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Abstract. Let R be a ring with identity and M be a unitary left R-module. The co-intersection graph of proper submodules of M, denoted by $\Omega(M)$, is an undirected simple graph whose vertex set $V(\Omega)$ is a set of all nontrivial submodules of M and two distinct vertices N and K are adjacent if and only if $N+K\neq M$. We study the connectivity, the core and the clique number of $\Omega(M)$. Also, we provide some conditions on the module M, under which the clique number of $\Omega(M)$ is infinite and $\Omega(M)$ is a planar graph. Moreover, we give several examples for which n the graph $\Omega(\mathbb{Z}_n)$ is connected, bipartite and planar.

Keywords: co-intersection graph; core; clique number; planarity

Classification: 05C15, 05C25, 05C69, 16D10

1. Introduction

The concept of the intersection graph of algebraic structures, first introduced in [7] by J. Bosak, was defined for the intersection graph of proper subsemigroups of a semigroup in 1964. Inspired by his work, many mathematicians have been attracted to this topic and considered the intersection graph of various algebraic structures. The intersection graph related to the subspaces of a finite dimensional vector space over a finite field and graphs associated with the group and ring structures have been studied extensively by several authors, for example see [1], [8], [11], [12], [13], [14], [17], [18] and [20]. Recently various constructions of intersection graphs associated with the module structure are found in [2], [3], [4], [15] and [19]. The idea of studying the co-intersection graph of submodules of a module first appeared in [15] as dual graph of the intersection graph of submodules of a module in [2]. In this paper, our main goal is to study some results on the co-intersection graph of submodules of a module. From an algebraic point of view, the goal of such an endeavor is to determine what algebraic information can be gleaned from analyzing the associated graph. In this paper, we intend to investigate the interplay between combinatorial properties of the co-intersection graph of submodules of a module and algebraic properties of the module. Throughout this paper all rings are commutative with identity, unless otherwise specified and all modules are unitary. Let R be a ring, the term R-module, will always signify

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a left R-module. Let M be an R-module. By a nontrivial submodule of M, we mean a nonzero proper left submodule of M. The co-intersection graph of submodules of M, denoted by $\Omega(M)$, is the undirected simple graph whose vertices are in one to one correspondence with all nontrivial submodules of M and two distinct vertices are adjacent if and only if the sum of corresponding submodules of M is not equal to M. For a ring R, $\Omega(R)$ is the co-intersection graph of ideals of R, where R is regarded as a left R-module. A submodule N of an R-module M is called *superfluous* or *small* in M (we write $N \ll M$), if $N + L \neq M$ for every proper submodule L of M. A nonzero R-module M is called *hollow*, if every proper submodule of M is small in M. An R-module M is called *uniserial*, if any two submodules are comparable. The heart of M is defined as the intersection of all nontrivial submodules of M and is denoted by $H(M)$. If M is simple, then we put $H(M) = M$. Clearly, when $H(M) \neq (0)$, $H(M)$ can be generated by any of its nonzero elements. A nonzero R-module M is called *local*, if it has a unique maximal submodule that contains all other proper submodules. A module M is called *coatomic*, if every proper submodule of M is contained in a maximal submodule of M. The module M is called *semisimple*, if it is a direct sum of simple submodules. An R-module M is called *co-semisimple*, if every proper submodule of M is the intersection of maximal submodules. It is well-known that co-semisimple modules and finitely generated modules are coatomic. For an R-module M, the *length* of M is the length of composition series of M, denoted by $l_R(M)$. An R-module M has finite length, if $l_R(M) < \infty$, i.e., M is Noetherian and Artinian. The ring of all endomorphisms of an R-module M is denoted by $\mathrm{End}_R(M)$. The *radical* of an R-module M, denoted by $\mathrm{Rad}(M)$, is the intersection of all maximal submodules of M. The *socle* of an R-module M, denoted by $\mathrm{Soc}(M)$, is the sum of all simple submodules of M. We use the notations $\mathrm{Max}(M)$ and $\mathrm{Min}(M)$ to denote the set of all maximal submodules and the set of all minimal submodules of M, respectively. By $\mathrm{ann}(M)$, we mean the set of all elements $r \in R$, with the property that $rx = 0$ for every $x \in M$. An R-module M is called *faithful*, if $\mathrm{ann}(M) = (0)$.

