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# SOME CONVERGENCE, STABILITY AND DATA DEPENDENCY RESULTS FOR A PICARD-S ITERATION METHOD OF QUASI-STRICTLY CONTRACTIVE OPERATORS

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*Abstract.* We study some qualitative features like convergence, stability and data dependency for Picard-S iteration method of a quasi-strictly contractive operator under weaker conditions imposed on parametric sequences in the mentioned method. We compare the rate of convergence among the Mann, Ishikawa, Noor, normal-S, and Picard-S iteration methods for the quasi-strictly contractive operators. Results reveal that the Picard-S iteration method converges fastest to the fixed point of quasi-strictly contractive operators. Some numerical examples are given to validate the results obtained herein. Our results substantially improve many other results available in the literature.

*Keywords*: iteration method; quasi-strictly contractive operator; convergence; rate of convergence; stability; data dependency

*MSC 2010*: 47H09, 47H10, 54H25

## 1. Introduction

Many problems arising from various branches of science can be modeled by a fixed point equation of the type $Tx = x$, where $T$ is an appropriate operator defined on an ambient space. One can encounter situations where the solution of this equation cannot be obtained analytically. In such a case, fixed point iteration methods play a very important role to locate the fixed point of $T$.

Let $T$ be a self-map of a nonempty closed convex subset $C$ of a real normed space $X$, and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty} \subset [0,1]$ be real sequences satisfying certain control condition(s).

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For arbitrarily chosen $w_0, x_0 \in C$, construct two iterative sequences $\{w_n\}_{n=0}^{\infty}$ and $\{x_n\}_{n=0}^{\infty}$ by

$$
\begin{cases} w_{n+1} = (1-\alpha_n)w_n + \alpha_n T\overline{w}_n, \\ \overline{w}_n = (1-\beta_n)w_n + \beta_n T\varrho_n, \\ \varrho_n = (1-\gamma_n)w_n + \gamma_n T w_n \quad \forall\, n \in \mathbb{N}, \end{cases} \tag{1.1}
$$

and

$$
\begin{cases} x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)Tx_n + \alpha_n T z_n, \\ z_n = (1-\beta_n)x_n + \beta_n T x_n \quad \forall\, n \in \mathbb{N}, \end{cases} \tag{1.2}
$$

where the iteration methods defined by (1.1) and (1.2) are called Noor, see [30], and Picard-S, see [10], iteration methods, respectively.

Remark 1.1. The Noor iteration method reduces to:

(i) Picard iteration method, see [26], if $\alpha_n = 1$, $\beta_n = \gamma_n = 0$, $\alpha_n = 1$ for all $n \in \mathbb{N}$;
(ii) Mann iteration method, see [22], if $\beta_n = \gamma_n = 0$ for all $n \in \mathbb{N}$;
(iii) Ishikawa iteration method, see [15], if $\gamma_n = 0$ for all $n \in \mathbb{N}$;
(iv) normal-S iteration method, see [27] and also [18], if $\alpha_n = 1$, $\gamma_n = 0$ for all $n \in \mathbb{N}$.

However, the Picard-S iteration method is independent of all the Noor, Ishikawa, Mann, Picard, and normal-S iteration methods.

The above-mentioned iteration methods have been intensively investigated in view of convergence, rate of convergence, stability, and data dependency in the literature (see, e.g., [4], [2], [7]–[17], [19]–[22], [24]–[30]) for the different classes of mappings including the class of contraction mappings satisfying:

$$
\|Tx - Ty\| \leqslant \delta\|x - y\|, \quad \delta \in [0,1)\ \forall\, x, y \in X. \tag{1.3}
$$

In 2010, Bosede and Rhoades in [6] proved some stability results for the Picard and Mann iteration methods of quasi-strictly contractive operators satisfying the following condition:

$$
\|x^* - Ty\| \leqslant \delta\|x^* - y\|, \quad \delta \in [0,1)\ \forall\, y \in X, \tag{1.4}
$$

where $x^*$ is a fixed point of $T$.

The class of operators satisfying (1.4) was introduced by Scherzer in [28] and called quasi-strictly contractive operators (see also [5]).

The following example shows that the class of quasi-strictly contractive operators properly includes the class of contraction operators.

