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STRONG ENDOMORPHISM KERNEL PROPERTY FOR MONOUNARY ALGEBRAS

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Abstract. All monounary algebras which have strong endomorphism kernel property are described.

Keywords: (strong) endomorphism; congruence; kernel; connected monounary algebra; cycle

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1. Introduction

Let $\mathcal{A}$ be an algebra. An endomorphism φ of $\mathcal{A}$ is strong if φ is compatible with all congruences on the algebra $\mathcal{A}$. If every congruence on $\mathcal{A}$ is a kernel of some strong endomorphism of $\mathcal{A}$, then we say that the algebra $\mathcal{A}$ has a strong endomorphism kernel property. We will write that $\mathcal{A}$ has SEKP for short.

There are many papers dealing with this property, e.g. for semilattices in [4], distributive lattices in [6], p-algebras in [1], [3], [6], Ockham algebras in [2] and Stone algebras in [5]. The definition of SEKP omits the universal congruence in these papers. We will denote this weaker property by wSEKP and we will handle it in the last section.

All monounary algebras with SEKP are described in Theorem 4.1 and all monounary algebras with wSEKP are described in Theorem 5.1.

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The set of all positive integers is denoted by $\mathbb{N}$, the set of all nonnegative integers is denoted by $\mathbb{N}_0$. If ψ is a mapping from a set A into a set B, then $\mathrm{Ker}(\psi)$ denotes the kernel of ψ.

Let $\mathcal{A}$ be an algebra. We denote by

- ▷ $\mathbf{I}(\mathcal{A})$ the class of all algebras which are isomorphic to $\mathcal{A}$,
- ▷ $\mathbf{S}(\mathcal{A})$ the class of all algebras which are isomorphic to a subalgebra of $\mathcal{A}$,
- ▷ $\mathbf{H}(\mathcal{A})$ the class of all algebras which are isomorphic to a homomorphic image of $\mathcal{A}$,
- ▷ $\mathrm{End}(\mathcal{A})$ the set of all endomorphisms of $\mathcal{A}$,
- ▷ $\mathrm{Con}(\mathcal{A})$ the set of all congruences on $\mathcal{A}$.

Let $\mathcal{A}$, $\mathcal{B}$ be algebras of the same type. We will repeatedly use the following fact.

Lemma 2.1. *Let ψ be a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then the following statements are equivalent:*

(1) *there exists $\varphi \in \mathrm{End}(\mathcal{A})$ such that $\mathrm{Ker}(\varphi) = \mathrm{Ker}(\psi)$;*
(2) $\mathcal{B} \in \mathbf{S}(\mathcal{A})$.

Proof. Let (1) hold. Then $\varphi(\mathcal{A})$ is a subalgebra of $\mathcal{A}$ and it is isomorphic to $\mathcal{B}$.

Suppose that (2) is fulfilled. Let $\mathcal{D}$ be a subalgebra of $\mathcal{A}$ such that Ψ is an isomorphism from the algebra $\mathcal{B}$ onto $\mathcal{D}$. Take $\varphi(a) = \Psi(\psi(a))$ for every $a \in A$. □

We deal with monounary algebras. The fundamental operation is denoted by f.

For monounary terminology see, e.g. [7], [8]. One-element monounary algebra will be called *trivial*.

Let $\mathcal{A} = (A, f)$ be a monounary algebra. If $b \in A$, then we denote

$$\begin{aligned} f^{-1}(b) &= \{d \in A\colon\ f(d) = b\},\\ {\downarrow} b &= \{d \in A\colon\ f^k(d) = b \ \text{ for some } k \in \mathbb{N}_0\},\\ {\uparrow} b &= \{f^k(b), k \in \mathbb{N}\}. \end{aligned}$$

Let $(A_i, i \in I)$ be a partition of the set A. We will write

$$\tau = [A_i]_{i \in I}$$

if τ is an equivalence relation on A determined by $(A_i, i \in I)$. We will write $\tau = [A_1][A_2]$ for $I = \{1, 2\}$ and we will use the notation $\tau = [A_i][A_j]_{j \in J}$ if $I = J \cup \{i\}$. Let $\iota_A = [A]$ be the *universal congruence* on $\mathcal{A}$.