Let Ω be a graph. By order of Ω, we mean the number of vertices of Ω and we denoted it by $|\Omega|$. A vertex u is called *universal*, if it is adjacent to all other vertices. A vertex v is called *isolated*, if $\deg(v) = 0$. A vertex w is called *end vertex*, if $\deg(w) = 1$. A path with n vertices is denoted by P_n. A cycle of n vertices is denoted by C_n and is called an n-cycle. The *core* of a graph Ω is the subgraph induced on all vertices of cycles of Ω, i.e., the union of the cycles in Ω. A graph is said to be *null*, if it has no edge. A graph is said to be *disconnected*, if it is not connected. A *star graph* is a tree consisting of one universal vertex. Graph Ω is said to be *r-regular*, if $\deg(v) = r$ for any vertex v in Ω. A complete graph of order n is denoted by K_n. A *complete bipartite graph* with two part sizes m and n is denoted by $K_{m,n}$. The *complement graph* of Ω is denoted by $\overline{\Omega}$. By a *clique* in a graph Ω, we mean a complete subgraph of Ω. The number of vertices in a largest clique of Ω, is called the clique number of Ω and is denoted by $\omega(\Omega)$. For a graph Ω, let $\chi(\Omega)$ denote the *chromatic number* of Ω, i.e., the minimum number of colors which can be assigned to the vertices of Ω such that every two adjacent vertices have different colors. A graph is said to be *planar*, if it has a drawing in a plane without crossings.

2. Main results of $\Omega(M)$

Let R be a ring and M be an R-module. In this section, we show that, if $\Omega(M)$ contains an edge, then $\Omega(M)$ is a connected graph and $l_R(M) \geq 3$. Also, if $\Omega(M)$ is a k-regular connected graph, where $k > 0$, then $\Omega(M)$ is complete. We prove that, if M is a finitely generated R-module and $\Omega(M)$ is a connected graph which contains a cycle, then the core of $\Omega(M)$ is a union of 3-cycles and also every vertex of $\Omega(M)$ is either an end vertex or a vertex of the core. Moreover, it is proved that, if $\omega(\Omega(M)) = \infty$, then $\Omega(M)$ contains an infinite clique. We determine some conditions on the module M, under which $\Omega(M)$ is a planar graph.

A fundamental theorem about the connectivity of the co-intersection graph of submodules of a module was proved in [15] and says that for an R-module M, the co-intersection graph $\Omega(M)$ is disconnected if and only if M is a direct sum of two simple R-modules. The following corollaries are immediate consequences of this theorem.

Corollary 2.1. *Let M be an R-module. If $\Omega(M)$ contains an edge, then $\Omega(M)$ is a connected graph and $l_R(M) \geq 3$.*

Proof: On the contrary, suppose that $\Omega(M)$ is disconnected or $l_R(M) \leq 2$. If $\Omega(M)$ is disconnected, then by [15, Theorem 2.1], any nontrivial submodule of M is simple. Hence, $\Omega(M)$ is a null graph, a contradiction. Now, if $l_R(M) = 1$, then M is simple and $\Omega(M)$ is empty, a contradiction. Also, if $l_R(M) = 2$, then any nontrivial submodule of M is minimal and maximal, and so $\Omega(M)$ is connected, which contradicts part 2 of Corollary 2.3 of [15]. □

Corollary 2.2. *Let M be an R-module and $\Omega(M)$ contains an edge. Then the number of edges of $\Omega(M)$ is finite if and only if $\Omega(M)$ is finite.*

Proof: Suppose that $\Omega(M)$ has finitely many edges. Then by Corollary 2.1, $\Omega(M)$ is connected. Since every edge determines two vertices of this graph, hence $\Omega(M)$ is finite. The converse is straightforward. □

Lemma 2.3. *Let M be a uniserial R-module. Then $\Omega(M)$ is a complete graph.*

