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OSCILLATIONS OF NONLINEAR DIFFERENCE EQUATIONS WITH DEVIATING ARGUMENTS

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Abstract. This paper is concerned with the oscillatory behavior of first-order nonlinear difference equations with variable deviating arguments. The corresponding difference equations of both retarded and advanced type are studied. Examples illustrating the results are also given.

Keywords: infinite sum condition; retarded argument; advanced argument; oscillatory solution; nonoscillatory solution

MSC 2010: 39A10, 39A21

1. Introduction

In this paper we establish sufficient conditions for the oscillation of all solutions of the nonlinear retarded difference equations

$$\Delta x(n) + \sum_{i=1}^{m} f_i(n, x(\tau_i(n))) = 0, \quad \tau_i(n) < n, \ n \geqslant n_0 \tag{1.1}$$

and the (dual) nonlinear advanced difference equations

$$\nabla x(n) - \sum_{i=1}^{m} f_i(n, x(\sigma_i(n))) = 0, \quad \sigma_i(n) > n, \ n \geqslant n_0. \tag{1.2}$$

Here, f_i are real-valued functions, τ_i and σ_i are integer-valued sequences, Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ and ∇ corresponds to the backward difference operator $\nabla x(n) = x(n) - x(n-1)$.

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By a *solution* of (1.1), we mean a sequence of real numbers $\{x(n)\}_n$ which satisfies (1.1). When the initial data

$$x(n) = \varphi(n) \quad \text{for } \inf_{s \geqslant n_0} \min_{1 \leqslant i \leqslant m} \tau_i(s) \leqslant n \leqslant n_0$$

is given, we can obtain a unique solution to (1.1) by using the method of steps.

By a solution of (1.2), we mean a sequence of real numbers $\{x(n)\}_n$ which satisfies (1.2). Existence of solutions to (1.2) is usually proved using a fixed point argument.

We shall assume the existence of solutions, and concentrate on the study of their oscillatory behavior.

A solution $\{x(n)\}_n$ of (1.1) or (1.2) is called *oscillatory*, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

First, we state some known results for particular cases of (1.1) and (1.2). When $f_i(n, x) = p_i(n)x$, equations (1.1) and (1.2) become the linear equations:

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) = 0 \tag{1.3}$$

and

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\sigma_i(n)) = 0, \tag{1.4}$$

respectively.

Under the assumption that the retarded arguments $\tau_i(n)$, $1 \leqslant i \leqslant m$ are non-decreasing, Chatzarakis et al. [3], Theorem 2.2 proved that if

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p_i(n) > 0, \quad \liminf_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau_i(n)}^{n-1} p_i(j) > \frac{1}{\mathrm{e}}, \tag{1.5}$$

then all solutions of (1.3) oscillate.

In the same paper [3], Theorem 3.2, they proved that if

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p_i(n) > 0, \quad \liminf_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n+1}^{\sigma_i(n)} p_i(j) > \frac{1}{\mathrm{e}}, \tag{1.6}$$

where the advanced arguments $\sigma_i(n)$, $1 \leqslant i \leqslant m$ are non-decreasing, then all solutions of (1.4) oscillate.

Yan et al. [17] proved that if

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}\sum_{i=1}^{m}p_i(j)\Big(\frac{n-\tau_i(j)+1}{n-\tau_i(j)}\Big)^{n-\tau_i(j)+1}>1, \tag{1.7}$$

where $\tau(n)=\max\limits_{1\leqslant i\leqslant m}\tau_i(n)$, then all solutions of (1.3) oscillate.

If $\tau_i(n)=n-k_i$, condition (1.7) is expressed as

$$\liminf_{n\to\infty}\sum_{i=1}^{m}\Big(\frac{k_i+1}{k_i}\Big)^{k_i+1}\sum_{j=n-k}^{n-1}p_i(j)>1 \tag{1.8}$$

and was used by Li and Zhou [12]. When $m=1$, condition (1.8) reduces to

$$\liminf_{n\to\infty}\sum_{j=n-k}^{n-1}p(j)>\Big(\frac{k}{k+1}\Big)^{k+1}, \tag{1.9}$$

which was used by Ladas et al. [9]. For more information about difference and differential equations, the reader may refer to the references [1]–[18].

Note that the summations in conditions (1.5)–(1.9) have finitely many terms and relate to equations wih monotone deviating arguments. Our goal is to find an infinite summation condition that applies to some cases where (1.5) cannot be applied. In addition, we consider nonlinear difference equations, and do not assume that the deviating arguments are monotone. To establish an infinite sum condition, we adapt on the steps used by Li [11] for the linear delay differential equation

$$x'(t)+\sum_{i=1}^{m}p_i(t)x(t-\tau_i)=0, \tag{1.10}$$

where $p_i(t)\geqslant 0$ are continuous and τ_i are positive constants. For this equation, Li proved the following theorem:

Theorem 1.1 ([11], Theorem 2)**.** *Let* $\tau_*=\max\{\tau_1,\tau_2,\dots,\tau_m\}$. *Suppose that*

$$\sum_{i=1}^{m}\int_t^{t+\tau_i}p_i(s)\,\mathrm{d}s>0\quad \text{for } t\geqslant t_0,\ t_0>0$$

and

$$\limsup_{t\to\infty}\int_t^{t+\tau_*}p_*(s)\,\mathrm{d}s>0.$$

If, in addition,

$$\int_{t_0}^{\infty}\Big(\sum_{i=1}^{m}p_i(t)\Big)\ln\Big(\mathrm{e}\sum_{i=1}^{m}\int_t^{t+\tau_i}p_i(s)\,\mathrm{d}s\Big)\,\mathrm{d}t=\infty,$$

then all solutions of (1.10) *oscillate.*

Strong interest in the nonlinear retarded difference equation (1.1) is motivated by the fact that it represents a discrete analogue of the nonlinear delay differential equation

$$x'(t) + \sum_{i=1}^{m} f_i(t, x(\tau_i(t))) = 0, \quad t \geqslant t_0, \tag{1.11}$$

where

$$\tau_i(t) < t \quad \text{for } t \geqslant t_0 \quad \text{and} \quad \lim_{t\to\infty} \tau_i(t) = \infty, \quad 1 \leqslant i \leqslant m. \tag{1.12}$$

Recently, Dix et al. [5] proved the following theorem:

