Vikas Kumar Mishra; Nutan Kumar Tomar

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ALTERNATE CHECKING CRITERIA FOR REACHABLE CONTROLLABILITY OF RECTANGULAR DESCRIPTOR SYSTEMS

Vikas Kumar Mishra and Nutan Kumar Tomar

Contrary to state space systems, there are different notions of controllability for linear time invariant descriptor systems due to the non smooth inputs and inconsistent initial conditions. A comprehensive study of different notions of controllability for linear descriptor systems is performed. Also, it is proved that reachable controllability for general linear time invariant descriptor system is equivalent to the controllability of some matrix pair under an assumption milder than impulse controllability. The whole theory has been developed by coining two new decompositions for system matrices. Examples are given to illustrate the presented theory.

Keywords: descriptor systems, controllability, reachable controllability

Classification: 93B05, 93B25

1. INTRODUCTION

Consider linear time-invariant continuous descriptor systems of the form

$$E\dot{x}(t) \quad = \quad Ax(t) + Bu(t), \tag{1}$$

where $E, A \in \mathbb{R}^{m\times n}$, and $B \in \mathbb{R}^{m\times r}$. The vectors $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^r$ represent the semistate vector and the control (input) vector for the system (1), respectively. The set of systems of the form (1) is denoted by $\Sigma_{m,n,r}$. A square system $[E\ A\ B] \in \Sigma_{n,n,r}$ is called regular if there exists $\lambda \in \mathbb{C}$ such that the matrix pencil $(\lambda E - A)$ is invertible, where $\mathbb{C}$ denotes the set of complex numbers. A square system $[E\ A\ B] \in \Sigma_{n,n,r}$ is called state space if the matrix $E = I_n$, the identity matrix of size n.

Descriptor systems arise naturally in various real world applications [5, 6, 10, 22, 23, 24, 31] as these are general enough to describe the intrinsic properties of underlying physical systems. However, the analysis of descriptor systems is more delicate than state space systems in the sense that the solutions may have impulses if the input is not sufficiently smooth or the initial condition is not suitably chosen. These properties give rise to the two important concepts for descriptor systems, viz. index and consistent initialization. These concepts also led to different kind of controllability properties for

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descriptor systems. See the works [9, 11, 13, 14, 29, 30] for regular descriptor systems and [1, 2, 17, 25, 32] for general descriptor systems in context of solvability and controllability.

Now, we recall the following concepts which are useful in the development of the paper. See [1] for details.

Definition 1.1. A trajectory $(x, u) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^r$ is said to be solution of (1) if and only if it belongs to the behavior $\mathfrak{B}_{[E\ A\ B]}$ of (1) that is defined as follows:

$$\mathfrak{B}_{[E\ A\ B]} := \{(x, u) \in W^{1,1}_{loc}(\mathbb{R}, \mathbb{R}^n) \times L^1_{loc}(\mathbb{R}, \mathbb{R}^r) : (x, u) \text{ satisfies } (1) \text{ for almost all } t \in \mathbb{R}\}, \tag{2}$$

where

$$L^1_{loc}(\mathbb{R}, \mathbb{R}^r) := \text{Locally Lebesgue integrable functions } u : \mathbb{R} \to \mathbb{R}^r,$$

and

$$W^{1,1}_{loc}(\mathbb{R}, \mathbb{R}^n) := \{x : \mathbb{R} \to \mathbb{R}^n : x, \dot{x} \in L^1_{loc}(\mathbb{R}, \mathbb{R}^n)\}.$$

Definition 1.2. The set of all consistent initial vectors for the system (1) is defined as follows

$$\mathcal{V} = \{x_0 \in \mathbb{R}^n : \exists\ (x, u) \in \mathfrak{B}_{[E\ A\ B]} : x(0) = x_0\}.$$

Definition 1.3. Two systems $[E_i\ A_i\ B_i] \in \Sigma_{m,n,r},\ i = 1, 2$, are called restricted system equivalent (r.s.e.) if and only if there exist invertible matrices $W \in \mathbb{R}^{m\times m}$ and $T \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix}\lambda E_1 - A_1 & B_1\end{bmatrix} = W \begin{bmatrix}\lambda E_2 - A_2 & B_2\end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_r \end{bmatrix}.$$

To analyze various controllability concepts for any system $[E\ A\ B] \in \Sigma_{m,n,r}$, we write the following conditions [1, 25].

(C1) $\operatorname{rank}\begin{bmatrix}\lambda E - A & B\end{bmatrix} = \operatorname{rank}\begin{bmatrix}E & A & B\end{bmatrix}, \quad \forall\ \lambda \in \mathbb{C};$

(C2) $\operatorname{rank}\begin{bmatrix}\lambda E - A & B\end{bmatrix} = \operatorname{rank}_{\mathbb{R}(s)}\begin{bmatrix}sE - A & B\end{bmatrix}, \quad \forall\ \lambda \in \mathbb{C}$, where RHS (right-hand side) represents the maximum rank of the matrix over $\mathbb{R}(s)$: the quotient field of the ring of polynomials with coefficients in $\mathbb{R}$;

(C3) $\operatorname{rank}\begin{bmatrix}E & B\end{bmatrix} = \operatorname{rank}\begin{bmatrix}E & A & B\end{bmatrix};$

(C4) $\operatorname{rank}\begin{bmatrix}E & AV_\infty & B\end{bmatrix} = \operatorname{rank}\begin{bmatrix}E & A & B\end{bmatrix}$, where V_∞ spans the null space of E;

(C5) $\operatorname{rank}\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = \operatorname{rank}\begin{bmatrix}E & A & B\end{bmatrix} + \operatorname{rank} E.$

The following comments on the above conditions are warrant.