Proof: Suppose that M is a uniserial R-module. Let X and Y be two nontrivial submodules of M. Hence $X \subseteq Y$ or $Y \subseteq X$. This implies that $X + Y \neq M$ and the graph $\Omega(M)$ is complete. □

Theorem 2.4. *Let R be a ring and M be an R-module. Then $\Omega(M)$ is a complete graph, if one of the following conditions holds:*

(1) *if R has the only one left maximal ideal and every finitely generated submodule of M is cyclic;*
(2) *if $\Omega(M)$ is a k-regular connected graph for some $k > 0$.*

Proof: (1) In order to establish this part, we claim that each two distinct submodules of M are comparable. On the contrary, suppose that M_1 and M_2 are two distinct submodules of M such that $M_1 \nsubseteq M_2$ and $M_2 \nsubseteq M_1$. Then there exist $m_1 \in M_1 \backslash M_2$ and $m_2 \in M_2 \backslash M_1$. Since $Rm_1 + Rm_2$ is a finitely generated submodule of M, there is $m \in M$ such that $Rm = Rm_1 + Rm_2$. However, Rm_1 is a proper submodule of Rm, thus there is a maximal submodule N_1 of Rm such that $Rm_1 \subseteq N_1 \subset Rm$. Similarly, there is a maximal submodule N_2 of Rm such that $Rm_2 \subseteq N_2 \subset Rm$. Hence Rm/N_1 and Rm/N_2 are two simple left R-module and since R has only one left maximal ideal I, by Proposition 9.1 of [5, page 116], $Rm/N_1 \cong R/I$ and $Rm/N_2 \cong R/I$. So $Rm/N_1 \cong Rm/N_2$, thus $N_1 = N_2$. Now, we have $Rm = Rm_1 + Rm_2 \subseteq N_1 \subset Rm$, which is a contradiction. Consequently, M_1 and M_2 are comparable and M is a uniserial R-module. Thus by Lemma 2.3, $\Omega(M)$ is a complete graph.

(2) Let N be a nontrivial submodule of M. Since $\Omega(M)$ is a k-regular graph, $\deg(N) = k < \infty$ and by [15, Lemma 3.4], $l_R(M) < \infty$. Hence, M is Noetherian. On the contrary, suppose that $\Omega(M)$ is not complete. Then by [15, Theorem 2.9], M has at least two maximal submodules. Assume that M_1 and M_2 be two maximal submodules of M. Since $\Omega(M)$ is connected, by [15, Theorem 2.5], $\mathrm{diam}(\Omega(M)) \leq 3$ and since M_1 and M_2 are not adjacent vertices in $\Omega(M)$, then there exists at least a vertex X in $\Omega(M)$ such that $M_1 - X - M_2$ is a path in $\Omega(M)$. Since $M_i \subseteq M_i + X \neq M$ for $i = 1, 2$, the maximality of M_i implies that $X \subseteq M_i$. Hence for every vertex Y of $\Omega(M)$, if $M_1 + Y \neq M$, then $X + Y \neq M$. Therefore, $\deg(M_1) < \deg(X)$ and this is a contradiction. Consequently, $\Omega(M)$ is a complete graph. □

Proposition 2.5. *Let M be an R-module and $\Omega(M)$ be a connected graph. If M has at least three minimal submodules, then $\Omega(M)$ is not bipartite graph.*

Proof: Suppose M_1, M_2 and M_3 are three minimal submodules of M and $\Omega(M)$ is a connected graph. Then by part 2 of [15, Corollary 2.3], (M_1, M_2, M_3) is a 3-cycle of $\Omega(M)$. We know that a simple graph is bipartite if and only if it has no odd cycle. Hence, $\Omega(M)$ is not bipartite graph. □

Theorem 2.6. *Let M be a finitely generated R-module and $\Omega(M)$ be a connected graph which contains a cycle. Then the following statements hold.*

(1) *The core of $\Omega(M)$ is a union of 3-cycles.*
(2) *Every vertex of $\Omega(M)$ is either an end vertex or a vertex of the core.*