Let $\varphi \in \mathrm{End}(\mathcal{A})$. We will say that φ is *strong* if it is compatible with all congruences of $\mathcal{A}$, i.e. if $\theta \in \mathrm{Con}(\mathcal{A})$ and $(a, b) \in \theta$, then $(\varphi(a), \varphi(b)) \in \theta$.

E x a m p l e 2.1. The identity mapping is a strong endomorphism of $\mathcal{A}$.

If $\mathcal{A}$ contains a one-element cycle $\{a\}$, then the constant mapping $\varphi(x) = a$ for each $x \in A$ is a strong endomorphism of $\mathcal{A}$.

Lemma 2.2. *Let* $\{a\}$, $\{b\}$, $\{c\}$ *be distinct one-element cycles of* $\mathcal{A}$. *Let* $\varphi \in \mathrm{End}(\mathcal{A})$ *be such that* $\mathrm{Ker}(\varphi) = [{\downarrow}a \cup {\downarrow}b][A - ({\downarrow}a \cup {\downarrow}b)]$. *Then* φ *is not strong.*

P r o o f. We have that there exist $u, v \in A$ such that $\varphi(A) = \{u, v\}$, $u \neq v$ and $f(u) = u$, $f(v) = v$, because u, v are homomorphic images of one-element cycles. Suppose that $\varphi(a) = u$. Then $\varphi(b) = u$ and $\varphi(c) = v$. We will analyse all possible cases.

Assume that $\{u, v\} = \{a, b\}$. Take

$$\theta = [{\downarrow}a \cup {\downarrow}c][A - ({\downarrow}a \cup {\downarrow}c)].$$

Then $\theta \in \mathrm{Con}(\mathcal{A})$. We have $(a, c) \in \theta$. Further, $(u, v) \notin \theta$ according to $(a, b) \notin \theta$. Conclude φ does not preserve θ.

Assume that $u, v \notin \{a, b\}$. Then $\mathcal{A}$ has at least four one-element cycles. Take

$$\theta = [{\downarrow}a \cup {\downarrow}u][A - ({\downarrow}a \cup {\downarrow}u)].$$

We have $\theta \in \mathrm{Con}(\mathcal{A})$. As $u \notin \{a, b\}$ and u is a one-element cycle, $u \in A - ({\downarrow}a \cup {\downarrow}b)$ and therefore $\varphi(u) = v$. The endomorphism φ does not preserve θ, since $(a, u) \in \theta$ and $(\varphi(a), \varphi(u)) = (u, v) \notin \theta$.

Finally, assume that $u \notin \{a, b\}$ and $v = a$. Then $a \notin {\downarrow}u \cup {\downarrow}b$. Take

$$\theta = [{\downarrow}u \cup {\downarrow}b][A - ({\downarrow}u \cup {\downarrow}b)].$$

We have $\theta \in \mathrm{Con}(\mathcal{A})$, $(u, b) \in \theta$ and $(\varphi(u), \varphi(b)) = (v, u) = (a, u) \notin \theta$.

Cases

(1) $u \notin \{a, b\}$, $v = b$,
(2) $u = a$, $v \notin \{a, b\}$,
(3) $u = b$, $v \notin \{a, b\}$

can be proved analogously. □

We say that an algebra $\mathcal{A}$ has a *strong endomorphism kernel property* if every congruence relation on $\mathcal{A}$ is a kernel of some strong endomorphism of $\mathcal{A}$, i.e.

$$\mathrm{Con}(\mathcal{A}) = \{\mathrm{Ker}(\varphi)\colon \varphi \text{ is a strong endomorphism of } \mathcal{A}\}.$$

We will write that $\mathcal{A}$ has SEKP for short.