Example 1.1. Let $X = \ell_\infty$, $B = \{x \in \ell_\infty \colon \|x\| \leqslant 1\}$ and let $T \colon X \to B \subseteq X$ be defined by

$$Tx = \begin{cases} \dfrac{11}{12}(0, x_1^2, x_2^2, x_3^2, \ldots), & \text{if } \|x\|_\infty \leqslant 1, \\ \dfrac{11}{12\|x\|_\infty^2}(0, x_1^2, x_2^2, x_3^2, \ldots), & \text{if } \|x\|_\infty > 1 \end{cases}$$

for $x = (x_1, x_2, x_3, \ldots) \in \ell_\infty$. Then, $Tp = p$ if and only if $p = 0$. We compute as follows:

$$\|Tx - p\|_\infty = \begin{cases} \dfrac{11}{12}\|(0, x_1^2, x_2^2, x_3^2, \ldots)\|_\infty, & \text{if } \|x\|_\infty \leqslant 1, \\ \dfrac{11}{12\|x\|_\infty^2}\|(0, x_1^2, x_2^2, x_3^2, \ldots)\|_\infty, & \text{if } \|x\|_\infty > 1, \end{cases}$$

so that

$$\|Tx - p\|_\infty \leqslant \begin{cases} \dfrac{11}{12}\|x\|_\infty^2 \leqslant \dfrac{11}{12}\|x\|_\infty, & \text{if } \|x\|_\infty \leqslant 1, \\ \dfrac{11}{12} \cdot 1, & \text{if } \|x\|_\infty > 1. \end{cases}$$

Hence, we obtain that

$$\|Tx - p\|_\infty \leqslant \frac{11}{12}\|x - p\|_\infty \quad \forall\, x \in \ell_\infty,\ p = 0.$$

Hence, $T$ satisfies contractive condition (1.4). But the map $T$ is not a contraction. To see this, take $x = \left(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \ldots\right)$ and $y = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots\right)$. Then,

$$\|x - y\|_\infty = \frac{1}{4}, \quad \|Tx - Ty\|_\infty = \frac{11}{12}\left\|\left(0, \frac{5}{16}, \frac{5}{16}, \ldots\right)\right\|_\infty = \frac{55}{192}.$$

Suppose there exists $\delta \in [0, 1)$ such that $\|Tx - Ty\|_\infty \leqslant \delta\|x - y\|_\infty$ for all $x, y \in \ell_\infty$, then we must have $\frac{55}{192} \leqslant \frac{1}{4}\delta$, which yields that $\delta \geqslant \frac{220}{192} > 1$, a contradiction. So, $T$ is not a contraction map.

Although Akewe and Okeke seemed to have introduced Example 1.1 above in [1], actually, Example 1.1 was introduced in [7] for the first time by Chidume and Olaleru. Furthermore, Akewe and Okeke in [1] proved some convergence and stability results for the normal-S iteration method of quasi-strictly contractive operators. More precisely, they proved the following theorems.

**Theorem 1.1.** *Let $X$ be a real normed linear space and $T \colon X \to X$ be a map satisfying* (1.4) *with a fixed point $x^*$. For arbitrary $u_0 \in X$, let $\{u_n\}_{n=0}^\infty$ be an iterative sequence defined by the normal-S iteration method* [27] *with the real sequence $\{\alpha_n\}_{n=0}^\infty \subset [0, 1]$ such that $\sum\limits_{n=0}^\infty \alpha_n = \infty$. Then $\{u_n\}_{n=0}^\infty$ converges strongly to $x^*$.*

**Theorem 1.2.** *Let $X$ be a real normed linear space and $T\colon X \to X$ be a map satisfying* (1.4) *with a fixed point $x^*$. Then the normal-S iteration method* [27] *with the real sequence $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$ satisfying*

$$0 < \alpha \leqslant \alpha_n \quad \forall\, n \in \mathbb{N} \tag{1.5}$$

*is $T$-stable.*

Remark 1.2. Theorems 1.1 and 1.2 are corrected forms of Theorems 2.1 and 3.1 of [1], respectively. In Theorems 2.1 and 3.1 of [1], no assumption has been introduced for the sequence $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$. But, we observed that the conditions $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$ and $0 < \alpha \leqslant \alpha_n$ for all $n \in \mathbb{N}$ were used in the proofs of Theorems 2.1 and 3.1 of [1], respectively.