Proof: (1) Suppose that $M_1 - M_2 - \cdots - M_n - M_1$ is a cycle. We claim that each edge of this cycle is an edge of a 3-cycle. By symmetric property of cycle, it is enough to prove that $M_1 - M_2$ is an edge of a 3-cycle. First, we can assume that $n \geq 4$, such that $M_1 + M_3 = M = M_2 + M_n = M$, otherwise we have 3-cycle $M_1 - M_2 - M_3 - M_1$ or $M_1 - M_2 - M_n - M_1$. Then $M_1 \nsubseteq M_2$, nor $M_2 \nsubseteq M_1$, which follows from the observation $M_1 \subseteq M_2$ and $M_1 + M_3 = M \Rightarrow M_2 + M_3 = M$, a contradiction. Hence $M_1 - M_2 - M_1 + M_2 - M_1$ is a 3-cycle. Therefore, the core of the graph $\Omega(M)$ is a union of 3-cycles.

(2) We should prove that if X is not a vertex in any cycle, then X is an end vertex. Since $\Omega(M)$ contains a cycle, the order of $\Omega(M)$ is at least 3. We claim that there is only one edge adjacent to X. On the contrary, then there is a path $L - X - N$. Let $L + N = K$. If $K \neq M$, then $L - X - N - L$ is a cycle, a contradiction. Also, if $K = M$, then there exist two maximal submodules $X^\star$ and $N^\star$ such $X \subseteq X^\star$ and $N \subseteq N^\star$. Hence $X + X^\star \neq M$ and $N + N^\star \neq M$. Now, if $X^\star \cap N^\star = (0)$, then $M = X^\star \oplus N^\star$ and $X^\star \cong M/N^\star$ and thus $X^\star$ is simple and similarly, $N^\star$ is simple. Hence, by [15, Theorem 2.1], $\Omega(M)$ is not connected, a contradiction. So $X^\star \cap N^\star \neq (0)$ and $X - X^\star - X^\star \cap N^\star - N^\star - N - X$ is another cycle, a contradiction. Therefore, X is an end vertex and the proof is complete. □

Corollary 2.7. *Let M be an R-module and $N \ll M$. If $\Omega(M)$ is a connected graph which contains a cycle, then N is a vertex of the core not an end vertex of $\Omega(M)$.*

Example 2.8. Suppose that p and q are two distinct primes. We consider $\mathbb{Z}_{pq^2}$ as $\mathbb{Z}_{pq^2}$-module. The nontrivial submodules of $\mathbb{Z}_{pq^2}$ are $\langle p\rangle$, $\langle q\rangle$, $\langle q^2\rangle$ and $\langle pq\rangle$ such that $\langle pq\rangle$ is the only nontrivial small submodule of $\mathbb{Z}_{pq^2}$ and a vertex of the core of the graph $\Omega(\mathbb{Z}_{pq^2})$ and also $\langle p\rangle$ is an end vertex of this graph.

Example 2.9. Consider $\mathbb{Z}_{pqr}$ as $\mathbb{Z}$-module, where p, q and r are three distinct primes. We know $\langle p\rangle = p\mathbb{Z}_{pqr}$, $\langle q\rangle = q\mathbb{Z}_{pqr}$ and $\langle r\rangle = r\mathbb{Z}_{pqr}$ are the only maximal submodules of $\mathbb{Z}_{pqr}$. Also, $\langle pq\rangle = pq\mathbb{Z}_{pqr}$, $\langle pr\rangle = pr\mathbb{Z}_{pqr}$ and $\langle qr\rangle = qr\mathbb{Z}_{pqr}$ are the other submodules and $\mathbb{Z}_{pqr} = \langle pq\rangle \oplus \langle qr\rangle \oplus \langle pr\rangle$ is semisimple and finitely generated. Hence, $\Omega(\mathbb{Z}_{pqr})$ is a connected graph and any of its vertices is a vertex of the core.

In the following theorem, we provide the condition under which $\omega(\Omega(R))$ is finite if $\omega(\Omega(M))$ is finite.