We say that an algebra $\mathcal{A}$ has a *weaker strong endomorphism kernel property* if every congruence relation on $\mathcal{A}$ different from the universal congruence ι_A is a kernel of some strong endomorphism of $\mathcal{A}$, i.e.

$$\mathrm{Con}(\mathcal{A}) = \{\iota_{\mathrm{A}}\} \cup \{\mathrm{Ker}(\varphi)\colon\ \varphi \text{ is a strong endomorphism of } \mathcal{A}\}.$$

We will write that $\mathcal{A}$ has wSEKP for short.

Lemma 2.3. *Let $\mathcal{A}$ be a monounary algebra. Then the following statements are equivalent:*

(1) *$\mathcal{A}$ has* SEKP*;*

(2) *$\mathcal{A}$ has* wSEKP *and $\mathcal{A}$ contains a one-element cycle.*

P r o o f. It follows from Lemma 2.1 and Example 2.1. □

Lemma 2.4. *Let $\mathcal{A}$ have* wSEKP.

(a) *Then $\mathcal{A}$ consists of at most two components.*

(b) *If $\mathcal{A}$ is not connected, then every component of $\mathcal{A}$ has a one-element cycle.*

P r o o f. Let $\mathcal{A}$ consist of κ components, $\kappa > 1$. Take $\mathcal{B}$ a κ-element algebra with the identity operation. The algebra $\mathcal{B}$ consists of κ one-element cycles and it is a homomorphic image of $\mathcal{A}$. Let ψ be a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. We have $\mathrm{Ker}(\psi) \neq \iota_{\mathrm{A}}$. Further, condition (1) from Lemma 2.1 is satisfied by wSEKP of $\mathcal{A}$. Therefore $\mathcal{A}$ contains κ one-element cycles according to Lemma 2.1.

Suppose that $\kappa > 2$. Take distinct one-element cycles $\{a\}$, $\{b\}$, $\{c\}$ of $\mathcal{A}$. Then $\theta = [{\downarrow}a \cup {\downarrow}b][A - ({\downarrow}a \cup {\downarrow}b)]$ is a congruence of $\mathcal{A}$. In view of Lemma 2.2 we have that θ is not a kernel of any strong endomorphism of $\mathcal{A}$, a contradiction. Therefore $\kappa = 2$. □

3. Algebras $\mathcal{P}_{k,s}$

Let $k \in \mathbb{N}_0$ and $s \in \mathbb{N}$. Put

$$P_{k,s} = \{-s+1, -s+2, \ldots, k\}.$$

We define algebras

$$\mathcal{P}_{k,s} = (P_{k,s}, f), \quad \mathcal{P}_{\infty,s} = (P_{0,s} \cup \mathbb{N}, f),$$

such that $f(-s+1) = 0$ and $f(i) = i-1$ for every $i \neq -s+1$.

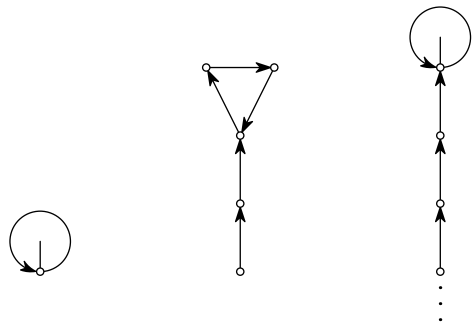

Figure 1.

Lemma 3.1. *Let $\mathcal{A} = (A, f)$ be a connected monounary algebra such that it contains a cycle C. Then the following conditions are equivalent:*

(a) *There exist $k \in \mathbb{N}_0 \cup \{\infty\}$ and $s \in \mathbb{N}$ such that $\mathcal{A}$ is isomorphic to $\mathcal{P}_{k,s}$.*

(b) *For every $a, b \in A$ there is $i \in \mathbb{N}_0$ such that $f^i(a) = b$ or $f^i(b) = a$.*

Lemma 3.2. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $s \in \mathbb{N}$ and $\varphi \in \mathrm{End}(\mathcal{P}_{\mathrm{k,s}})$. Then φ is strong.*

P r o o f. An endomorphism of a monounary algebra maps a cyclic element to a cyclic element.