Theorem 1.2 ([5], Theorem 4)**.** *Assume that* (1.12) *holds,* $xf_i(t,x) \geqslant 0$ *and there exist continuous functions* $p_i(t) \geqslant 0$ *and* $g_i(x)$ *such that*

$$|f_i(t,x)| \geqslant p_i(t)|g_i(x)| \quad \forall\, x \in \mathbb{R},\ t \geqslant t_0,$$

where $xg_i(x) > 0$ *for* $x \neq 0$ *and* $\limsup_{x\to 0} x/g_i(x) < \infty$*. If, in addition,*

$$\int_s^{\tau_{\mathrm{inv}}(s)} \sum_{j=1}^{m} p_j(r)\,\mathrm{d}r > 0 \quad \forall\, s \geqslant t_0$$

and

$$\int_{t_0}^{\infty} \sum_{i=1}^{m} p_i(s)\Big(1 + \ln \int_s^{\tau_{\mathrm{inv}}(s)} \sum_{j=1}^{m} p_j(r)\,\mathrm{d}r\Big)\,\mathrm{d}s = \infty,$$

where $\tau_{\mathrm{inv}}(s) = \max\{t\colon\ \tau(t) = s\}$, $\tau(t) = \max_{t_0\leqslant s\leqslant t} \tau_0(s)$, $\tau_0(t) = \max_{1\leqslant i\leqslant m} \tau_i(t)$, *then all solutions of* (1.11) *oscillate.*

An interesting question then arises whether there exists a discrete analogue of Theorem 1.2 for (1.1), and consequently for (1.2).

In the present paper an affirmative answer to the above question is given.

In this section we make the following assumptions:

(A1) The retarded arguments satisfy $\tau_i(n) < n$ and $\lim\limits_{n\to\infty} \tau_i(n) = \infty$ for $1 \leqslant i \leqslant m$ and $n \geqslant n_0$;

(A2) $x f_i(n,x) \geqslant 0$ and there exist non-negative functions p_i such that $|f_i(n,x)| \geqslant p_i(n)|x|$ for all $x \in \mathbb{R}$, $n \geqslant n_0$.

Having made these assumptions, we define the sequences

$$\widetilde{\tau}_i(n) = \max\{\tau_i(s)\colon\ 1 \leqslant s \leqslant n\} \tag{2.1}$$

and

$$\widetilde{\tau}(n) = \max_{1\leqslant i\leqslant m} \widetilde{\tau}_i(n). \tag{2.2}$$

Clearly, the sequences $\widetilde{\tau}_i(n)$ and $\widetilde{\tau}(n)$ are non-decreasing and

$$\tau_i(n) \leqslant \widetilde{\tau}_i(n) \leqslant \widetilde{\tau}(n) < n, \quad 1 \leqslant i \leqslant m,\ \forall\, n \geqslant n_0. \tag{2.3}$$

Lemma 2.1. *Assume* (A1) *and* (A2) *hold and x is a nonoscillatory solution of* (1.1). *Then there exists a positive integer n_1 such that: When x is eventually positive, then $x(n) > 0$, $x(\tau_i(n)) > 0$, $x(\widetilde{\tau}_i(n)) > 0$, $x(\widetilde{\tau}(n)) > 0$ for all $n \geqslant n_1$ and $x(n)$ is non-increasing. When x is eventually negative, then $x(n) < 0$, $x(\tau_i(n)) < 0$, $x(\widetilde{\tau}_i(n)) < 0$, $x(\widetilde{\tau}(n)) < 0$ for all $n \geqslant n_1$ and $x(n)$ is non-decreasing.*

Proof. First, assume that x is eventually positive. Then there exists n^* such that $x(n) > 0$ for all $n \geqslant n^*$. Since $\lim\limits_{n\to\infty} \tau_i(n) = \infty$, there exists $n_1 \geqslant n^*$ such that $\tau_i(n) \geqslant n^*$ for all $n \geqslant n_1$. By (2.3) the result follows. In view of (A2), (1.1) gives

$$\Delta x(n) = -\sum_{i=1}^{m} f_i(n, x(\tau_i(n))) \leqslant -\sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \leqslant 0;$$

therefore $x(n)$ is non-increasing.

For eventually negative solutions, the corresponding statements are shown similarly. □

From the definition of $\widetilde{\tau}$ it follows that $\widetilde{\tau}(n)$ is non-decreasing, but it may not be one-to-one. We define an "inverse" function by taking the largest element in the set $\{n\colon\ \widetilde{\tau}(n) \leqslant s\}$. This set is bounded because $\lim\limits_{n\to\infty} \widetilde{\tau}(n) = \infty$. Let

$$\widetilde{\tau}^{\text{inv}}(s) = \max\{n\colon\ \widetilde{\tau}(n) \leqslant s\}.$$

This function is defined for all $s \geqslant \widetilde{\tau}(n_0)$, and is non-decreasing. In addition, by (A1), it satisfies

$$\widetilde{\tau}(n) < n < \widetilde{\tau}^{\text{inv}}(n).$$

Lemma 2.2. *Assume* (A1) *and* (A2) *hold and* x *is a nonoscillatory solution of* (1.1). *Then*

$$\sum_{j=n}^{\widetilde{\tau}^{\text{inv}}(n)} \sum_{i=1}^{m} p_i(j) < 1 \quad \text{for } n \geqslant n_1.$$

P r o o f. First, assume that x is eventually positive. By Lemma 2.1, x is non-increasing. In view of (A2), (1.1) gives

$$\begin{aligned} \Delta x(n) &= -\sum_{i=1}^{m} f_i(n, x(\tau_i(n))) \leqslant -\sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \\ &\leqslant -\sum_{i=1}^{m} p_i(n) x(\widetilde{\tau}_i(n)) \leqslant -x(\widetilde{\tau}(n)) \sum_{i=1}^{m} p_i(n). \end{aligned} \tag{2.4}$$

To abbreviate the notation, let $\mathcal{P}(n) = \sum\limits_{i=1}^{m} p_i(n)$. Summing this inequality from n to $\widetilde{\tau}^{\text{inv}}(n)$ yields

$$x(\widetilde{\tau}^{\text{inv}}(n)+1) - x(n) \leqslant -\sum_{j=n}^{\widetilde{\tau}^{\text{inv}}(n)} x(\widetilde{\tau}(j))\mathcal{P}(j) \leqslant -x(n) \sum_{j=n}^{\widetilde{\tau}^{\text{inv}}(n)} \mathcal{P}(j).$$

Since $x(\cdot) > 0$, we have

$$0 < x(\widetilde{\tau}^{\text{inv}}(n)+1) \leqslant x(n)\biggl(1 - \sum_{j=n}^{\widetilde{\tau}^{\text{inv}}(n)} \mathcal{P}(j)\biggr)$$

and the result follows for eventually positive solutions.