- Conditions (C1) and (C3) hold if and only if the system is completely controllable, that is, $\exists\ T > 0\ \forall\ x_0, x_f \in \mathbb{R}^n\ \exists\ (x, u) \in \mathfrak{B}_{[E\ A\ B]} : x(0) = x_0$ and $x(T) = x_f$.
- Conditions (C1) and (C4) hold if and only if the system is strongly controllable, that is, $\exists\ T > 0\ \forall\ x_0, x_f \in \mathbb{R}^n\ \exists\ (x, u) \in \mathfrak{B}_{[E\ A\ B]} : Ex(0) = Ex_0$ and $Ex(T) = Ex_f$.

- Condition (C2) holds if and only if the system is reachable controllable (R-controllable), that is, $\forall\ x_0, x_f \in \mathcal{V}\ \exists\ T > 0\ \exists\ (x, u) \in \mathfrak{B}_{[E\ A\ B]} : x(0) = x_0$ and $x(T) = x_f$.
- Condition (C3) holds if and only if the system is controllable at infinity, that is, $\forall\ x_0 \in \mathbb{R}^n\ \exists\ (x, u) \in \mathfrak{B}_{[E\ A\ B]} : x(0) = x_0$.
- Condition (C4) holds if and only if the system is impulse controllable (I-controllable), that is, $\forall\ x_0 \in \mathbb{R}^n\ \exists\ (x, u) \in \mathfrak{B}_{[E\ A\ B]} : Ex(0) = Ex_0$.

One or more of the above conditions have been assumed in designing some appropriate controllers such that the closed loop system satisfies certain properties [4, 7, 15, 20, 21, 26]. In case the system (1) is regular, the term rank $\begin{bmatrix} E & A & B \end{bmatrix}$ appeared in conditions (C1) – (C5) is replaced by n, i. e. the order of the matrix E or A [13, 14]. Furthermore, for regular descriptor systems conditions (C1) and (C2) turn out to be the same and are equivalent to the reachable controllability of the system [30]. But, if the system (1) is not regular, conditions (C1) and (C2) are obviously different. To the best of our knowledge, the condition (C2) first appeared in the article [1] where it is proved to be equivalent to the reachable controllability of the descriptor system (1). It should be noted that in [1] the reachable controllability and behavioral controllability are the same concepts.

Now, the following questions arise naturally for any general system $[E\ A\ B] \in \Sigma_{m,n,r}$: (i) Is (C1) weaker or stronger than (C2)? (ii) When do conditions (C1) and (C2) coincide? Such questions are answered in this article by developing a new numerically reliable decomposition of the system matrices. In establishing these results, we have proven a novel result on impulse controllability of the system. Thereafter, we study R-controllability. By using the Weierstrass canonical form [16], it can be checked easily that for any regular descriptor system, R-controllability is equivalent to the controllability of some matrix pair. We prove analogous result for any general descriptor system (1) by coining another numerically reliable decomposition of system matrices under some mild condition. Since the provided condition is in terms of the controllability of matrix pair, it may be useful to obtain other necessary and sufficient conditions for various control applications. In this direction, it is important to mention the work of Stefanovski where optimal control problems for descriptor systems have been transformed to optimal control problems for state space systems [28]. In summary, we propose two novel decompositions for the system matrices. The first is used to derive a result on impulse controllability and the second on R controllability.

The rest of the paper is organized as follows: In the next section, we recall some simple properties of matrix theory. Section 3 presents a mathematical discussion on the conditions (C1) – (C5). A new decomposition of system matrices is proposed and is used in deriving some equivalences for different types of controllability properties. Section 4 is devoted to the study of R-controllability of descriptor systems $[E\ A\ B] \in \Sigma_{m,n,r}$. Examples are provided to illustrate the effectiveness of the developed theory in Section 5. Section 6 concludes the paper.

2. PRELIMINARY RESULTS FROM MATRIX THEORY

Throughout the development of the presented theory, we make frequent use of the singular value decomposition (SVD) of a matrix $T \in \mathbb{R}^{m\times n}$ of rank α, which is given as

$$T = U \begin{bmatrix} \Sigma_T & 0 \\ 0 & 0 \end{bmatrix} V^T,$$

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal matrices and $\Sigma_T \in \mathbb{R}^{\alpha \times \alpha}$ is a diagonal positive definite matrix. Here, V^T denotes the transpose of the matrix V.

Apart from the SVD of the matrices, the following simple properties of matrix theory will be incredibly employed in the subsequent development of the paper.

(P1) The rank of a matrix is unaltered if it is pre- and/or post- multiplied by an invertible matrix.

(P2) Let A, B, and C be any matrices of compatible dimensions. Then, the following inequality holds

$$\operatorname{rank} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \geq \operatorname{rank}(A) + \operatorname{rank}(C).$$

Moreover, the equality holds if A is full row rank and/or C is full column rank.

(P3) Let $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank} A = \alpha$. Then there exist orthogonal matrices $P_1 \in \mathbb{R}^{m \times m}$ and $Q_1 \in \mathbb{R}^{n \times n}$ such that

$$P_1 A = \begin{bmatrix} \Sigma_\alpha \\ 0 \end{bmatrix} \text{ and } AQ_1 = \begin{bmatrix} \hat{\Sigma}_\alpha & 0 \end{bmatrix},$$

where $\Sigma_\alpha \in \mathbb{R}^{\alpha \times n}$ is a full row rank matrix and $\hat{\Sigma}_\alpha \in \mathbb{R}^{m \times \alpha}$ is a full column rank matrix. Matrices P_1 and Q_1 may be calculated using the SVD of the matrix A. Matrices P_1 and Q_1 are called row compression and column compression of the matrix A, respectively.