Theorem 2.10. *Let M be a faithful R-module with the graph $\Omega(M)$ and $\omega(\Omega(M)) < \infty$. If $\Omega(M)$ is null, then $\omega(\Omega(R)) < \infty$.*

Proof: Assume that M is faithful and $\Omega(M)$ is null. Then by part 2 of [15, Lemma 3.1], either $|\Omega(M)| = 1$ or $|\Omega(M)| \geq 2$ and M is a direct sum of two simple R-modules. Let $|\Omega(M)| = 1$. Then M has a unique minimal and maximal submodule. Thus M is cyclic. Therefore, $M \cong R$ and so $\omega(\Omega(R)) < \infty$. Now, suppose that $|\Omega(M)| \geq 2$ and $M = M_1 \oplus M_2$, where M_1 and M_2 are simple R-modules. We consider two possible cases.
Case 1. If $M_1 \cong M_2$, then $\operatorname{ann}(M_1) = \operatorname{ann}(M_2)$. Since M is faithful, $\operatorname{ann}(M) = \operatorname{ann}(M_1) \cap \operatorname{ann}(M_2) = \operatorname{ann}(M_1) = (0)$. As M_1 is simple and cyclic, we have $M_1 \cong R/\operatorname{ann}(M_1) \cong R$. Thus R is a field and so $\omega(\Omega(R)) = 0$.
Case 2. If $M_1 \ncong M_2$, then $\operatorname{ann}(M_1) \ncong \operatorname{ann}(M_2)$. Clearly, $\operatorname{ann}(M_1)$ and $\operatorname{ann}(M_2)$ are maximal ideals of R and $R = \operatorname{ann}(M_1) + \operatorname{ann}(M_2)$. Now, by Chinese remainder theorem, we have $R \cong R/\operatorname{ann}(M_1) \oplus R/\operatorname{ann}(M_2) \cong M_1 \oplus M_2 = M$. Therefore $\Omega(R)$ is finite and we are done. □

In [15], it was proved that, if $1 < \omega(\Omega(M)) < \infty$, then $|\operatorname{Min}(M)| < \infty$ and $|\operatorname{Max}(M)| = \infty$. Now, we prove that if $\omega(\Omega(M))$ is infinite, then there is an infinite clique in $\Omega(M)$.

Theorem 2.11. *Let M be an R-module. If $\omega(\Omega(M)) = \infty$, then $\Omega(M)$ contains an infinite clique.*

Proof: First assume that $l_R(M)$ is infinite. Then M contains an infinitely increasing or decreasing chain of submodules and the assertion holds. Hence, we assume that $l_R(M) < \infty$. Now, since M is Noetherian, it possesses at least one maximal submodule. Moreover, every nonzero submodule of M is contained in a maximal submodule. As the sum of every pair of maximal submodules is equal to M, our assumption $\omega(\Omega(M)) = \infty$ implies that the number of non-maximal submodules of M is infinite. Now, if the number of maximal submodules is finite, then there exists a maximal submodule, say U, which contains infinitely many submodules. These submodules, induce an infinite clique in $\Omega(M)$, as desired. If the number of maximal submodules is infinite, then we define $T_n = \{X \lneq M\colon l_R(M/X) = n\}$ and $n_0 = \max\{n\colon \operatorname{Card}(T_n) = \infty\}$. Since $T_1 = \{X \lneq M\colon l_R(M/X) = 1\}$, then M/X is a simple R-module, thus X is a maximal submodule of M. Hence, $T_1 = \{X \lneq M\colon X \leq^{\max} M\}$ is infinite and clearly, $1 \leq n_0 < l_R(M)$. However, since $l_R(M) < \infty$, Theorem 5 of [16, page 19] implies that every proper submodule of length n_0 is contained in a submodule of length $n_0 + 1$. Moreover, by the definition of n_0, the number of submodules of length $n_0 + 1$ is finite. Hence there exists a submodule N of M such that $l_R(M/N) = n_0+1$ and N is contained in an infinite number of submodules $\{N_i\}_{i\in I}$ of M, where $l_R(M/N_i) = n_0$ for all $i \in I$. Now, if the sum of every pair of these submodules containing N is not equal to M, then we obtain an infinite clique. Otherwise, assume that there exist submodules K and L of M, with $N \subseteq K, L$ such that $K + L = M$ and $l_R(M/K) = l_R(M/L) = n_0$. Since by [9, Corollary 1.30], $M/(K \cap L) \cong M/K \oplus M/L$, $n_0 + 1 = l_R(M/N) \geq l_R(M/(K \cap L)) = l_R(M/K \oplus M/L) = l_R(M/K) + l_R(M/L) = 2n_0$ and so $n_0 = 1$. Therefore, the number of non-maximal submodules is finite, a contradiction. Consequently, the sum of every pair of non-maximal submodules of M of length n_0, which are containing N is not equal to M and this completes the proof. $\square$