The following definitions and lemmas will be needed to realize our goals.

**Definition 1.1** ([3]). Let $T$, $\widetilde{T}\colon X \to X$ be operators. $\widetilde{T}$ is called an approximate operator for $T$ if there exists some $\varepsilon > 0$ such that

$$\|Tx - \widetilde{T}x\| < \varepsilon \quad \forall\, x \in X.$$

**Definition 1.2** ([2]). Suppose that two fixed point iteration methods $\{\varphi_n\}_{n=0}^{\infty}$ and $\{\varphi_n\}_{n=0}^{\infty}$ both converge to the same fixed point $x^*$. Assume further that the error estimates

$$\|\varphi_n - x^*\| \leqslant \tau_n^1, \quad \|\varphi_n - x^*\| \leqslant \tau_n^2$$

are available (and these estimates are the best possible, see [4]), where $\tau_n^i$ , $i = 1,2$ are two sequences of positive numbers (converging to zero). If $\{\tau_n^1\}_{n=0}^{\infty}$ converges faster than $\{\tau_n^2\}_{n=0}^{\infty}$, then we shall say that $\{\varphi_n\}_{n=0}^{\infty}$ converges faster than $\{\varphi_n\}_{n=0}^{\infty}$ to $x^*$.

**Definition 1.3** ([25]). Let $\{\tau_n^i\}_{n=0}^{\infty}$, $i = 1,2$ be two sequences converging to the same point $\eta^*$. We say that $\{\tau_n^1\}_{n=0}^{\infty}$ converges faster than $\{\tau_n^2\}_{n=0}^{\infty}$ to $\eta^*$ if

$$\lim_{n\to\infty} \frac{\|\tau_n^1 - \eta^*\|}{\|\tau_n^2 - \eta^*\|} = 0.$$

**Definition 1.4** ([14]). Let $X$ be a normed space, $T\colon X \to X$ be an operator, and $\{x_n\}_{n=0}^{\infty}$ be a sequence generated by the iteration method $x_{n+1} = f(T, x_n)$, $x_0 \in X$ with limit point $x \in F_T = \{x\colon Tx = x\}$. Let $\{q_n\}_{n=0}^{\infty}$ be an arbitrary sequence in $X$ and set

$$\varepsilon_n = \|q_{n+1} - f(T, q_n)\|.$$

Then the iteration method $\{x_n\}_{n=0}^{\infty}$ is said to be $T$-stable or stable w.r.t. $T$ if and only if

$$\lim_{n\to\infty} \varepsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} q_n = x.$$

**Lemma 1.1** ([3]). *Let $\{u_n\}$, $\{\varepsilon_n\}$ be nonnegative sequences of real numbers satisfying*

$$u_{n+1} \leqslant \delta u_n + \varepsilon_n \quad \forall\, n \in \mathbb{N},\ \delta \in [0,1)$$

*and $\lim\limits_{n\to\infty} \varepsilon_n = 0$. Then we have $\lim\limits_{n\to\infty} u_n = 0$.*

**Lemma 1.2** ([29]). *Let $\{\mu_n\}_{n=0}^{\infty}$, $\{\nu_n\}_{n=0}^{\infty}$ and $\{\xi_n\}_{n=0}^{\infty}$ be nonnegative real sequences with $\nu_n \in (0,1)$ for all $n \in \mathbb{N}$, $\sum\limits_{k=0}^{\infty} \nu_n = \infty$. Suppose there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ one has the inequality*

$$\mu_{n+1} \leqslant (1-\nu_n)\mu_n + \nu_n \xi_n.$$

*Then the following inequality holds:*

$$0 \leqslant \limsup_{n\to\infty} \mu_{n+1} \leqslant \limsup_{n\to\infty} \xi_n.$$

## 2. Main results

**Theorem 2.1.** *Let $X$ be a real normed linear space, $T\colon X \to X$ be a quasi-strictly contractive operator satisfying* (1.4) *with a fixed point $x^*$ and $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by* (1.2) *with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in* $[0,1]$. *Then $\{x_n\}_{n=0}^{\infty}$ strongly converges to $x^*$.*