The discussion on matrix theory presented in this section may be looked in [3, 18, 27].

3. DISCUSSION ON CONDITIONS (C1) – (C5)

The following facts are important for subsequent discussion.

Fact 1. Conditions (C4) and (C5) are equivalent.

Fact 2. Condition (C3) $\Rightarrow$ condition (C4).

Fact 3. Condition (C1) $\Rightarrow$ condition (C2).

The proof of Fact 1 is a straightforward generalization of the proof given for regular descriptor systems in [14, Section 4.5.3]. The proofs of Fact 2 and Fact 3 are consequences of the following inequalities, respectively.

$$\operatorname{rank} \begin{bmatrix} E & B \end{bmatrix} \leq \operatorname{rank} \begin{bmatrix} E & AV_\infty & B \end{bmatrix} \leq \operatorname{rank} \begin{bmatrix} E & A & B \end{bmatrix}. \tag{3}$$

$$\operatorname{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} \leq \operatorname{rank}_{\mathbb{R}(s)} \begin{bmatrix} sE - A & B \end{bmatrix} \leq \operatorname{rank} \begin{bmatrix} E & A & B \end{bmatrix}, \quad \forall\, \lambda \in \mathbb{C}. \tag{4}$$

The following example indicates that the converse of the Fact 3 need not be true in general.

Example 3.1. Let the following matrices represent the system (1)

$$E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5}$$

It can be checked that condition (C2) holds but (C1) does not hold.

In order to derive the sufficient condition under which condition (C2) implies condition (C1), we need to develop some background material.

Theorem 3.2. Consider the system $[E\ A\ B] \in \Sigma_{m,n,r}$. Then, there exist orthogonal matrices $P \in \mathbb{R}^{m\times m}$, $Q \in \mathbb{R}^{n\times n}$ and $U \in \mathbb{R}^{r\times r}$ such that

$$PEQ = \begin{array}{c} \begin{array}{ccc} n_0 & h_0 & g \end{array} \\ \begin{bmatrix} \Sigma_E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ n_0 \\ v_0 \\ h_0 \\ d \end{array}, \quad PBU = \begin{array}{c} \begin{array}{cc} v_0 & f \end{array} \\ \begin{bmatrix} B_{11} & B_{12} \\ \Sigma_{B_2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ n_0 \\ v_0 \\ h_0 \\ d \end{array}, \text{ and}$$

$$PAQ = \begin{array}{c} \begin{array}{ccc} n_0 & h_0 & g \end{array} \\ \begin{bmatrix} A_1 & A_{21} & A_{22} \\ A_{31} & A_{411} & A_{412} \\ A_{321} & \Sigma_{A_{42}} & 0 \\ A_{322} & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ n_0 \\ v_0 \\ h_0 \\ d \end{array}.$$

Here, matrix partitions are compatible. The matrices Σ_E, Σ_{B_2}, and $\Sigma_{A_{42}}$ are diagonal positive definite matrices. Moreover, $n_0 + h_0 + g = n$ and $n_0 + v_0 + h_0 + d = m$.

Proof. Let $\operatorname{rank} E = n_0$. Then, there exist orthogonal matrices $P_1 \in \mathbb{R}^{m\times m}$ and $Q_1 \in \mathbb{R}^{n\times n}$ such that

$$P_1EQ_1 = \begin{array}{c} \begin{array}{cc} n_0 & e \end{array} \\ \begin{bmatrix} \Sigma_E & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ n_0 \\ v \end{array}, \quad P_1AQ_1 = \begin{array}{c} \begin{array}{cc} n_0 & e \end{array} \\ \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \end{array} \begin{array}{c} \\ n_0 \\ v \end{array}, \text{ and } P_1B = \begin{array}{c} r \\ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \end{array} \begin{array}{c} \\ n_0 \\ v \end{array}.$$

Here, $n_0 + e = n$ and $n_0 + v = m$.
Let us assume that $\operatorname{rank} B_2 = v_0$. Then, there exist orthogonal matrices $P_2 \in \mathbb{R}^{v\times v}$ and $Q_2 \in \mathbb{R}^{r\times r}$ such that

$$P_2B_2Q_2 = \begin{bmatrix} \Sigma_{B_2} & 0 \\ 0 & 0 \end{bmatrix}.$$

This implies that for

$$\tilde{P}_2 = \begin{bmatrix} I_{n_0} & 0 \\ 0 & P_2 \end{bmatrix} P_1,$$

the following hold:

$$\tilde{P}_2 E Q_1 = \begin{array}{cc} n_0 & e \end{array} \atop \begin{bmatrix} \Sigma_E & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{array}{c} n_0 \\ v_0 \\ h \end{array}, \quad \tilde{P}_2 B Q_2 = \begin{array}{cc} v_0 & f \end{array} \atop \begin{bmatrix} B_{11} & B_{12} \\ \Sigma_{B_2} & 0 \\ 0 & 0 \end{bmatrix} \begin{array}{c} n_0 \\ v_0 \\ h \end{array}, \text{ and}$$

$$\tilde{P}_2 A Q_1 = \begin{array}{cc} n_0 & e \end{array} \atop \begin{bmatrix} A_1 & A_2 \\ A_{31} & A_{41} \\ A_{32} & A_{42} \end{bmatrix} \begin{array}{c} n_0 \\ v_0 \\ h \end{array}.$$

Here,

$$B_1 Q_2 = \begin{bmatrix} B_{11} & B_{12} \end{bmatrix}, \; P_2 A_3 = \begin{bmatrix} A_{31} \\ A_{32} \end{bmatrix}, \text{ and } P_2 A_4 = \begin{bmatrix} A_{41} \\ A_{42} \end{bmatrix}.$$