Let $a \in A$ and $b = \varphi(a)$. Take $i \in \mathbb{N}_0$ from Lemma 3.1 (b).

Assume that a is not cyclic and $k \in \mathbb{N}$ is such that $f^k(a)$ is cyclic and $f^{k-1}(a)$ is not cyclic. Further, let $i \geqslant 1$ and $f^i(\varphi(a)) = a$. If $k < i$, then $\varphi(f^k(a)) \in \uparrow\varphi(a) \setminus \uparrow a$ is not cyclic, a contradiction. If $k \geqslant i$, then $\varphi(f^k(a)) = f^k(\varphi(a)) = f^{k-1}(a)$ is not cyclic, a contradiction.

Therefore $\varphi(a) = f^i(a)$. Then $\varphi(d) = f^i(d)$ for every $d \in \uparrow a$.

Suppose that $\theta \in \mathrm{Con}(\mathcal{A})$ and $(a, a') \in \theta$, $a \neq a'$. Then $a \in \uparrow a'$ or $a' \in \uparrow a$. If $a' \in \uparrow a$, then $(f^i(a), f^i(a')) \in \theta$ and thus $(\varphi(a), \varphi(a')) \in \theta$. $\square$

Lemma 3.3. *The following properties are equivalent:*

(1) *s is a prime;*

(2) *the algebra $\mathcal{P}_{0,s}$ has* wSEKP.

P r o o f. If s is a prime, then $\mathcal{P}_{0,s}$ is simple and every endomorphism of $\mathcal{P}_{0,s}$ is an automorphism.

Suppose that s is not a prime. Take t a nontrivial divisor of s. Then

$$\mathcal{P}_{0,t} \in \mathbf{H}(\mathcal{P}_{0,s}) - \mathbf{S}(\mathcal{P}_{0,s}).$$

Let ψ be a homomorphism from $\mathcal{P}_{0,s}$ onto $\mathcal{P}_{0,t}$. Then

$$\mathrm{Ker}(\psi) \neq \iota_{\mathrm{P}_{0,\mathrm{s}}}.$$

In view of Lemma 2.1 we have $\mathrm{Ker}(\varphi) \neq \mathrm{Ker}(\psi)$ for all $\varphi \in \mathrm{End}(\mathcal{P}_{0,\mathrm{s}})$. $\square$

Lemma 3.4. *Let* $k \in \mathbb{N}_0 \cup \{\infty\}$. *Then*

$$\mathrm{Con}(\mathcal{P}_{\mathrm{k},1}) = \{\mathrm{Ker}(\varphi)\colon\ \varphi \in \mathrm{End}(\mathcal{P}_{\mathrm{k},1})\}.$$

P r o o f. We have

$$\mathbf{H}(\mathcal{P}_{\infty,1}) = \mathbf{I}(\mathcal{P}_{\infty,1}) \cup \mathbf{I}(\mathcal{P}_{0,1}) \subset \mathbf{S}(\mathcal{P}_{\infty,1}).$$

Further, for $k \neq \infty$ we have

$$\mathbf{H}(\mathcal{P}_{k,1}) = \bigcup_{l \leqslant k} \mathbf{I}(\mathcal{P}_{l,1}) = \mathbf{S}(\mathcal{P}_{k,1}).$$

Take $\theta \in \mathrm{Con}(\mathcal{P}_{\mathrm{k},1})$. Then $\mathcal{P}_{k,1}/\theta \in \mathbf{S}(\mathcal{P}_{k,1})$. Therefore there exists $\varphi \in \mathrm{End}(\mathcal{P}_{\mathrm{k},1})$ such that $\mathrm{Ker}(\varphi) = \theta$ according to Lemma 2.1. $\square$

Corollary 3.1. *Let* $k \in \mathbb{N}_0 \cup \{\infty\}$. *The algebra* $\mathcal{P}_{k,1}$ *has* SEKP.