Now, assume that x is eventually negative. By Lemma 2.1, x is non-decreasing. In view of (A2), (1.1) gives

$$\begin{aligned} \Delta x(n) &= -\sum_{i=1}^{m} f_i(n, x(\tau_i(n))) \geqslant -\sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \\ &\geqslant -\sum_{i=1}^{m} p_i(n) x(\widetilde{\tau}_i(n)) \geqslant -x(\widetilde{\tau}(n)) \sum_{i=1}^{m} p_i(n). \end{aligned} \tag{2.5}$$

Summing this inequality from n to $\widetilde{\tau}^{\rm inv}(n)$ yields

$$x(\widetilde{\tau}^{\rm inv}(n)+1)-x(n)\geqslant -\sum_{j=n}^{\widetilde{\tau}^{\rm inv}(n)} x(\widetilde{\tau}(j))\mathcal{P}(j)\geqslant -x(n)\sum_{j=n}^{\widetilde{\tau}^{\rm inv}(n)}\mathcal{P}(j).$$

Since $x(\cdot)<0$, we have

$$0>x(\widetilde{\tau}^{\rm inv}(n)+1)\geqslant x(n)\Bigl(1-\sum_{j=n}^{\widetilde{\tau}^{\rm inv}(n)}\mathcal{P}(j)\Bigr)$$

and the result follows for eventually negative solutions. □

From Lemma 2.2, using a contradiction argument we can show that if

$$\limsup_{n\to\infty}\sum_{j=n}^{\widetilde{\tau}^{\rm inv}(n)}\sum_{i=1}^{m}p_i(j)>1,$$

then all solutions of (1.1) are oscillatory.

Lemma 2.3. *Assume* (A1) *and* (A2) *hold and x is a nonoscillatory solution of* (1.1). *Furthermore, assume that*

$$\limsup_{n\to\infty}\sum_{j=n+1}^{\widetilde{\tau}^{\rm inv}(n)}\sum_{i=1}^{m}p_i(j)>0. \tag{2.6}$$

Then

$$\liminf_{n\to\infty}\frac{x(\widetilde{\tau}(n))}{x(n)}<\infty.$$

P r o o f. From (2.6), it is clear that there exist a positive constant d and a sequence $\{n_k\}_{k=1}^{\infty}$, converging to ∞, such that

$$\sum_{j=n_k+1}^{\widetilde{\tau}^{\rm inv}(n_k)}\mathcal{P}(j)\geqslant d>0\quad\text{for } k=1,2,\ldots, \tag{2.7}$$

where $\mathcal{P}(j)=\sum\limits_{i=1}^{m}p_i(j)$. Then in each interval $[n_k+1,\widetilde{\tau}^{\rm inv}(n_k)]$ there exists an integer θ_k such that

$$\sum_{j=n_k+1}^{\theta_k}\mathcal{P}(j)\geqslant\frac{d}{2}\quad\text{and}\quad\sum_{j=\theta_k}^{\widetilde{\tau}^{\rm inv}(n_k)}\mathcal{P}(j)\geqslant\frac{d}{2}. \tag{2.8}$$

Note that $\mathcal{P}(\theta_k)$ is included in both the summations.

To find θ_k, partial sums are considered recursively: If $\mathcal{P}(n_k+1) \geqslant \frac{1}{2}d$, we set $\theta_k = n_k+1$; otherwise, we proceed with the next term. If $\mathcal{P}(n_k+1)+\mathcal{P}(n_k+2) \geqslant \frac{1}{2}d$, we set $\theta_k = n_k+2$; otherwise proceed with the next term. This process will eventually stop, after finitely many steps, since the whole summation is greater than $\frac{1}{2}d$.

In the first part of this proof, we assume that the solution x is eventually positive. Summing (2.4) from n_k to θ_k, and using that x is non-increasing, we have

$$0 \geqslant \sum_{j=n_k}^{\theta_k} \Delta x(j) + \sum_{j=n_k}^{\theta_k} x(\widetilde{\tau}(j))\mathcal{P}(j) \geqslant x(\theta_k+1) - x(n_k) + x(\widetilde{\tau}(\theta_k)) \sum_{j=n_k}^{\theta_k} \mathcal{P}(j).$$

Since $x(\theta_k+1) > 0$, by omitting this term and using (2.8), we have

$$x(n_k) > x(\widetilde{\tau}(\theta_k)) \sum_{j=n_k}^{\theta_k} \mathcal{P}(j) \geqslant x(\widetilde{\tau}(\theta_k))\frac{d}{2}.$$

Thus

$$0 < \frac{x(\widetilde{\tau}(\theta_k))}{x(n_k)} < \frac{2}{d}. \tag{2.9}$$

Summing (2.4) from θ_k to $\widetilde{\tau}^{\mathrm{inv}}(n_k)$, and using that x is non-increasing, we have

$$\begin{aligned} 0 &\geqslant \sum_{j=\theta_k}^{\widetilde{\tau}^{\mathrm{inv}}(n_k)} \Delta x(j) + \sum_{j=\theta_k}^{\widetilde{\tau}^{\mathrm{inv}}(n_k)} x(\widetilde{\tau}(j))\mathcal{P}(j) \\ &\geqslant x(\widetilde{\tau}^{\mathrm{inv}}(n_k)+1) - x(\theta_k) + x(n_k) \sum_{j=\theta_k}^{\widetilde{\tau}^{\mathrm{inv}}(n_k)} \mathcal{P}(j). \end{aligned}$$

Since $x(\widetilde{\tau}^{\mathrm{inv}}(n_k)+1) > 0$, by omitting this term and using (2.8), we have

$$x(\theta_k) > x(n_k) \sum_{j=\theta_k}^{\widetilde{\tau}^{\mathrm{inv}}(n_k)} \mathcal{P}(j) \geqslant x(n_k)\frac{d}{2}.$$

Thus

$$0 < \frac{x(n_k)}{x(\theta_k)} < \frac{2}{d}. \tag{2.10}$$

Combining the inequalities (2.9) and (2.10), we obtain

$$\frac{x(\widetilde{\tau}(\theta_k)}{x(\theta_k)} < \Big(\frac{2}{d}\Big)^2.$$

Note that the right-hand side is independent of k. Therefore, the conclusion follows for eventually positive solutions.