Corollary 2.12. *Let M be an R-module and $|\Omega(M)| = \infty$. Then there is an infinite clique in $\Omega(M)$, if one of the following holds.*

1. *The module M is uniserial.*
2. *The ring R is local and such that either the set of all submodules of M is totally ordered by inclusion or every 2-generated submodule is cyclic.*
3. *The module M is hollow or local.*
4. *The module M is self-projective and $\operatorname{End}_R(M)$ is a local ring.*

Proof: (1) Use Lemma 2.3.

(2) Since R is a local ring, R has the only one left maximal ideal and since the set of all submodules of M is totally ordered by inclusion or every 2-generated

submodule is cyclic, by Exercise 9 of [10, page 83], every finitely generated submodule of M is cyclic and the rest follows from part 1 of Theorem 2.4.

(3) Use [15, Proposition 2.11] and part 2 of [15, Corollary 2.16].

(4) Use part 3 of [15, Corollary 2.16]. $\square$

Example 2.13. (1) For every prime number p, we consider the graph $\Omega(\mathbb{Z}_{p^\infty})$. Since the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is hollow, by part 3 of Corollary 2.12, $\Omega(\mathbb{Z}_{p^\infty})$ contains an infinite clique.

(2) Suppose that $R = F[x,y]/(x,y)^2$, where F is an infinite field and x and y are indeterminates. Then $I = \overline{(x,y)}$, $I_x = \overline{(x)}$, $I_y = \overline{(y)}$, and $I_a = \{\overline{(ax+y)}\colon\ 0 \neq a \in F\}$ are all nontrivial ideals of R. Also, I is the only maximal ideal, i.e., $J \subseteq I$ for every proper ideal J of R. Then $I = J(R)$ and since R is a finitely generated, $J(R) \ll R$. Hence $J \ll R$ for every proper ideal J of R and R is hollow. Consequently, $\Omega(R)$ is an infinite complete graph and it has an infinite clique.

Lemma 2.14. *Let M be a non-simple R-module with $H(M) \neq (0)$. Then the following hold.*

(1) *$\Omega(H(M))$ is an empty graph.*
(2) *$H(M)$ is a universal vertex of $\Omega(M)$.*
(3) *There exists at least a path in $\Omega(M)$ such that it passes from $H(M)$.*
(4) *If M is coatomic, then $H(M) - \mathrm{Rad}(M)$ is an edge in $\Omega(M)$.*
(5) *If M is co-semisimple, then $\Omega(\mathrm{Rad}(M))$ is an empty graph.*

Proof: (1) As $H(M)$ is a minimal submodule of M, then $H(M)$ is simple and $\Omega(H(M))$ is empty.

(2) As $H(M) \subseteq N$ for any submodule N of M, then $H(M) + N = N \neq M$ and thus $H(M)$ is a universal vertex of $\Omega(M)$.

(3) As $H(M) \subseteq \mathrm{Rad}(M)$ and $H(M) \subseteq \mathrm{Soc}(M)$, then $\mathrm{Rad}(M) - H(M) - \mathrm{Soc}(M)$ is a path in $\Omega(M)$.

(4) It is clear.