P r o o f. It follows from Lemmas 3.4 and 3.2. $\square$

4. SEKP

Lemma 4.1. *If* $\mathcal{A}$ *has* SEKP, *then* $\mathcal{A}$ *consists of at most two components and every component of* $\mathcal{A}$ *has a one-element cycle.*

P r o o f. It follows from Lemmas 2.3 and 2.4. $\square$

Lemma 4.2. *Let* $\{a\}$, $\{c\}$ *be distinct one-element cycles of* $\mathcal{A}$ *and* $\mathcal{A}$ *consists of two components. Let* $\varphi \in \mathrm{End}(\mathcal{A})$ *be such that*

(1) $(b,d) \in \mathrm{Ker}(\varphi)$ *for some* $b \in f^{-1}(a) - \{a\}$, $d \in f^{-1}(c) - \{c\}$;
(2) $(a,x) \notin \mathrm{Ker}(\varphi)$ *for every* $x \in A - \{a,c\}$.

Then $\mathcal{A}$ *does not have* SEKP.

P r o o f. We have $(a,c)\in \operatorname{Ker}(\varphi)$ according to (1). Thus $\varphi(A)\subseteq {\downarrow}\varphi(a)$ since $\mathcal{A}$ has two components. Further, $\varphi(b)$ is not cyclic by (2). Let x be a non-cyclic and y a cyclic elements of $\mathcal{A}$. Condition (2) yields that $(x,y)\notin \operatorname{Ker}(\varphi)$.

Consider $\theta = [{\downarrow}a][d]_{d\in{\downarrow}c}$. Then $\theta\in \operatorname{Con}(\mathcal{A})$. Suppose that $\psi\in \operatorname{End}(\mathcal{A})$ is such that $\theta = \operatorname{Ker}(\psi)$. We have that $\psi(b)=\psi(a)$ is a cyclic element of $\mathcal{A}$. Moreover, the algebra $(\psi({\downarrow}c), f)$ is isomorphic to the algebra $({\downarrow}c, f)$. Thus, $\psi(d)$ is a non-cyclic element of $\mathcal{A}$. It means that the endomorphism ψ does not preserve $\operatorname{Ker}(\varphi)$ since $(b,d)\in \operatorname{Ker}(\varphi)$ and $(\psi(b),\psi(d))\notin \operatorname{Ker}(\varphi)$. □

Let us remind the notion of *degree* $s(a)$ of an element $a\in A$, cf. [7], page 12.

A subset B of A is called a *chain of* $\mathcal{A}$ if for every $a,b\in B$ there is $n\in\mathbb{N}_0$ such that either $f^n(a)=b$ or $f^n(b)=a$.

Let us denote by $A^{(\infty)}$ the set of all elements $a\in A$ such that the set ${\downarrow}a$ contains a cycle or an infinite chain of $\mathcal{A}$.

Further, let $A^{(0)} = \{a\in A\colon\ f^{-1}(a)=\emptyset\}$. We can now define a set $A^{(\lambda)}\subseteq A$ for each ordinal λ by induction. Assume that we have defined $A^{(\alpha)}$ for each ordinal $\alpha<\lambda$. Then we put

$$A^{(\lambda)} = \Big\{a\in A - \bigcup_{\alpha<\lambda} A^{(\alpha)}\colon\ f^{-1}(a)\subseteq \bigcup_{\alpha<\lambda} A^{(\alpha)}\Big\}.$$

The sets $A^{(\lambda)}$ are pairwise disjoint. For each $a\in A$, either $a\in A^{(\infty)}$ or there is an ordinal λ with $a\in A^{(\lambda)}$. In the former case we put $s(a)=\infty$, in the latter we set $s(a)=\lambda$.