Proof. It follows by (1.2) and (1.4) that

$$\|x_{n+1} - x^*\| = \|Ty_n - x^*\| \leqslant \delta\|y_n - x^*\|, \tag{2.1}$$

$$\begin{aligned} \|y_n - x^*\| &\leqslant \|(1-\alpha_n)Tx_n + \alpha_n Tz_n - x^*\| \\ &\leqslant (1-\alpha_n)\|Tx_n - x^*\| + \alpha_n\|Tz_n - x^*\| \\ &\leqslant (1-\alpha_n)\delta\|x_n - x^*\| + \alpha_n\delta\|z_n - x^*\|, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \|z_n - x^*\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - x^*\| \\ &\leqslant (1-\beta_n)\|x_n - x^*\| + \beta_n\|Tx_n - x^*\| \\ &\leqslant (1-\beta_n)\|x_n - x^*\| + \beta_n\delta\|x_n - x^*\| \\ &= (1-\beta_n(1-\delta))\|x_n - x^*\|. \end{aligned} \tag{2.3}$$

Combining (2.1)–(2.3), we get

$$\|x_{n+1} - x^*\| \leqslant (1 - \alpha_n\beta_n(1-\delta))\delta^2\|x_n - x^*\|. \tag{2.4}$$

Using the fact $1 - \alpha_n\beta_n(1-\delta) < 1$, we have

$$\|x_{n+1} - x^*\| \leqslant \delta^2\|x_n - x^*\| \leqslant \ldots \leqslant \delta^{2(n+1)}\|x_0 - x^*\|. \tag{2.5}$$

Taking the limit of both sides of inequality (2.5), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x^*\| = 0.$$

□

**Theorem 2.2.** *Let $X$ be a real normed linear space, $T\colon X \to X$ be a quasi-strictly contractive operator satisfying* (1.4) *with a fixed point $x^*$ and $\{x_n\}_{n=0}^{\infty}$ be the iterative sequence generated by* (1.2) *with real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$. Let $\{q_n\}_{n=0}^{\infty} \subset X$ be any sequence and define a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ in $\mathbb{R}^+$ by*

$$\begin{cases} \varepsilon_n = \|q_{n+1} - Tu_n\|, \\ u_n = (1-\alpha_n)Tq_n + \alpha_n Tv_n, \\ v_n = (1-\beta_n)q_n + \beta_n Tq_n \quad \forall\, n \in \mathbb{N}. \end{cases}$$

*Then the Picard-S iteration method* (1.2) *is $T$-stable.*

Proof. Assume that $\lim\limits_{n\to\infty} \varepsilon_n = 0$. In order to prove that the sequence $\{x_n\}_{n=0}^{\infty}$ is stable with respect to $T$, it suffices to prove that $\lim\limits_{n\to\infty} q_n = x^*$. It follows from (1.2) and (1.4) that

$$\|q_{n+1} - x^*\| \leqslant \|q_{n+1} - Tu_n\| + \|Tu_n - x^*\| \leqslant \varepsilon_n + \delta\|u_n - x^*\|, \tag{2.6}$$

$$\begin{aligned} \|u_n - x^*\| &= \|(1-\alpha_n)Tq_n + \alpha_n Tv_n - x^*\| \\ &\leqslant (1-\alpha_n)\|Tq_n - x^*\| + \alpha_n\|Tv_n - x^*\| \\ &\leqslant (1-\alpha_n)\delta\|q_n - x^*\| + \alpha_n\delta\|v_n - x^*\|, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \|v_n - x^*\| &= \|(1-\beta_n)q_n + \beta_n Tq_n - x^*\| \\ &\leqslant (1-\beta_n)\|q_n - x^*\| + \beta_n\|Tq_n - x^*\| \\ &\leqslant (1-\beta_n)\|q_n - x^*\| + \beta_n\delta\|q_n - x^*\| \\ &= (1-\beta_n(1-\delta))\|q_n - x^*\|. \end{aligned} \tag{2.8}$$

Combining (2.6)–(2.8), we obtain

$$\|q_{n+1} - x^*\| \leqslant \delta^2(1 - \alpha_n\beta_n(1-\delta))\|q_n - x^*\| + \varepsilon_n. \tag{2.9}$$