Now, assume that x is eventually negative. Summing (2.5) from n_k to θ_k, and using that x is non-decreasing, we have

$$0 \leqslant \sum_{j=n_k}^{\theta_k} \Delta x(j) + \sum_{j=n_k}^{\theta_k} x(\widetilde{\tau}(j))\mathcal{P}(j) \leqslant x(\theta_k + 1) - x(n_k) + x(\widetilde{\tau}(\theta_k)) \sum_{j=n_k}^{\theta_k} \mathcal{P}(j).$$

Since $x(\theta_k + 1) < 0$, by omitting this term and using (2.8), we have

$$x(n_k) < x(\widetilde{\tau}(\theta_k)) \sum_{j=n_k}^{\theta_k} \mathcal{P}(j) \leqslant x(\widetilde{\tau}(\theta_k))\frac{d}{2}.$$

Thus

$$0 < \frac{x(\widetilde{\tau}(\theta_k))}{x(n_k)} < \frac{2}{d}. \tag{2.11}$$

Summing (2.5) from θ_k to $\widetilde{\tau}^{\text{inv}}(n_k)$, and using that x is non-decreasing, we have

$$\begin{aligned} 0 &\leqslant \sum_{j=\theta_k}^{\widetilde{\tau}^{\text{inv}}(n_k)} \Delta x(j) + \sum_{j=\theta_k}^{\widetilde{\tau}^{\text{inv}}(n_k)} x(\widetilde{\tau}(j))\mathcal{P}(j) \\ &\leqslant x(\widetilde{\tau}^{\text{inv}}(n_k) + 1) - x(\theta_k) + x(n_k) \sum_{j=\theta_k}^{\widetilde{\tau}^{\text{inv}}(n_k)} \mathcal{P}(j). \end{aligned}$$

Since $x(\widetilde{\tau}^{\text{inv}}(n_k) + 1) < 0$, by omitting this term and using (2.8), we have

$$x(\theta_k) > x(n_k) \sum_{j=\theta_k}^{\widetilde{\tau}^{\text{inv}}(n_k)} \mathcal{P}(j) \geqslant x(n_k)\frac{d}{2}.$$

Thus

$$0 < \frac{x(n_k)}{x(\theta_k)} < \frac{2}{d}. \tag{2.12}$$

Combining the inequalities (2.11) and (2.12), we obtain

$$\frac{x(\widetilde{\tau}(\theta_k)}{x(\theta_k)} < \Big(\frac{2}{d}\Big)^2.$$

Note that the right-hand side is independent of k. Therefore, the conclusion follows for eventually negative solutions. □

Theorem 2.4. *Assume* (A1), (A2), (2.6) *hold, and*

$$\sum_{n=1}^{\infty}\Big(\sum_{i=1}^{m}p_i(n)\Big)\ln\Big(\mathrm{e}\sum_{j=n+1}^{\widetilde{\tau}^{\mathrm{inv}}(n)}\sum_{i=1}^{m}p_i(j)\Big)=\infty. \tag{2.13}$$

Then all solutions of (1.1) *are oscillatory.*

P r o o f. To reach a contradiction, suppose that x is an eventually positive solution of (1.1). By Lemma 2.1, $x(n)$ is positive and non-increasing for $n\geqslant n_1$. Recall that $\ln(r)\leqslant r-1$ for $r>0$. Defining a new variable ξ and using this inequality, we have

$$\xi(n):=-\frac{\Delta x(n)}{x(n)}=-\Big(\frac{x(n+1)}{x(n)}-1\Big)\leqslant-\ln\Big(\frac{x(n+1)}{x(n)}\Big).$$

Summing from $\widetilde{\tau}(n)$ to $n-1$, we have

$$\begin{aligned}\sum_{j=\widetilde{\tau}(n)}^{n-1}\xi(j)&\leqslant-\sum_{j=\widetilde{\tau}(n)}^{n-1}\ln\Big(\frac{x(j+1)}{x(j)}\Big)\\&=-\ln\Big(\frac{x(\widetilde{\tau}(n)+1)}{x(\widetilde{\tau}(n))}\times\ldots\times\frac{x(n)}{x(n-1)}\Big)=\ln\Big(\frac{x(\widetilde{\tau}(n))}{x(n)}\Big).\end{aligned} \tag{2.14}$$

Dividing by $-x(n)$ in (2.4) and using the notation $\mathcal{P}=\sum\limits_{i=1}^{m}p_i(j)$, we have

$$\xi(n)=-\frac{\Delta(n)}{x(n)}\geqslant\frac{x(\widetilde{\tau}(n))}{x(n)}\mathcal{P}(n),$$

or

$$\frac{\xi(n)}{\mathcal{P}(n)}\geqslant\frac{x(\widetilde{\tau}(n))}{x(n)}. \tag{2.15}$$

Applying logarithms on both sides and using (2.14), we obtain

$$\ln\Big(\frac{\xi(n)}{\mathcal{P}(n)}\Big)\geqslant\ln\Big(\frac{x(\widetilde{\tau}(n))}{x(n)}\Big)\geqslant\sum_{j=\widetilde{\tau}(n)}^{n-1}\xi(j).$$

Therefore,

$$\xi(n)\geqslant\mathcal{P}(n)\exp\Big(\sum_{j=\widetilde{\tau}(n)}^{n-1}\xi(j)\Big). \tag{2.16}$$

Next we use the inequality $e^{by} \geqslant y + \ln(eb)/b$ which holds for $b > 0$. Indeed, the function $g(y) = e^{by} - y - \ln(eb)/b$ attains its absolute minimum $g_{\min} = 0$ at $y = -\ln(b)/b$. Using this inequality, with

$$b(n) = \sum_{j=n+1}^{\widetilde{\tau}^{\mathrm{inv}}(n)} \mathcal{P}(j) \quad \text{and} \quad y = \frac{1}{b(n)} \sum_{j=\widetilde{\tau}(n)}^{n-1} \xi(j)$$

in (2.16), we obtain

$$\xi(n) \geqslant \mathcal{P}(n)\left(\frac{1}{b(n)} \sum_{j=\widetilde{\tau}(n)}^{n-1} \xi(j) + \frac{1}{b(n)} \ln(eb(n))\right).$$

Then

$$\xi(n)b(n) - \mathcal{P}(n) \sum_{j=\widetilde{\tau}(n)}^{n-1} \xi(j) \geqslant \mathcal{P}(n) \ln(eb(n)).$$

We select a positive integer v such that $\widetilde{\tau}(v-1) < \widetilde{\tau}(v)$. Note that there are infinitely many integers with this property because $\lim_{n\to\infty} \widetilde{\tau}(n) = \infty$. By selecting such an integer, we aim at interchanging the order, in the double summation below.