Corollary 4.1. *Let $\mathcal{A}$ consist of two components. If $\mathcal{A}$ has* SEKP*, then there exists $a\in A$ such that $\{a\}$ is a component of $\mathcal{A}$.*

P r o o f. Let $a,c\in A$ be such that $f(a)=a$, $f(c)=c$ and $a\neq c$. Further, let $f^{-1}(a)-\{a\}\neq\emptyset$ and $f^{-1}(c)-\{c\}\neq\emptyset$. Consider

$$M = \{s(b)\colon\ b\in f^{-1}(a)-\{a\}\},$$
$$M' = \{s(d)\colon\ d\in f^{-1}(c)-\{c\}\}.$$

We have that M, M' are sets of ordinal numbers. Suppose that a supremum of M is less or equal than a supremum of M'. Then there exists a homomorphism φ from $({\downarrow}a, f)$ into $({\downarrow}c, f)$ such that $\varphi({\downarrow}a-\{a\})\subseteq {\downarrow}c-\{c\}$ according to Theorem 1.2 of [7]. Put $\varphi(x)=x$ for $x\in{\downarrow}c$. Now we have $\varphi\in \operatorname{End}(\mathcal{A})$. Further, φ has properties (1), (2) from the previous lemma, a contradiction with $\mathcal{A}$ having SEKP. Therefore either $\{a\}$ is a component of $\mathcal{A}$ or $\{c\}$ is a component of $\mathcal{A}$. □

Lemma 4.3. *Let $\mathcal{A} = (A, f)$ and $a \in A$. Suppose that*

(1) *$\mathcal{A}$ is connected,*
(2) *$\{a\}$ is a one-element cycle of $\mathcal{A}$,*
(3) *$A - \{a\} \neq \emptyset$ and the operation f is not injective on $A - \{a\}$.*

Then $\mathcal{A}$ does not have SEKP.

P r o o f. Consider ψ a homomorphism from $\mathcal{A}$ into $\mathcal{P}_{\infty,1}$ such that $\psi(x) \neq 0$ for every $x \in A - \{a\}$. Let θ be a kernel of ψ. Take $b_1, b_2 \in A - \{a\}$ such that $b_1 \neq b_2$, $f(b_1) = f(b_2)$ and $f|(\{f^k(b_1)\colon\ k \in \mathbb{N}\} - \{a\})$ is injective. We have $(b_1, b_2) \in \theta$ and $(a, f^k(b_1)) \notin \theta$ for $f^k(b_1) \neq a$.

Suppose that $\varphi \in \mathrm{End}(\mathcal{A})$ is such that $\mathrm{Ker}(\varphi) = \theta$. Let $\varphi(b_1) \neq b_1$. Denote $D = \{f^k(b_1)\colon\ k \in \mathbb{N}\} \cup {\downarrow} b_1$. We have $a \in D$ and $\varphi(b_1) \notin D$, since φ keeps distances from the cycle. Take $\tau = [D][d]_{d \in A-D}$. Then $\tau \in \mathrm{Con}(\mathcal{A})$ in view of D is a subalgebra of $\mathcal{A}$. We have $(a, b_1) \in \tau$ and $(\varphi(a), \varphi(b_1)) = (a, \varphi(b_1)) \notin \tau$. Therefore φ does not preserve τ.

Let $\varphi(b_1) = b_1$. Then $\varphi(b_2) \neq b_2$, because $\mathrm{Ker}(\varphi) = \theta$ implies that $\varphi(b_1) = \varphi(b_2)$. We can use the previous argument for b_2 instead of b_1. Therefore $\mathcal{A}$ does not have SEKP. $\square$

Corollary 4.2. *Let $\mathcal{B} = (B, f)$ be such that it contains a one-element component $\{c\}$ such that the algebra $\mathcal{A} = (B - \{c\}, f)$ satisfies assumptions of Lemma* 4.3. *Then $\mathcal{B}$ does not have* SEKP.