Also, $v_0 + h = v$, $v_0 + f = r$ and $n_0 + v_0 + h = m$. Again, let us assume that rank $A_{42} = h_0$. Then, there exist orthogonal matrices $P_3 \in \mathbb{R}^{h \times h}$ and $Q_3 \in \mathbb{R}^{e \times e}$ such that

$$P_3 A_{42} Q_3 = \begin{bmatrix} \Sigma_{A_{42}} & 0 \\ 0 & 0 \end{bmatrix}.$$

Now, we obtain the desired decomposition by setting the matrices P, Q, and U as follows

$$P = \begin{bmatrix} I_{n_0} & 0 & 0 \\ 0 & I_{v_0} & 0 \\ 0 & 0 & P_3 \end{bmatrix} \tilde{P}_2, \; Q = Q_1 \begin{bmatrix} I_{n_0} & 0 \\ 0 & Q_3 \end{bmatrix}, \text{ and } U = Q_2,$$

with the following decompositions,

$$A_2 Q_3 = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix}, \; A_{41} Q_3 = \begin{bmatrix} A_{411} & A_{412} \end{bmatrix}, \text{ and } P_3 A_{32} = \begin{bmatrix} A_{321} \\ A_{322} \end{bmatrix}.$$

□

Theorem 3.2 is proved constructively by using the SVD of system matrices. Hence, the above theorem not only presents an existence result on the decomposition but the decomposition is also computable in a numerically efficient way. Theorem 3.2 as well as Theorem 4.1 (given in next Section) seem to be an extension of the works of Bunse-Gerstner et al. [4] to rectangular systems. In [4], a repetitive number of SVDs for square system matrices have been performed to get decompositions to regularize the system. But, in the present work, we have performed limited number of SVDs for rectangular system matrices in a different fashion than [4] to get the desired decompositions for our special purposes in particular to analyze condition (C2).

The next result provides an equivalent condition to the condition (C5).

Theorem 3.3. The system is impulse controllable if and only if the block matrix A_{322} is identically zero.

Proof. Using (P1), condition (C5) is equivalent to the following

$$\operatorname{rank}\begin{bmatrix} PEQ & 0 & 0 \\ PAQ & PEQ & PBU \end{bmatrix} = \operatorname{rank}\begin{bmatrix} PEQ & PAQ & PBU \end{bmatrix} + \operatorname{rank} E.$$

Applying Theorem 3.2, we obtain

$$\operatorname{rank}\begin{bmatrix} \Sigma_E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_1 & A_{21} & A_{22} & \Sigma_E & 0 & 0 & B_{11} & B_{12} \\ A_{31} & A_{411} & A_{412} & 0 & 0 & 0 & \Sigma_{B_2} & 0 \\ A_{321} & \Sigma_{A_{42}} & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{322} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \operatorname{rank}\begin{bmatrix} \Sigma_E & 0 & 0 & A_1 & A_{21} & A_{22} & B_{11} & B_{12} \\ 0 & 0 & 0 & A_{31} & A_{411} & A_{412}^2 & \Sigma_{B_2} & 0 \\ 0 & 0 & 0 & A_{321} & \Sigma_{A_{42}} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{322} & 0 & 0 & 0 & 0 \end{bmatrix} + \operatorname{rank} E,$$

which is equivalent to

$$A_{322} = 0, \quad \text{(using (P2))}.$$

Hence, the theorem is proved. □

Remark 3.4. The above theorem provides a novel criterion to check the impulse controllability of the system (1). It should also be noted that the matrix A_{322} being identically zero includes the possibility of the matrix being empty and this convention will be used throughout the paper.

The next result presents a sufficient condition under which both the conditions (C1) and (C2) are equivalent.

Theorem 3.5. If condition (C5) holds, then condition (C2) implies condition (C1).

Proof. It is sufficient to prove that

$$\operatorname{rank}\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} - \operatorname{rank} E = \operatorname{rank}_{\mathbb{R}(s)} \begin{bmatrix} sE - A & B \end{bmatrix}. \tag{6}$$

Applying Theorem 3.2 and Theorem 3.3, the RHS of (6) becomes

$$\begin{aligned}
&\operatorname{rank}_{\mathbb{R}(s)}\begin{bmatrix} s\Sigma_E - A_1 & -A_{21} & -A_{22} & B_{11} & B_{12} \\ -A_{31} & -A_{411} & -A_{412} & \Sigma_{B_2} & 0 \\ -A_{321} & -\Sigma_{A_{42}} & 0 & 0 & 0 \\ -A_{322} & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \operatorname{rank}_{\mathbb{R}(s)}\begin{bmatrix} s\Sigma_E - A_1 + B_{11}\Sigma_{B_2}^{-1}A_{31} & -A_{21} + B_{11}\Sigma_{B_2}^{-1}A_{411} & \mathfrak{B} & 0 & B_{12} \\ -A_{31} & -A_{411} & -A_{412} & \Sigma_{B_2} & 0 \\ -A_{321} & -\Sigma_{A_{42}} & 0 & 0 & 0 \end{bmatrix} \\
&= v_0 + \operatorname{rank}_{\mathbb{R}(s)}\begin{bmatrix} s\Sigma_E - A_1 + B_{11}\Sigma_{B_2}^{-1}A_{31} & -A_{21} + B_{11}\Sigma_{B_2}^{-1}A_{411} & \mathfrak{B} & B_{12} \\ -A_{321} & -\Sigma_{A_{42}} & 0 & 0 \end{bmatrix} \\
&= v_0 + h_0 + \operatorname{rank}_{\mathbb{R}(s)}\begin{bmatrix} s\Sigma_E - \mathfrak{A} & \mathfrak{B} & B_{12} \end{bmatrix}, \quad \text{(using (P2))} \\
&= v_0 + h_0 + n_0,
\end{aligned}$$

where $\mathfrak{A} = A_1 - B_{11}\Sigma_{B_2}^{-1}A_{31} - A_{21}\Sigma_{A_{42}}^{-1}A_{321} + B_{11}\Sigma_{B_2}^{-1}A_{411}\Sigma_{A_{42}}^{-1}A_{321}$ and $\mathfrak{B} = -A_{22} + B_{11}\Sigma_{B_2}^{-1}A_{412}$. Again, applying Theorem 3.2, the LHS of (6) turns out to be $v_0 + h_0 + n_0$. Hence the theorem is proved. □