Summing the above inequality from u to $v-1$ (with $u < \widetilde{\tau}(v-1)$), we have

$$\sum_{n=u}^{v-1} \xi(n)b(n) - \sum_{n=u}^{v-1} \mathcal{P}(n) \sum_{j=\widetilde{\tau}(n)}^{n-1} \xi(j) \geqslant \sum_{n=u}^{v-1} \mathcal{P}(n) \ln(eb(n)). \tag{2.17}$$

Interchanging the order in the double summation, and shortening the area of summation, we have

$$\begin{aligned}\sum_{n=u}^{v-1} \mathcal{P}(n) \sum_{j=\widetilde{\tau}(n)}^{n-1} \xi(j) &\geqslant \sum_{j=u-1}^{\widetilde{\tau}(v-1)} \xi(j) \sum_{n=j+1}^{\widetilde{\tau}^{\mathrm{inv}}(j)} \mathcal{P}(n) = \sum_{n=u-1}^{\widetilde{\tau}(v-1)} \xi(n) \sum_{j=n+1}^{\widetilde{\tau}^{\mathrm{inv}}(n)} \mathcal{P}(j) \\ &= \sum_{n=u-1}^{\widetilde{\tau}(v-1)} \xi(n)b(n) \geqslant \sum_{n=u}^{\widetilde{\tau}(v-1)} \xi(n)b(n).\end{aligned}$$

From this inequality and (2.17), we obtain

$$\sum_{n=u}^{v-1} \xi(n)b(n) - \sum_{n=u}^{\widetilde{\tau}(v-1)} \xi(n)b(n) \geqslant \sum_{n=u}^{v-1} \mathcal{P}(n) \ln(eb(n)).$$

By Lemma 2.2, we have $b(n) < 1$ (with one term to spare). Therefore

$$\sum_{n=\widetilde{\tau}(v-1)+1}^{v-1} \xi(n) \geqslant \sum_{n=u}^{v-1} \mathcal{P}(n) \ln(eb(n)).$$

Applying (2.14) with $v-1$ instead of n, it follows that

$$\ln\Big(\frac{x(\widetilde{\tau}(v-1))}{x(v-1)}\Big) \geqslant \sum_{n=\widetilde{\tau}(v-1)+1}^{v-1} \xi(n) \geqslant \sum_{n=u}^{v-1} \mathcal{P}(n)\ln(\mathrm{e}b(n)).$$

Since (2.13) holds, the last inequality yields

$$\lim_{v\to\infty} \frac{x(\widetilde{\tau}(v-1))}{x(v-1)} = \infty. \tag{2.18}$$

Using a contradiction argument, from (2.13) we can show the existence of a sequence $\{\theta_k\}$ converging to ∞, such that

$$\sum_{j=\theta_k+1}^{\widetilde{\tau}^{\mathrm{inv}}(\theta_k)} \mathcal{P}(j) > \frac{1}{\mathrm{e}} \quad \text{for } k \geqslant 1.$$

Therefore (2.6) is satisfied and Lemma 2.3 implies

$$\liminf_{k\to\infty} \frac{x(\widetilde{\tau}(\theta_k))}{x(\theta_k)} < \infty.$$

This contradicts (2.18), and shows that a solution cannot be eventually positive.

Now we assume that x is an eventually negative solution of (1.1). By Lemma 2.1, x is negative and non-decreasing for $n \geqslant n_1$. Dividing by $x(n)$ in (2.5), we have

$$\xi(n) := -\frac{\Delta(n)}{x(n)} \geqslant \frac{x(\widetilde{\tau}(n))}{x(n)}\mathcal{P}(n).$$

Then we follow the steps in the first part of this proof to show that a solution cannot be eventually negative. □

Next we compare our oscillation condition with those stated in the introduction. First we show that (1.5) implies (2.13). It is easy to see that (1.5) implies the existence of a constant $\alpha > 1/\mathrm{e}$ such that

$$\sum_{i=1}^{m} \sum_{j=\tau_i(n)}^{n-1} p_i(j) \geqslant \alpha > \frac{1}{\mathrm{e}}.$$

Using this constant, we can show (2.13).

E x a m p l e 2.5. In this example (2.13) is satisfied, but (1.5) and (1.8) are not. Consider the retarded difference equation

$$\Delta x(n) + \frac{1}{\mathrm{e}}\mathrm{e}^{\sin n}x(n-1) + \frac{3}{8\mathrm{e}}\mathrm{e}^{\sin n}x(n-2) = 0, \quad n \geqslant 2. \tag{2.19}$$

Here $\widetilde{\tau}(n) = n - 1$, $\widetilde{\tau}^{\text{inv}}(n) = n + 1$, and $\sum_{i=1}^{2} p_i(n) = \frac{11}{8}\exp(\sin n)/\mathrm{e} > 0$. Using that $\ln(\cdot)$ is a non-decreasing function, we obtain

$$\begin{aligned}\sum_{n=2}^{\infty}\Big(\sum_{i=1}^{2} p_i(n)\Big)\ln\Big(\mathrm{e}\sum_{j=n+1}^{n+1}\sum_{i=1}^{2} p_i(j)\Big) &\geqslant \frac{11}{8\mathrm{e}}\sum_{n=2}^{\infty}\mathrm{e}^{\sin n}\ln\Big(\mathrm{e}\sum_{j=n+1}^{n+1} p_1(j)\Big)\\ &= \frac{11}{8\mathrm{e}}\sum_{n=2}^{\infty}\mathrm{e}^{\sin n}\ln\Big(\mathrm{e}\frac{1}{\mathrm{e}}\sum_{j=n+1}^{n+1}\mathrm{e}^{\sin j}\Big)\\ &\geqslant \frac{11}{8\mathrm{e}}\sum_{n=2}^{\infty}\mathrm{e}^{\sin n}\sum_{j=n+1}^{n+1}\sin j\\ &= \frac{11}{8\mathrm{e}}\sum_{n=2}^{\infty}\mathrm{e}^{\sin n}\sin(n+1).\end{aligned}$$

To estimate the above series, we make groups of 7 summands

$$\sum_{n=2}^{\infty}\varphi(n) = \sum_{n=0}^{6}\varphi(n+2) + \sum_{n=0}^{6}\varphi(n+9) + \sum_{n=0}^{6}\varphi(n+16) + \ldots$$

Using that $\mathrm{e}^{\sin n}\sin(n+1)$ is 2π-periodic, each summation can be written as $\sum_{n=0}^{6}\mathrm{e}^{\sin(n+t)}\sin(n+1+t)$ for certain values of $t \in [0, 2\pi)$. Computations (using Mathematica) show that

$$\inf_{0\leqslant t\leqslant 2\pi}\sum_{n=0}^{6}\mathrm{e}^{\sin(n+t)}\sin(n+1+t) = 1.26288.$$

Therefore $\sum_{n=2}^{\infty}\mathrm{e}^{\sin n}\sin(n+1) = \infty$, and consequently all solutions of (2.19) are oscillatory.