P r o o f. Put $A = B - \{c\}$. Consider ψ a homomorphism from $\mathcal{B}$ into $\mathcal{P}_{\infty,1}$ such that $\psi(x) \neq 0$ for every $x \in A - \{a\}$. We continue analogously as in the proof of Lemma 4.3. Take $\varphi \in \mathrm{End}(\mathcal{B})$ and $\tau = [D][d]_{d \in B-D}$. $\square$

Theorem 4.1. *Let $\mathcal{A} = (A, f)$ be a monounary algebra. Then $\mathcal{A}$ has* SEKP *if and only if the following three conditions are satisfied:*

(i) *$\mathcal{A}$ contains at most two components,*
(ii) *$\mathcal{A}$ contains at most one nontrivial component,*
(iii) *if $\mathcal{B}$ is a component of $\mathcal{A}$, then $\mathcal{B}$ is isomorphic to $\mathcal{P}_{k,1}$ for some $k \in \mathbb{N}_0 \cup \{\infty\}$.*

P r o o f. Let $\mathcal{A}$ have SEKP. Then $\mathcal{A}$ has wSEKP and (i) is satisfied according to Lemma 2.4 (a). Condition (ii) is valid according to Corollary 4.1. Suppose that $\mathcal{B} = (B, f)$ is a component of $\mathcal{A}$. If $|B| = 1$, then $\mathcal{B} \cong \mathcal{P}_{0,1}$. Let $|B| > 1$. Then $\mathcal{B}$ contains one-element cycle $\{b\}$ by Lemma 2.4, because $\mathcal{A}$ has wSEKP. Further, $\mathcal{B}$ does not satisfy assumptions of Lemma 4.3. That means $f|B - \{b\}$ is injective. Therefore $\mathcal{B} \cong \mathcal{P}_{k,1}$ for some $k \in \mathbb{N} \cup \{\infty\}$ according to Lemma 3.1.

Let $\mathcal{A}$ satisfy (i)–(iii). If $\mathcal{A}$ is connected, then $\mathcal{A}$ has SEKP according to Corollary 3.1.

Let $\mathcal{A} = (A, f)$ be an algebra which consists of two components. Let one of them be $\mathcal{P}_{k,1}$ for some $k \in \mathbb{N}_0 \cup \{\infty\}$ and let the second one be a one-element cycle $\{a\}$.

The equality

$$\{\mathrm{Ker}(\varphi)\colon\ \varphi \in \mathrm{End}(\mathcal{A})\} = \mathrm{Con}(\mathcal{A})$$

follows from Lemma 2.1 since $\mathbf{H}(\mathcal{A}) \subseteq \mathbf{S}(\mathcal{A})$. Let $\varphi \in \mathrm{End}(\mathcal{A})$. We will prove that φ is strong. Assume that $\theta \in \mathrm{Con}(\mathcal{A})$ and $(x, y) \in \theta,\ x \neq y$. We have three possibilities for x, y.

(1) Suppose $x = a$. Then $y \in \mathbb{N}_0$. Consider $0 \leqslant i, j \leqslant y$. Let $k, l \in \mathbb{N}$ be such that $f^k(y) = i$ and $f^l(y) = j$. We obtain $(a, i) = (f^k(a), f^k(y)) \in \theta$. Analogously $(a, j) \in \theta$. Further, $(i, j) \in \theta$ by the transitivity.

(2) Suppose $y = a$. Then by symmetry $(a, x) \in \theta$ and therefore

$$(i, j) \in \theta \quad \text{for } i, j \text{ such that } \ 0 \leqslant i, j \leqslant x.$$

(3) Suppose $a \notin \{x, y\}$. Then $x, y \in \mathbb{N}_0$. If $x < y$, then

$$(i, j) \in \theta \quad \text{for } 0 \leqslant i, j \leqslant y.$$

We will discuss two cases: $\varphi(0) = a$ and $\varphi(0) = 0$.