Remark 3.6. In view of Fact 2, it is clear that when condition (C3) holds, again the conditions (C1) and (C2) are equivalent.

In summary, in this section, both the questions (i) – (ii) raised in the Introduction section have been answered.

4. REACHABLE CONTROLLABILITY

First, we present the following theorem on the decomposition of system matrices which will be used in deriving the main result on R-controllability.

Theorem 4.1. Consider the system $[E\ A\ B] \in \Sigma_{m,n,r}$. Then, there exist orthogonal matrices $M \in \mathbb{R}^{m\times m}$ and $N \in \mathbb{R}^{n\times n}$ such that

$$MEN = \begin{array}{c} \begin{array}{ccc} s_0 & q_0 & k_0 \end{array} \\ \begin{bmatrix} E_{11} & E_{121} & E_{122} \\ \Sigma_{E_2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ r_0 \\ s_0 \\ q_0 \\ q_1 \end{array}, \quad MB = \begin{array}{c} r \\ \begin{bmatrix} B_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \\ r_0 \\ s_0 \\ q_0 \\ q_1 \end{array}, \text{ and}$$

$$MAN = \begin{array}{c} \begin{array}{ccc} s_0 & q_0 & k_0 \end{array} \\ \begin{bmatrix} A_1^1 & A_{21}^1 & A_{22}^1 \\ A_1^2 & A_{21}^2 & A_{22}^2 \\ A_{31}^2 & \Sigma_{A_4^2} & 0 \\ A_{32}^2 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ r_0 \\ s_0 \\ q_0 \\ q_1 \end{array}.$$

where the matrices Σ_{E_2} and $\Sigma_{A_4^2}$ are diagonal positive definite while matrix B_1 has full row rank. Moreover, $s_0 + q_0 + k_0 = n$ and $r_0 + s_0 + q_0 + q_1 = m$.

Proof. Let rank $B = r_0$. Then, there exists, using (P3), an orthogonal matrix $M_1 \in \mathbb{R}^{m\times m}$ such that

$$M_1E = \begin{array}{c} n \\ \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \end{array} \begin{array}{c} r_0 \\ s \end{array}, \quad M_1A = \begin{array}{c} n \\ \begin{bmatrix} A^1 \\ A^2 \end{bmatrix} \end{array} \begin{array}{c} r_0 \\ s \end{array}, \text{ and } M_1B = \begin{array}{c} r \\ \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} r_0 \\ s \end{array},$$

where $r_0 + s = m$. Let us assume that rank $E_2 = s_0$. Then, there exist orthogonal matrices $M_2 \in \mathbb{R}^{s\times s}$ and $N_2 \in \mathbb{R}^{n\times n}$ such that

$$M_2E_2N_2 = \begin{bmatrix} \Sigma_{E_2} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } M_2A^2N_2 = \begin{bmatrix} A_1^2 & A_2^2 \\ A_3^2 & A_4^2 \end{bmatrix}.$$

This implies that for

$$\tilde{M}_2 = \begin{bmatrix} I_{r_0} & 0 \\ 0 & M_2 \end{bmatrix} M_1,$$

the following hold:

$$\tilde{M}_2EN_2 = \begin{array}{c} \begin{array}{cc} s_0 & k \end{array} \\ \begin{bmatrix} E_{11} & E_{12} \\ \Sigma_{E_2} & 0 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} r_0 \\ s_0 \\ q \end{array}, \quad \tilde{M}_2B = \begin{array}{c} r \\ \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} r_0 \\ s_0 \\ q \end{array}, \text{ and}$$

$$\tilde{M}_2AN_2 = \begin{array}{c} \begin{array}{cc} s_0 & k \end{array} \\ \begin{bmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \\ A_3^2 & A_4^2 \end{bmatrix} \end{array} \begin{array}{c} r_0 \\ s_0 \\ q \end{array}.$$

Here,

$$E_1N_2 = \begin{bmatrix} E_{11} & E_{12} \end{bmatrix} \text{ and } A^1N_2 = \begin{bmatrix} A_1^1 & A_2^1 \end{bmatrix}.$$

Also $s_0 + k = n$ and $r_0 + s_0 + q = m$. Again, let us assume that rank $A_4^2 = q_0$. Then, there exist orthogonal matrices $M_3 \in \mathbb{R}^{q\times q}$ and $N_3 \in \mathbb{R}^{k\times k}$ such that

$$M_3A_4^2N_3 = \begin{bmatrix} \Sigma_{A_4^2} & 0 \\ 0 & 0 \end{bmatrix}.$$

Now, we obtain the desired decomposition by setting the matrices M and N as follows

$$M = \begin{bmatrix} I_{r_0} & 0 & 0 \\ 0 & I_{s_0} & 0 \\ 0 & 0 & M_3 \end{bmatrix} \tilde{M}_2 \text{ and } N = N_2 \begin{bmatrix} I_{s_0} & 0 \\ 0 & N_3 \end{bmatrix},$$

with the following decompositions

$$M_3A_3^2 = \begin{bmatrix} A_{31}^2 \\ A_{32}^2 \end{bmatrix}, \ A_2^1N_3 = \begin{bmatrix} A_{21}^1 & A_{22}^1 \end{bmatrix}, \ A_2^2N_3 = \begin{bmatrix} A_{21}^2 & A_{22}^2 \end{bmatrix}, \text{ and } E_{12}N_3 = \begin{bmatrix} E_{121} & E_{122} \end{bmatrix}.$$

□

Before we relate the R-controllability of general descriptor system to some suitably designed matrix pair, the following basic definition is recalled.