To show that (1.5) and (1.8) are not satisfied, note that

$$\begin{aligned}\liminf_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau_i(n)}^{n-1} p_i(j) &= \liminf_{n\to\infty}\Big(\sum_{j=n-1}^{n-1} p_1(j) + \sum_{j=n-2}^{n-1} p_2(j)\Big)\\ &= \liminf_{n\to\infty}\Big(\frac{11}{8\mathrm{e}}\mathrm{e}^{\sin(n-1)} + \frac{3}{8\mathrm{e}}\mathrm{e}^{\sin(n-2)}\Big) = \frac{7}{4\mathrm{e}^2} < \frac{1}{\mathrm{e}}\end{aligned}$$

and

$$
\begin{aligned}
\liminf_{n\to\infty} \sum_{i=1}^{m} \Bigl(\frac{k_i+1}{k_i}\Bigr)^{k_i+1} & \sum_{j=n-1}^{n-1} p_i(j) \\
&= \liminf_{n\to\infty} \biggl(\Bigl(\frac{2}{1}\Bigr)^2 \sum_{j=n-1}^{n-1} p_1(j) + \Bigl(\frac{3}{2}\Bigr)^3 \sum_{j=n-1}^{n-1} p_2(j)\biggr) \\
&= \liminf_{n\to\infty} \Bigl(\frac{4}{\mathrm{e}}\mathrm{e}^{\sin(n-1)} + \frac{81}{64\mathrm{e}}\mathrm{e}^{\sin(n-1)}\Bigr) \\
&= \liminf_{n\to\infty} \Bigl(\frac{337}{64\mathrm{e}}\mathrm{e}^{\sin(n-1)}\Bigr) \frac{337}{64\mathrm{e}^2} < 1.
\end{aligned}
$$

The next two examples show that conditions (1.9) and (2.13) are independent of each other.

Example 2.6. Consider the retarded difference equation

$$
\Delta x(n) + \frac{1}{\mathrm{e}}\mathrm{e}^{\sin n} x(n-1) = 0, \quad n \geqslant 2. \tag{2.20}
$$

Clearly

$$
\begin{aligned}
\liminf_{n\to\infty} \sum_{j=n-k}^{n-1} p(j) = \liminf_{n\to\infty} \sum_{j=n-1}^{n-1} p(j) &= \liminf_{n\to\infty} \frac{1}{\mathrm{e}}\mathrm{e}^{\sin(n-1)} \\
&= \frac{1}{\mathrm{e}^2} < \Bigl(\frac{k}{k+1}\Bigr)^{k+1} = \Bigl(\frac{1}{2}\Bigr)^2 = \frac{1}{4},
\end{aligned}
$$

which means that (1.9) is not satisfied.

Observe, however, that (2.13) holds. Indeed, as in the previous example

$$
\sum_{n=2}^{\infty} p(n) \ln \biggl(\mathrm{e} \sum_{j=n+1}^{n+1} p(j)\biggr) \geqslant \frac{1}{\mathrm{e}} \sum_{n=2}^{\infty} \mathrm{e}^{\sin n} \sin(n+1) = \infty
$$

and therefore all solutions of (2.20) are oscillatory.

Example 2.7. Let $k > 3$ be an integer and α a real number such that

$$
\Bigl(\frac{k}{k+1}\Bigr)^{k+1} < \alpha < \frac{1}{\mathrm{e}}.
$$

Also let $\tau(n) = n - k$, $m = 1$, and

$$
p(j) = \begin{cases} 0 & \text{if } j \text{ is a multiple of } k, \\ \dfrac{\alpha}{k-1} & \text{otherwise.} \end{cases}
$$

We consider the nonlinear equation

$$\Delta x(n) = -p(n)\Big(\frac{11}{10}x(n-k) + \frac{1}{10}\sin(n-k)\Big).$$

Then for the corresponding linear equation (1.3), condition (1.9) is satisfied because

$$\sum_{i=1}^{1}\sum_{j=n-k}^{n-1} p(j) = \alpha > \Big(\frac{k}{k+1}\Big)^{k+1}.$$

To show that (2.13) is not satisfied, we have $\widetilde{\tau}^{\mathrm{inv}}(n) = n+k$, and

$$\sum_{j=n+1}^{n+k}\sum_{i=1}^{1} p(j) = \alpha < \frac{1}{\mathrm{e}}.$$

Therefore, $\ln(\cdot) < 0$ in (2.13), and the series cannot approach ∞.

3. Nonlinear advanced difference equations

In this section we make the following assumptions:

(A2) $xf_i(n,x) \geqslant 0$ and there exist non-negative functions p_i such that $|f_i(n,x)| \geqslant p_i(n)|x|$ for all $x \in \mathbb{R}$, $n \geqslant n_0$.

(A3) The advanced arguments satisfy $\sigma_i(n) > n$ for $1 \leqslant i \leqslant m$ and $n \geqslant n_0$;

Having made these assumptions, we define the sequences:

$$\widetilde{\sigma}_i(n) = \min\{\sigma_i(s)\colon\ s \geqslant n\} \tag{3.1}$$

and

$$\widetilde{\sigma}(n) = \min_{1\leqslant i\leqslant m} \widetilde{\sigma}_i(n). \tag{3.2}$$

Clearly, the sequences $\widetilde{\sigma}_i(n)$ and $\widetilde{\sigma}(n)$ are non-decreasing and

$$n < \widetilde{\sigma}(n) \leqslant \widetilde{\sigma}_i(n) \leqslant \sigma_i(n), \quad 1 \leqslant i \leqslant m,\ \forall\, n \geqslant n_0. \tag{3.3}$$

Lemma 3.1. *Assume* (A2) *and* (A3) *hold and x is a nonoscillatory solution of* (1.2). *Then there exists a positive integer n_1 such that: When x is eventually positive, then $x(n) > 0$, $x(\sigma_i(n)) > 0$, $x(\widetilde{\sigma}_i(n)) > 0$, $x(\widetilde{\sigma}(n)) > 0$ for all $n \geqslant n_1$ and $x(n)$ is non-decreasing. When x is eventually negative, then $x(n) < 0$, $x(\sigma_i(n)) < 0$, $x(\widetilde{\sigma}_i(n)) < 0$, $x(\widetilde{\sigma}(n)) < 0$ for all $n \geqslant n_1$ and $x(n)$ is non-increasing.*

The proof of this lemma is similar to that of Lemma 2.1, so we omit it.

