i_6$ are distinct elements. Then it can be easily seen that $(\sigma - 1)(\tau - 1) \neq 0$ and $(\sigma - 1)(\tau - 1)$ belongs to the ideal $I = Ann_{RG}V$, but $(\sigma - 1) \notin I$. So, $I$ is a non-trivial proper ideal of $RG$. We claim that $RG/I$ is not Boolean. Because if it would had been so then $\sigma^2 - \sigma \in I$, and thus $\sigma - 1 \in I$. But as $\sigma - 1 \notin I$. Hence $RG$ is not restricted Boolean.

We characterize, below, non-prime restricted Boolean group rings.

**Theorem 2.7.** *Let $G$ be a non-trivial group. A non-prime group ring $RG$ is restricted Boolean, but not Boolean if and only if $R \cong \mathbb{Z}_2$ and $G \cong C_2$.*

**Proof.** Suppose $RG$ is restricted Boolean but not Boolean, then by Theorem 2.4, $R \cong \mathbb{Z}_2$ and $G$ is a simple group.

First we show that $\omega G$ is the only non-zero ideal of $RG$. Since $RG/I$ is Boolean for all nontrivial ideals $I$ of $RG$, so $g - g^2 \in I$ for all $g \in G$. Then $(1 - g) \in I$ for all $g \in G$. So $\omega G \subseteq I$ for all nontrivial ideals $I$ of $RG$. But $\omega G$ is maximal ideal because $RG/\omega G \cong \mathbb{Z}_2$. Thus $\omega G$ is the only non-zero ideal of $RG$.

Since $RG$ is not a prime ring, $RG$ has two non-zero two-sided ideals $I_1$ and $I_2$ with $I_1 I_2 = 0$. From above we have $I_1 = \omega G$ as well as $I_2 = \omega G$. This proves that $(\omega G)^2 = 0$. By Connell [2, Theorem 9], $G$ is a finite 2-group. Since $G$ is a simple group, $G \cong C_2$.

Conversely, let $R \cong \mathbb{Z}_2$ and $G \cong C_2$. In this case we have $J(RG) = \omega G$, and it is the only proper ideal of $RG$. Thus $RG$ is restricted Boolean but not Boolean. □

From the above it can be easily seen that $\omega G$ is the only non-zero ideal of $RG$ even if $RG$ is prime restricted Boolean but not Boolean. Thus, restricted Boolean group rings can be characterized in terms of simple augmentation ideal as follows.

**Corollary 2.8.** *The group ring $RG$ is restricted Boolean if and only if $R \cong \mathbb{Z}_2$ and $\omega G$ is the only proper ideal of $RG$.*

The *FC-subgroup* $\Delta(G)$ of a group $G$ is the set of all elements of $G$ which have finitely many conjugates in $G$, i.e. $\Delta(G) = \{x \in G | [G : C_G(x)] < \infty\}$.

**Corollary 2.9.** *Let $G$ be an FC group (abelian or finite in particular), then $RG$ is restricted Boolean but not Boolean if and only if $R \cong \mathbb{Z}_2$ and $G \cong C_2$.*

**Proof.** We prove that $RG$ can not be prime. Let us suppose on the contrary that $RG$ is prime, then by Connell [2, Theorem 8], $\Delta(G)$ is a torsion free abelian group. By Theorem 2.4, $G$ is a simple group and also $G$ is an FC group. Hence, two cases arise either $G = \Delta(G) = \{1\}$ or $G = \Delta(G) \neq \{1\}$. The first case is not possible because $RG$ is restricted Boolean, but not Boolean. Thus, $G$ is a non-trivial torsion free abelian group. But there is no simple torsion free abelian group possible. Thus, the second case is also not possible. Hence, our assumption is wrong and $RG$ is non-prime. Now the result follows from Theorem 2.7. □

The following example shows that the above Corollary does not hold when $G$ is locally finite.

**Example 2.10.** Let $R = \mathbb{Z}_2$ and $G$ be a universal locally finite group, then $G$ is a simple group, $\Delta(G) = \{1\}$ and $RG$ is prime ([6, Theorem 9.4.9]). By Passman [6] Corollary 9.4.10, $\omega G$ is the unique proper ideal of $RG$. Since $RG/\omega G \cong R$, so $R$ is the only proper homomorphic image of $RG$. Thus $RG$ is restricted Boolean, but not Boolean as $G$ is non-trivial. And a universal locally finite group need not be a 2-group ([6, Theorem 9.4.8]).

**Remark 2.11.** A complete characterization has been obtained if $RG$ is non-prime restricted Boolean. But if we take $RG$ to be prime then in view of example 2.6, example 2.10 and Corollary 2.8, a characterization of prime restricted Boolean group rings amounts to an old question due to I. Kaplansky [3] that for which groups $G$ and which fields $K$ the augmentation ideal $\omega G$ is the only proper two sided ideal in $KG$.

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