Definition 4.2. [3, 8, 19] A matrix pair $(\mathbf{A}, \mathbf{B})$ is controllable if and only if

$$\operatorname{rank}\begin{bmatrix}\lambda I - \mathbf{A} & \mathbf{B}\end{bmatrix} = p, \quad \forall\, \lambda \in \mathbb{C}, \tag{7}$$

where p is the order of square matrix $\mathbf{A}$.

Exploiting Theorem 4.1 and Definition 1.3, the system (1) is r.s.e. to the following system

$$\begin{aligned}
\begin{bmatrix}E_{11} & E_{121} & E_{122}\end{bmatrix}\dot{x} &= \begin{bmatrix}A_1^1 & A_{21}^1 & A_{22}^1\end{bmatrix}x + B_1 u && \text{(8a)}\\
\Sigma_{E_2}\dot{x}_1 &= A_1^2 x_1 + A_{21}^2 x_2 + A_{22}^2 x_3 && \text{(8b)}\\
0 &= A_{31}^2 x_1 + \Sigma_{A_4^2} x_2 && \text{(8c)}\\
0 &= A_{32}^2 x_1, && \text{(8d)}
\end{aligned}$$

where $N^{-1}x = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$.

Systems (8b) and (8c) can equivalently be written as:

$$\dot{x}_1 = \mathcal{A}x_1 + \mathcal{B}x_3, \tag{9}$$

where $\mathcal{A} = \Sigma_{E_2}^{-1}A_1^2 - \Sigma_{E_2}^{-1}A_{21}^2\Sigma_{A_4^2}^{-1}A_{31}^2$ and $\mathcal{B} = \Sigma_{E_2}^{-1}A_{22}^2$. Note that $\mathcal{A} \in \mathbb{R}^{s_0 \times s_0}$ and $\mathcal{B} \in \mathbb{R}^{s_0 \times k_0}$.

Now, we present the main theorem on R-controllability of the system (1).

Theorem 4.3. If the block matrix A_{32}^2 in Theorem 4.1 is identically zero, then controllability of the matrix pair $(\mathcal{A}, \mathcal{B})$ is equivalent to the R-controllability of the system (1).

Proof. Applying Theorem 4.1 to the condition (C2), we obtain

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix}\lambda E_{11} - A_1^1 & \lambda E_{121} - A_{21}^1 & \lambda E_{122} - A_{22}^1 & B_1\\ \lambda\Sigma_{E_2} - A_1^2 & -A_{21}^2 & -A_{22}^2 & 0\\ -A_{31}^2 & -\Sigma_{A_4^2} & 0 & 0\\ -A_{32}^2 & 0 & 0 & 0\end{bmatrix}\\
&= \operatorname{rank}_{\mathbb{R}(s)}\begin{bmatrix}sE_{11} - A_1^1 & sE_{121} - A_{21}^1 & sE_{122} - A_{22}^1 & B_1\\ s\Sigma_{E_2} - A_1^2 & -A_{21}^2 & -A_{22}^2 & 0\\ -A_{31}^2 & \Sigma_{A_4^2} & 0 & 0\\ -A_{32}^2 & 0 & 0 & 0\end{bmatrix},
\end{aligned}$$

which is, using (P2), the same as

$$\operatorname{rank}\begin{bmatrix}\lambda\Sigma_{E_2} - A_1^2 & -A_{21}^2 & -A_{22}^2\\ -A_{31}^2 & -\Sigma_{A_4^2} & 0\end{bmatrix} = \operatorname{rank}_{\mathbb{R}(s)}\begin{bmatrix}s\Sigma_{E_2} - A_1^2 & -A_{21}^2 & -A_{22}^2\\ -A_{31}^2 & -\Sigma_{A_4^2} & 0\end{bmatrix}, \tag{10}$$

which is, again using (P2), equivalent to

$$\operatorname{rank}\begin{bmatrix}\lambda I_{s_0} - \Sigma_{E_2}^{-1}A_1^2 + \Sigma_{E_2}^{-1}A_{21}^2\Sigma_{A_4^2}^{-1}A_{31}^2 & -\Sigma_{E_2}^{-1}A_{22}^2\end{bmatrix} = s_0. \tag{11}$$

In the notations of system (9),

$$\operatorname{rank}\begin{bmatrix}\lambda I_{s_0} - \mathcal{A} & \mathcal{B}\end{bmatrix} = s_0. \tag{12}$$

This completes the proof of the theorem. □

The above theorem relates the R-controllability of the system (1) to the controllability of the matrix pair $(\mathcal{A}, \mathcal{B})$. We now recall all the computational steps required to find the matrix pair $(\mathcal{A}, \mathcal{B})$ in the form of the following algorithm.

Algorithm 1 Computational steps to find the matrix pair $(\mathcal{A}, \mathcal{B})$.

Step 1. Convert the given system into the form of system (8) using the following substeps (based on the proof of Theorem 4.1)

(i) Obtain the matrices $\begin{bmatrix}E_1\\E_2\end{bmatrix}$ and $\begin{bmatrix}A^1\\A^2\end{bmatrix}$ in view of converting the matrix B into full row rank matrix.