From the definition of $\widetilde{\sigma}$ it follows that $\widetilde{\sigma}(n)$ is non-decreasing, but it may not be one-to-one. We define an "inverse" function by taking the smallest element in the set $\widetilde{\sigma}^{-1}(s) = \{n\colon\ \widetilde{\sigma}(n) \geqslant s\}$. This function is defined for $s \geqslant \widetilde{\sigma}(n_0)$ and is bounded below because $n < \widetilde{\sigma}(n)$. Let

$$\widetilde{\sigma}^{\rm inv}(s) = \min\{n\colon\ \widetilde{\sigma}(n) \geqslant s\}.$$

This sequence is non-decreasing. In addition, by (A3), it satisfies

$$\widetilde{\sigma}^{\rm inv}(n) < n < \widetilde{\sigma}(n).$$

Lemma 3.2. *Assume* (A2) *and* (A3) *hold and x is a nonoscillatory solution of* (1.2)*. Then*

$$\sum_{j=\widetilde{\sigma}^{\rm inv}(n)}^{n} \sum\nolimits_{i=1}^{m} p_i(j) < 1 \quad \text{for } n \geqslant n_1.$$

P r o o f. First assume that x is eventually positive. By Lemma 3.1, x is non-decreasing. Then by (A2) we have

$$\begin{aligned}\nabla x(n) &= \sum_{i=1}^{m} f_i(n, x(\sigma_i(n))) \geqslant \sum_{i=1}^{m} p_i(n) x(\sigma_i(n)) \\ &\geqslant \sum_{i=1}^{m} p_i(n) x(\widetilde{\sigma}_i(n)) \geqslant x(\widetilde{\sigma}(n)) \sum_{i=1}^{m} p_i(n).\end{aligned} \tag{3.4}$$

As before, we let $\mathcal{P}(n) = \sum\limits_{i=1}^{m} p_i(n)$. Summing this inequality from n to $\widetilde{\sigma}(n)$ yields

$$x(\widetilde{\sigma}(n)) - x(n-1) \geqslant \sum_{j=n}^{\widetilde{\sigma}(n)} x(\widetilde{\sigma}(j)) \mathcal{P}(j) \geqslant x(\widetilde{\sigma}(n)) \sum_{j=n}^{\widetilde{\sigma}(n)} \mathcal{P}(j).$$

Then

$$0 < x(n-1) \leqslant x(\widetilde{\sigma}(n)) \Bigl(1 - \sum_{j=n}^{\widetilde{\sigma}(n)} \mathcal{P}(j)\Bigr)$$

and the result follows for eventually positive solutions.

Now assume that x is eventually negative. By Lemma 3.1, x is non-increasing. Then by (A2) we have

$$\begin{aligned}\nabla x(n) &= \sum_{i=1}^{m} f_i(n, x(\sigma_i(n))) \leqslant \sum_{i=1}^{m} p_i(n) x(\sigma_i(n)) \\ &\leqslant \sum_{i=1}^{m} p_i(n) x(\widetilde{\sigma}_i(n)) \leqslant x(\widetilde{\sigma}(n)) \mathcal{P}(n).\end{aligned} \tag{3.5}$$

Summing this inequality from n to $\widetilde{\sigma}(n)$ yields

$$x(\widetilde{\sigma}(n)) - x(n-1) \leqslant \sum_{j=n}^{\widetilde{\sigma}(n)} x(\widetilde{\sigma}(j))\mathcal{P}(j) \leqslant x(\widetilde{\sigma}(n)) \sum_{j=n}^{\widetilde{\sigma}(n)} \mathcal{P}(j).$$

Then

$$0 > x(n-1) \geqslant x(\widetilde{\sigma}(n))\Bigl(1 - \sum_{j=n}^{\widetilde{\sigma}(n)} \mathcal{P}(j)\Bigr)$$

and the result follows for eventually negative solutions. □

From Lemma 3.2, using the contradiction argument we can show that if

$$\limsup_{n\to\infty} \sum_{j=\widetilde{\sigma}^{\mathrm{inv}}(n)}^{n} \sum_{i=1}^{m} p_i(j) > 1,$$

then all solutions of (1.2) are oscillatory.

Theorem 3.3. *Assume* (A2), (A3) *hold and that*

$$\sum_{n=1}^{\infty}\Bigl(\sum_{i=1}^{m} p_i(n)\Bigr) \ln\Bigl(\mathrm{e} \sum_{j=\widetilde{\sigma}^{\mathrm{inv}}(n)}^{n-1} \sum_{i=1}^{m} p_i(j)\Bigr) = \infty. \tag{3.6}$$

Then all solutions of (1.2) *are oscillatory.*

Proof. To reach a contradiction, suppose that x is an eventually positive solution of (1.2). By Lemma 3.1, $x(n)$ is positive and non-decreasing for $n \geqslant n_1$. Recall that $\ln(r) \leqslant r-1$ for $r > 0$. Defining a new variable ξ and using this inequality, we have

$$\xi(n) := \frac{\nabla x(n)}{x(n)} = -\Bigl(\frac{x(n-1)}{x(n)} - 1\Bigr) \leqslant -\ln\Bigl(\frac{x(n-1)}{x(n)}\Bigr).$$

Summing from $n+1$ to $\widetilde{\sigma}(n)$, we have

$$\sum_{j=n+1}^{\widetilde{\sigma}(n)} \xi(j) \leqslant -\sum_{j=n+1}^{\widetilde{\sigma}(n)} \ln\Bigl(\frac{x(j-1)}{x(j)}\Bigr) = \ln\Bigl(\frac{x(\widetilde{\sigma}(n))}{x(n)}\Bigr). \tag{3.7}$$

Dividing by $x(n)$ in (3.4) and using the notation $\mathcal{P} = \sum\limits_{i=1}^{m} p_i(j)$, we have

$$\xi(n) = \frac{\nabla(n)}{x(n)} \geqslant \frac{x(\widetilde{\sigma}(n))}{x(n)}\mathcal{P}(n),$$

or

$$\frac{\xi(n)}{\mathcal{P}(n)} \geqslant \frac{x(\widetilde{\sigma}(n))}{x(n)}. \tag{3.8}$$

Applying logarithms on both sides and using (3.7), we obtain

$$\ln\Big(\frac{\xi(n)}{\mathcal{P}(n)}\Big) \geqslant \ln\Big(\frac{x(\widetilde{\sigma}(n))}{x(n)}\Big) \geqslant \sum_{j=n+1}^{\widetilde{\sigma}(n)} \xi(j),$$

or

$$\xi(n) \geqslant \mathcal{P}(n) \exp\Big(\sum_{j=n+1}^{\widetilde{\sigma}(n)} \xi(j)\Big). \tag{3.9}$$

Using the inequality $\mathrm{e}^{by} \geqslant y + \ln(\mathrm{e}b)/b$, with

$$b(n) = \sum_{j=\widetilde{\sigma}^{\mathrm{inv}}(n)}^{n-1} \mathcal{P}(j) \quad \text{and} \quad y = \frac{1}{b(n)} \sum_{j=n+1}^{\widetilde{\sigma}(n)} \xi(j)$$

in (3.9), we obtain

$$\xi(n) \geqslant \mathcal{P}(n)\Big(\frac{1}{b(n)} \sum_{j=n+1}^{\widetilde{\sigma}(n)} \xi(j) + \frac{1}{b(n)} \ln(\mathrm{e}b(n))\Big).$$

Then

$$\xi(n)b(n) - \mathcal{P}(n) \sum_{j=n+1}^{\widetilde{\sigma}(n)} \xi(j) \geqslant \mathcal{P}(n) \ln(\mathrm{e}b(n)).$$

We select a positive integer u such that $\widetilde{\sigma}(u-1) < \widetilde{\sigma}(u)$. Note that there are infinitely many integers with this property because $\lim_{n\to\infty} \widetilde{\sigma}(n) = \infty$. By selecting such an integer, we aim at interchanging the order in the double summation below.