Let $\varphi(0) = a$. Then $\varphi(i) = a$ for every $i \in P_{k,1}$. If $x = a$, then

$$(\varphi(x), \varphi(y)) = (\varphi(a), a) \in \{(a, 0), (a, a)\} \subset \theta$$

according to (1). Analogous arguments can be used for $y = a$. If $a \notin \{x, y\}$, then $(\varphi(x), \varphi(y)) = (a, a) \in \theta$.

Now assume that $\varphi(0) = 0$. We have $\varphi(x) \leqslant x$ for all $x \in P_{k,1}$. If $x = a$, then $(\varphi(x), \varphi(y)) = (\varphi(a), \varphi(y)) \in \theta$ according to (1), since $\varphi(a) \in \{0, a\}$. Analogous arguments can be used for $y = a$. If $a \notin \{x, y\}$, then $(\varphi(x), \varphi(y)) \in \theta$ according to (3). □

SEKP is for some varieties of universal algebras preserved by finite direct products, cf. [5], Theorem 3.1. By Theorem 4.1 we see that this is not true for any variety of monounary algebras which contains a nontrivial monounary algebra with SEKP.

5. Weaker SEKP

Lemma 5.1. *Let $\mathcal{A}$ be connected without a cycle. Then $\mathcal{A}$ does not have* wSEKP.

Proof. The algebra $\mathcal{P}_{0,2}$ is a homomorphic image of $\mathcal{A}$ and it does not belong to $\mathbf{S}(\mathcal{A})$. The assertion follows from Lemma 2.1. $\square$

Lemma 5.2. *Let $\mathcal{A}$ be connected with a cycle of the length different from* 1*. If $\mathcal{A}$ has* wSEKP*, then the operation of $\mathcal{A}$ is injective.*

Proof. Suppose that the operation of $\mathcal{A}$ is not injective. Then there exists $k \in \mathbb{N} \cup \{\infty\}$ such that $\mathcal{P}_{k,1} \in \mathbf{H}(\mathcal{A})$. Let ψ be a homomorphism from $\mathcal{A}$ onto $\mathcal{P}_{k,1}$. We have $\mathrm{Ker}(\psi) \neq \iota_{\mathrm{A}}$. Further, $\mathcal{P}_{k,1} \notin \mathbf{S}(\mathcal{A})$ since $\mathcal{A}$ contains no one-element cycle. Therefore for every $\varphi \in \mathrm{End}(\mathcal{A})$ we obtain $\mathrm{Ker}(\varphi) \neq \mathrm{Ker}(\psi)$ according to Lemma 2.1. $\square$

Theorem 5.1. *Let $\mathcal{A} = (A, f)$ be a monounary algebra. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ has* wSEKP,

(ii) *$\mathcal{A}$ has* SEKP *or $\mathcal{A}$ is isomorphic to $\mathcal{P}_{0,s}$ for some $s \in \mathbb{N}$, s prime.*

Proof. Suppose that $\mathcal{A}$ has wSEKP. If $\mathcal{A}$ is not connected, then $\mathcal{A}$ has SEKP according to Lemmas 2.4 and 2.3.

Let $\mathcal{A}$ be connected. Then $\mathcal{A}$ has a cycle according to Lemma 5.1. If $\mathcal{A}$ has a one-element cycle, then $\mathcal{A}$ has SEKP according to Lemma 2.3. If the cycle of $\mathcal{A}$ has the length greater than 1, then the operation of $\mathcal{A}$ is injective according to Lemma 5.2. Therefore $\mathcal{A} \cong \mathcal{P}_{0,s}$ for some $s \in \mathbb{N}$. Then s is a prime according to Lemma 3.3.

The second implication follows from Lemma 3.3 and the fact that if $\mathcal{A}$ has SEKP, then it has wSEKP. $\square$

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