(ii) Performing the SVD of the matrix E_2, find the matrices Σ_{E_2}, $\begin{bmatrix}A_1^2 & A_2^2\\A_3^2 & A_4^2\end{bmatrix}$, and $\begin{bmatrix}A_1^1 & A_1^2\end{bmatrix}$.

(iii) Performing the SVD of the matrix A_4^2, calculate the matrices $\Sigma_{A_4^2}$, $\begin{bmatrix}A_{31}^2\\A_{32}^2\end{bmatrix}$, and $\begin{bmatrix}A_{21}^2 & A_{22}^2\end{bmatrix}$.

Step 2. The desired matrix pair $(\mathcal{A}, \mathcal{B})$ is given by:

$\mathcal{A} = \Sigma_{E_2}^{-1}A_1^2 - \Sigma_{E_2}^{-1}A_{21}^2\Sigma_{A_4^2}^{-1}A_{31}^2$,

$\mathcal{B} = \Sigma_{E_2}^{-1}A_{22}^2$.

Now, we show that the condition used as the assumption of Theorem 4.3 is milder than condition (C5).

Theorem 4.4. The condition (C5) implies that the block matrix A_{32}^2 is identically zero in Theorem 4.1.

Proof. Applying Theorem 4.1 to the condition (C5), we obtain that

$$\operatorname{rank}\begin{bmatrix} E_{11} & E_{121} & E_{122} & 0 & 0 & 0 & 0 \\ \Sigma_{E_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_1^1 & A_{21}^1 & A_{22}^1 & E_{11} & E_{121} & E_{122} & B_1 \\ A_1^2 & A_{21}^2 & A_{22}^2 & \Sigma_{E_2} & 0 & 0 & 0 \\ A_{31}^2 & \Sigma_{A_4^2} & 0 & 0 & 0 & 0 & 0 \\ A_{32}^2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} E_{11} & E_{121} & E_{122} & A_1^1 & A_{21}^1 & A_{22}^1 & B_1 \\ \Sigma_{E_2} & 0 & 0 & A_1^1 & A_{21}^1 & A_{22}^1 & 0 \\ 0 & 0 & 0 & A_{31}^2 & \Sigma_{A_4^2} & 0 & 0 \\ 0 & 0 & 0 & A_{32}^2 & 0 & 0 & 0 \end{bmatrix} + \operatorname{rank} E,$$

which is, using (P2), the same as

$$s_0 + \operatorname{rank} E_{122} = \operatorname{rank} E + \operatorname{rank} A_{32}^2,$$

which is equivalent to

$$s_0 + \operatorname{rank} E_{122} = s_0 + \operatorname{rank}\begin{bmatrix} E_{121} & E_{122} \end{bmatrix} + \operatorname{rank} A_{32}^2,$$

$$\Rightarrow A_{32}^2 = 0.$$

Hence, the theorem is proved. □

Remark 4.5. The converse of the above theorem need not be true. See Example 5.2 and Example 5.3 in the next Section. It can be seen that the converse holds if either block matrix E_{121} is identically zero or the columns of the matrix E_{121} are linear combinations of the columns of the matrix E_{122}.

5. ILLUSTRATING EXAMPLES

Example 5.1. Consider an electrical LR circuit as shown in Fig.1. Let I_1 and I_2 denote the currents flowing in the clockwise direction in first and second loop, respectively. Let L_i and R_i, for $i = 1, 2$, denote the inductances and resistances, respectively. The AC source voltage is denoted by u_s. Moreover, we denote by V_{L_i} and V_{R_i}, for $i = 1, 2$, the voltages across the inductors and resistors, respectively.

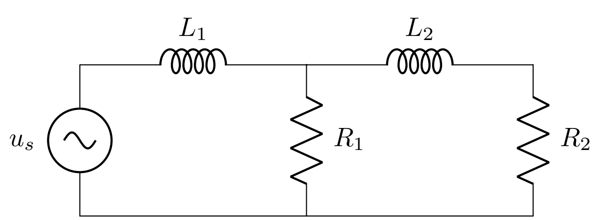

Fig. 1. LR circuit.

Applying basic circuit theory, we get the following equations

$$L_1\dot{I}_1 = V_{L_1}, \tag{13a}$$
$$0 = V_{R_1} - R_1 I_1 + R_1 I_2, \tag{13b}$$
$$0 = u_S - V_{L_1} - V_{R_1}, \tag{13c}$$
$$L_2\dot{I}_2 = V_{L_2}, \tag{13d}$$
$$0 = V_{R_2} - R_2 I_2, \tag{13e}$$
$$0 = V_{R_1} - V_{L_2} - V_{R_2}, \tag{13f}$$
$$0 = u_S - V_{L_1} - V_{L_2} - V_{R_2}. \tag{13g}$$

The system (13) can be written in the form of (1) by taking

$$E = \begin{bmatrix} L_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -R_1 & R_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -R_2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ x(t) = \begin{bmatrix} I_1 \\ I_2 \\ V_{L_1} \\ V_{R_1} \\ V_{L_2} \\ V_{R_2} \end{bmatrix}, \text{ and } u = u_s.$$

For numerical purposes, we take $L_1 = L_2 = 1$ unit and $R_1 = R_2 = 1$ unit. Now, applying Theorem 3.2, we obtain