(5) Since M is co-semisimple, $H(M) = \mathrm{Rad}(M)$ and it follows from part 1. $\square$

Now, we consider the conditions under which the graph $\Omega(M)$ is planar. Finally, we give several examples about the connectivity, the planarity and the bipartition of $\Omega(\mathbb{Z}_n)$, where n is an integer number greater than one except primes. In order to study the planarity of this graph, we state a celebrated theorem due to Kuratowski.

Theorem 2.15 ([6, Theorem 10.30])**.** *A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.*

Example 2.16. For every prime number p, we have:

(1) The number of all nontrivial submodules of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ as $\mathbb{Z}$-module is $p+1$, all are maximal and minimal submodules of order p and are isolated vertices of the graph $\Omega(\mathbb{Z}_p \oplus \mathbb{Z}_p)$. Consequently, $\Omega(\mathbb{Z}_p \oplus \mathbb{Z}_p) \cong \overline{K_{p+1}}$ is planar and bipartite graph.

(2) The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is hollow, then by [15, Proposition 2.11], $\Omega(\mathbb{Z}_{p^\infty})$ is complete and it is not planar and bipartite graph.

Lemma 2.17. *Let M be an R-module. Then the following hold.*

(1) *If $\Omega(M)$ is connected and planar graph, then $|\operatorname{Min}(M)| \leq 3$.*

(2) *If $M = \bigoplus_{i=1}^{n} M_i$ is a direct sum of n non-isomorphic simple R-modules, then for $n \leq 3$, $\Omega(M)$ is planar and for $n \geq 4$, $\Omega(M)$ is not planar.*

(3) *If M is coatomic and $\overline{\Omega(M)}$ is a planar graph, then $|\operatorname{Max}(M)| \leq 3$.*

(4) *If M is co-semisimple or finitely generated and $\overline{\Omega(M)}$ is a planar graph, then $|\operatorname{Max}(M)| \leq 3$.*

Proof: (1) On the contrary, suppose that $|\operatorname{Min}(M)| \geq 4$. If $|\operatorname{Min}(M)| \geq 5$, then $\Omega(M)$ contains a subgraph of the complete graph K_5 and if $|\operatorname{Min}(M)| = 4$, we can see easily that $\Omega(M)$ contains a subdivision of the complete bipartite graph $K_{3,3}$. Hence, by Theorem 2.15, $\Omega(M)$ is not planar, a contradiction.

(2) For $n \leq 3$ it is trivial and for $n \geq 4$ it follows from part 1.

(3) On the contrary, assume that $|\operatorname{Max}(M)| \geq 4$. If $|\operatorname{Max}(M)| \geq 5$, then $\overline{\Omega(M)}$ contains a subgraph of the complete graph K_5 and if $|\operatorname{Max}(M)| = 4$, we can see easily that $\overline{\Omega(M)}$ contains a subdivision of the complete bipartite graph $K_{3,3}$. Hence, by Theorem 2.15, $\overline{\Omega(M)}$ is not planar, a contradiction.

(4) As co-semisimple or finitely generated modules are coatomic, it follows from part 3. □

Lemma 2.18. *Let M be an R-module. Then the following hold.*

(1) *If $\Omega(M)$ is planar, then any chain of nontrivial proper submodules of M has length at most five.*

(2) *If M has a unique minimal submodule L such that every nontrivial submodule containing L is maximal submodule of M and $l_R(M) = 3$, then $\Omega(M)$ is planar.*

Proof: (1) Let $M_1 \subset M_2 \subset \cdots \subset M_5$ be a chain of nontrivial proper submodules of M. Since $M_i + M_j = M_j \neq M$ for $i < j$ and $1 \leq i, j \leq 5$, then the set $\{M_i \colon 1 \leq i \leq 5\}$ induces a complete subgraph K_5 in $\Omega(M)$. Hence, by Theorem 2.15, $\Omega(M)$ is not planar, a contradiction.