Summing the above inequality from u to $v-1$ (with $\widetilde{\sigma}(u) < v-1$), we have

$$\sum_{n=u}^{v-1} \xi(n)b(n) - \sum_{n=u}^{v-1} \mathcal{P}(n) \sum_{j=n+1}^{\widetilde{\sigma}(n)} \xi(j) \geqslant \sum_{n=u}^{v-1} \mathcal{P}(n) \ln(\mathrm{e}b(n)). \tag{3.10}$$

Interchanging the order in the double summation and shortening the summation area, we have

$$\begin{aligned}\sum_{n=u}^{v-1} \mathcal{P}(n) \sum_{j=n+1}^{\widetilde{\sigma}(n)} \xi(j) \geqslant \sum_{j=\widetilde{\sigma}(u)}^{v} \xi(j) \sum_{n=\widetilde{\sigma}^{\mathrm{inv}}(j)}^{j-1} \mathcal{P}(n) &= \sum_{n=\widetilde{\sigma}(u)}^{v} \xi(n) \sum_{j=\widetilde{\sigma}^{\mathrm{inv}}(n)}^{n-1} \mathcal{P}(j) \\ &= \sum_{n=\widetilde{\sigma}(u)}^{v} \xi(n)b(n) \geqslant \sum_{n=\widetilde{\sigma}(u)+1}^{v-1} \xi(n)b(n).\end{aligned}$$

From this inequality and (3.10) we obtain

$$\sum_{n=u}^{v-1} \xi(n)b(n) - \sum_{n=\widetilde{\sigma}(u)+1}^{v-1} \xi(n)b(n) \geqslant \sum_{n=u}^{v-1} \mathcal{P}(n)\ln(\mathrm{e}b(n)).$$

By Lemma 3.2, we have $b(n) < 1$ (with one term to spare). Therefore

$$\sum_{n=u}^{\widetilde{\sigma}(u)} \xi(n) \geqslant \sum_{n=u}^{v-1} \mathcal{P}(n)\ln(\mathrm{e}b(n)).$$

Note that the left-hand side is independent of v. By keeping u fixed and letting v approach ∞, we have a contradiction to (3.6).

Now we assume that x is an eventually negative solution of (1.2). By Lemma 3.1, x is negative and non-increasing for $n \geqslant n_1$. Dividing by $x(n)$ in (3.5), we have

$$\xi(n) := \frac{\nabla(n)}{x(n)} \geqslant \frac{x(\widetilde{\sigma}(n))}{x(n)}\mathcal{P}(n).$$

Then following the steps in the first part of the proof, we can show that a solution cannot be eventually negative. □

Example 3.4. Consider the advanced difference equation

$$\nabla x(n) - \frac{1}{\mathrm{e}}\mathrm{e}^{\sin n+\cos n}x(n+1) - \frac{1}{\mathrm{e}^2}\mathrm{e}^{\sin n/n}x(n+2) = 0, \quad n \geqslant 3. \tag{3.11}$$

Computations (using Mathematica) show that

$$\begin{aligned}
&\sum_{n=3}^{\infty}\left(\sum_{i=1}^{m} p_i(n)\right)\ln\left(\mathrm{e}\sum_{i=1}^{m}\sum_{j=n-k_i}^{n-1} p_i(j)\right)\\
&= \frac{1}{\mathrm{e}}\sum_{n=3}^{\infty}\left(\mathrm{e}^{\sin n+\cos n} + \frac{1}{\mathrm{e}}\mathrm{e}^{\sin n/n}\right)\ln\biggl(\sum_{j=n-1}^{n-1}\mathrm{e}^{\sin j+\cos j} + \frac{1}{\mathrm{e}}\sum_{j=n-2}^{n-1}\mathrm{e}^{\sin j/j}\biggr)\\
&= \frac{1}{\mathrm{e}}\sum_{n=3}^{\infty}\left(\mathrm{e}^{\sin n+\cos n} + \frac{1}{\mathrm{e}}\mathrm{e}^{\sin n/n}\right)\\
&\quad\times\ln\left(\mathrm{e}^{\sin(n-1)+\cos(n-1)} + \frac{1}{\mathrm{e}}(\mathrm{e}^{\sin(n-2)/(n-2)} + \mathrm{e}^{\sin(n-1)/(n-1)})\right) = \infty.
\end{aligned}$$

Thus

$$\sum_{n=3}^{\infty}\left(\sum_{i=1}^{m} p_i(n)\right)\ln\left(\mathrm{e}\sum_{i=1}^{m}\sum_{j=n-k_i}^{n-1} p_i(j)\right) = \infty,$$

that is (3.6) is satisfied, and therefore all solutions of (3.11) are oscillatory.

R e m a r k 3.5. A slight modification in the proofs of Theorems 2.4 and 3.3 leads to the following results about difference inequalities.

Theorem 3.6. *Assume that all the conditions of Theorem* 2.4 *hold. Then*

(i) *the retarded difference inequality*

$$\Delta x(n) + \sum_{i=1}^{m} f_i(n, x(\tau_i(n))) \leqslant 0 \quad \forall\, n \geqslant n_0$$

has no eventually positive solutions;

(ii) *the retarded difference inequality*

$$\Delta x(n) + \sum_{i=1}^{m} f_i(n, x(\tau_i(n))) \geqslant 0 \quad \forall\, n \geqslant n_0$$

has no eventually negative solutions.

Theorem 3.7. *Assume that all the conditions of Theorem* 3.3 *hold. Then*

(i) *the advanced difference inequality*

$$\nabla x(n) - \sum_{i=1}^{m} f_i(n, x(\sigma_i(n))) \geqslant 0 \quad \forall\, n \geqslant n_0$$

has no eventually positive solutions;

(ii) *the advanced difference inequality*

$$\nabla x(n) - \sum_{i=1}^{m} f_i(n, x(\sigma_i(n))) \leqslant 0 \quad \forall\, n \geqslant n_0$$

has no eventually negative solutions.

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