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0.7071 & 0 & 0 & 0 & 0.7071 \\ 0 & -0.2689 & 0.3776 & 0 & 0.2689 & -0.7551 & -0.3776 \\ 0 & 0.7071 & 0 & 0 & 0.7071 & 0 & 0 \\ 0 & 0.6540 & 0.1553 & 0 & -0.6540 & -0.3105 & -0.1553 \\ 0 & 0 & -0.5774 & 0 & 0 & -0.5774 & 0.5774 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -0.6139 & 0.7071 & 0.3508 & 0 \\ 0 & 0 & 0.4961 & 0 & 0.8682 & 0 \\ 0 & 0 & 0.6139 & 0.7071 & -0.3508 & 0 \end{bmatrix}, \text{ and } U = 1. \tag{14}$$

Also,

$$PEQ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; PBU = \begin{bmatrix} 0 \\ 0 \\ 1.4142 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\text{and } PAQ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -0.4961 & 0 & -0.8682 & 0 \\ 0 & 0 & -0.3508 & -1 & -0.6139 & 1.4142 \\ 0.2689 & 0.5378 & 2.2830 & 0 & 0 & 0 \\ -0.7071 & 0 & 0 & 1 & 0 & 0 \\ -0.6540 & -1.3080 & 0 & 0 & 0.5365 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{15}$$

Notice that the matrix A_{322} is a zero matrix. Hence, the system (13) is I-controllable by Theorem 3.3. It can also be verified that system (13) satisfies condition (C5). Moreover, It can be checked that system (13) is R-controllable by utilizing Theorems 4.1 and 4.3. This is because the desired system (9) is given by $\mathcal{A} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ is controllable. Note that here Theorem 4.4 ensures that the assumption required for the Theorem 4.3 is satisfied.

Example 5.2. Let the system (1) be represented by the following matrices

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 2 \\ 1 & 2 & 3 & 1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{16}$$

Clearly, condition (C5) is not satisfied. Applying Theorem 4.1, we obtain

$$M = \begin{bmatrix} 0.7071 & 0.7071 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -0.7071 & 0.7071 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.8156 & -0.5786 \\ 0 & 0 & 0 & 0.5786 & -0.8156 \end{bmatrix} \text{ and}$$

$$N = \begin{bmatrix} -0.7071 & -0.2223 & -0.6712 & 0 \\ 0 & -0.8491 & 0.2812 & -0.4472 \\ 0.7071 & -0.2223 & -0.6712 & 0 \\ 0 & -0.4245 & 0.1406 & 0.8944 \end{bmatrix}. \tag{17}$$

Also,

$$MEN = \begin{bmatrix} 0 & -0.3144 & -0.9493 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ MB = \begin{bmatrix} 0 & 1.4142 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\text{and } MAN = \begin{bmatrix} 0.5000 & -2.5872 & -1.7766 & 0.6325 \\ -1.4142 & -0.4447 & -1.3425 & 0 \\ 1.5000 & -1.6724 & -1.0262 & -0.6325 \\ -0.8183 & 5.2053 & 0 & 0 \\ -1.1534 & 0 & 2.4301 & 0 \end{bmatrix}. \tag{18}$$

Notice that the matrix A_{32}^2 is an empty matrix of order 0×1 and thus satisfying the requirement of the Theorem 4.3. Further, for system (9), the matrices are as follows $\mathcal{A} = 0.7500$ and $\mathcal{B} = -0.6325$ and clearly the system is controllable. Hence, by Theorem 4.3 the original system (18) is R-controllable.

Example 5.3. The present example is a prototype of constrained optimal control problem taken from the article [12]. The problem is to find an optimal control $u(t)$ such that a certain cost functional $J(x(t), u(t))$ is minimized, where $x(t)$ is governed by

$$\dot{x}(t) = Kx(t) + Lu(t), \tag{19}$$

with the path constraint

$$w = Mx(t) + Nu(t), \tag{20}$$

where K, L, M and N are constant matrices and w is a constant vector. The equation (19) and constraint (20) can be represented as

$$\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & 0 \\ M & -I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} L \\ 0 \\ N \end{bmatrix} u, \tag{21}$$

which is in the form of system (1). Therefore, an equivalent statement for the optimal control problem is to find optimal control $u(t)$ for the descriptor system (21) such that the cost functional $J(x(t), u(t))$ is minimized.

Now, we apply our theory to check the controllability of the system (21). For numerical purposes, we take

$$K = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \ M = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ L = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and } N = 0. \tag{22}$$

Then system (21) may be represented by (1) if we take

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{23}$$

Clearly, condition (C5) is not satisfied. Applying Theorem 4.1, we obtain

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{24}$$

Also,

$$MEN = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ MAN = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 4 & 3 \\ -1 & 0 & 1 \end{bmatrix}, \text{ and } MB = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{25}$$

Notice that the matrix A_{32}^2 is an empty matrix of order 0×2 and thus satisfying the requirement of Theorem 4.3. Further, for system (9), the matrices are as follows $\mathcal{A} = \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}$ and $\mathcal{B}$ is an empty matrix of order 2×0. Since the matrix $\mathcal{B}$ is empty, the system (9) is not controllable implying that the given system (21) is not R-controllable for numerical values as taken in (22).

6. CONCLUDING REMARKS

Various notions of controllability for general descriptor systems have been investigated. In particular, we have studied the R-controllability of general descriptor systems which has been less regarded in the existing literature. We have related the R-controllability of general linear time-invariant descriptor systems to the controllability of a matrix pair under some mild condition. Two numerically stable decompositions of system matrices have been developed which played a vital role in developing the whole theory. Physical as well as numerical examples are provided to illustrate the effectiveness of the presented theory.

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Vikas Kumar Mishra, Department of Mechanical Engineering, Ben-Gurion University of the Negev, Beer Sheva 8410501. Israel.
e-mail: vikas.mishra45@gmail.com

Nutan Kumar Tomar, Department of Mathematics, Indian Institute of Technology Patna, Patna 801103, Bihar. India.
e-mail: nktomar@iitp.ac.in