(2) Suppose that $l_R(M) = 3$ and M has a unique minimal submodule L such that for each $i \in I$, the nontrivial submodule L_i of M containing L is a maximal submodule. Then, $(0) \subsetneqq L \subsetneqq L_i \subsetneqq M$ for all $i \in I$, are composition series of M with length 3 such that $L_i + L = L_i \neq M$ and $L_i + L_j = M$ for $i \neq j$. Hence, $\Omega(M)$ is a star graph. Therefore, by Theorem 2.15, $\Omega(M)$ is planar. □

Example 2.19. Let n be an integer number greater than one except primes. Then the following hold.

(1) The graph $\Omega(\mathbb{Z}_n)$ is disconnected if and only if $n = pq$, where p and q are two distinct primes.

(2) The graph $\Omega(\mathbb{Z}_n)$ is complete if and only if $n = p^k$, where p is prime and $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 2$.

Example 2.20. (1) If $n = \prod_{i=1}^{m} p_i^{k_i}$, where p_i is prime and $k_i \geq 1$, then $|\Omega(\mathbb{Z}_n)| = \prod_{i=1}^{m}(k_i + 1) - 2$.

(2) The graph $\Omega(\mathbb{Z}_n)$ has a cycle if and only if $n = km$, where k is a positive integer and m is one of the forms: p^4, p^2q or pqr, where p, q and r are distinct primes.

Example 2.21. (1) The graph $\Omega(\mathbb{Z}_n)$ does not contain a cycle if and only if $n = pq$, p^2 or p^3 such that p and q are two distinct primes. In all other cases, it contains a 3-cycle.

(2) The graph $\Omega(\mathbb{Z}_n)$ is bipartite if and only if $n = pq$ or p^3, where p and q are two distinct primes.

Let p, q, r and s be distinct primes. The graphs $\Omega(\mathbb{Z}_{p^6})$, $\Omega(\mathbb{Z}_{p^2q^2})$, $\Omega(\mathbb{Z}_{p^3q})$, $\Omega(\mathbb{Z}_{p^2qr})$ and $\Omega(\mathbb{Z}_{pqrs})$ are not planar. As in Figure 1, it is shown that $\Omega(\mathbb{Z}_{p^6})$ is isomorphic to K_5 and also the subgraphs Ω_1, Ω_2, Ω_3 and Ω_4 are contained in graphs $\Omega(\mathbb{Z}_{p^2q^2})$, $\Omega(\mathbb{Z}_{p^3q})$, $\Omega(\mathbb{Z}_{p^2qr})$ and $\Omega(\mathbb{Z}_{pqrs})$, respectively. Clearly, all of this subgraphs are isomorphic to K_5 and have 3-cycle, hence they are not planar and bipartite.

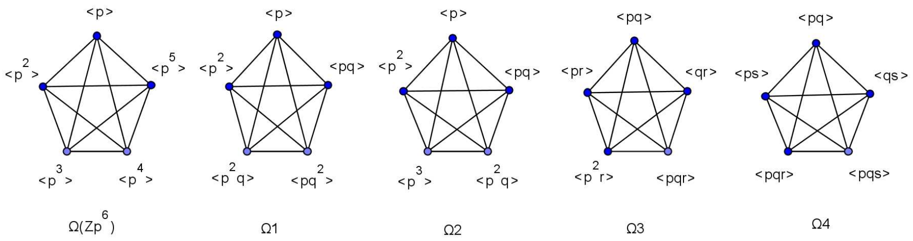

Figure 1.

Example 2.22. Let n be an integer number greater than one except primes. Then the graph $\Omega(\mathbb{Z}_n)$ is planar if and only if n is one of the forms: p^i, $2 \leq i \leq 5$, pq, p^2q, pqr, where p, q and r are distinct primes.

Example 2.23. Let n be an integer number greater than one except primes and p, q, r and s are distinct primes and k is a positive integer. Then the following hold:

(1) the graph $\Omega(\mathbb{Z}_n)$ is not planar if and only if $n = km$, where m is one of the forms: p^i, $i \geq 6$, p^2q, pqr, p^3q, pq or $pqrs$;

(2) the graph $\Omega(\mathbb{Z}_n)$ is not bipartite if and only if $n = km$, where m is one of the forms: p^i, $i \geq 4$, p^2q^2, p^3q or $pqrs$.

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