Applying the inequalities $1-\alpha_n\beta_n(1-\delta)<1$ for all $n\in\mathbb{N}$ and $\delta^2<\delta$ to (2.9), we have

$$\|q_{n+1}-x^*\|\leqslant \delta\|q_n-x^*\|+\varepsilon_n. \tag{2.10}$$

It is now easy to check that (2.10) satisfies the requirements in Lemma 1.1. So, we have $\lim\limits_{n\to\infty} q_n=x^*$. □

**Theorem 2.3.** *Let $X$ be a real normed linear space, $T\colon X\to X$ be a quasi-strictly contractive operator satisfying* (1.4) *with a fixed point $x^*$ and $\{x_n\}_{n=0}^\infty$, $\{w_n\}_{n=0}^\infty$, $\{w_n^{(1)}\}_{n=0}^\infty$, $\{w_n^{(2)}\}_{n=0}^\infty$, $\{w_n^{(3)}\}_{n=0}^\infty$ be the iterative sequences generated by Picard-S* (1.2), *Noor* (1.1), *Ishikawa* (*see* [15]), *Mann* (*see* [22]), *normal-S* (*see* [27]) *iteration methods, respectively, with real sequences $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ in* $[0,1]$ *satisfying $\lim\limits_{n\to\infty}\alpha_n=\lim\limits_{n\to\infty}\beta_n=0$. Then the Picard-S iteration method* (1.2) *converges to $x^*$ faster than Noor, Ishikawa, Mann and normal-S iteration methods, provided that the initial point is the same for all iterations.*

Proof. We know from Theorem 2.1 that

$$\|x_{n+1}-x^*\|\leqslant \delta^{2(n+1)}\prod_{k=0}^{n}(1-\alpha_k\beta_k(1-\delta))\|x_0-x^*\|. \tag{2.11}$$

By some simple calculations, we obtain the following estimates for Noor (1.1), Ishikawa [15], Mann [22], normal-S [27] iteration methods, respectively:

$$\|w_{n+1}-x^*\|\geqslant\prod_{k=0}^{n}(1-\alpha_k(1+\delta(1-\beta_k)+\beta_k\delta^2(1-\gamma_k(1-\delta))))\|w_0-x^*\|, \tag{2.12}$$

$$\|w_{n+1}^{(1)}-x^*\|\geqslant\prod_{k=0}^{n}(1-\alpha_k(1+\delta(1-\beta_k(1-\delta))))\|w_0^{(1)}-x^*\|, \tag{2.13}$$

$$\|w_{n+1}^{(2)}-x^*\|\geqslant\prod_{k=0}^{n}(1-\alpha_k(1+\delta))\|w_0^{(2)}-x^*\|, \tag{2.14}$$

$$\|w_{n+1}^{(3)}-x^*\|\leqslant\delta^{n+1}\prod_{k=0}^{n}(1-\beta_k(1-\delta))\|w_0^{(3)}-x^*\|. \tag{2.15}$$

Set

$$\tau_n^{(1)}=\delta^{2(n+1)}\prod_{k=0}^{n}(1-\alpha_k\beta_k(1-\delta))\|x_0-x^*\|,$$

$$\tau_n^{(2)}=\delta^{n+1}\prod_{k=0}^{n}(1-\beta_k(1-\delta))\|w_0^{(3)}-x^*\|\quad\forall\,n\in\mathbb{N}.$$

Since $\lim_{n\to\infty} \delta^{n+1} = \lim_{n\to\infty} \delta^{2(n+1)} = 0$, we have $\lim_{n\to\infty} \tau_n^{(1)} = 0$ and $\lim_{n\to\infty} \tau_n^{(2)} = 0$, that is, both the sequences $\{\tau_n^{(1)}\}_{n=0}^{\infty}$ and $\{\tau_n^{(2)}\}_{n=0}^{\infty}$ converge to zero as assumed in Definition 1.2. Now, using the assumption $x_0 = w_0^{(3)}$, we get

$$\theta_n^{(1)} = \frac{|\tau_n^{(1)} - 0|}{|\tau_n^{(2)} - 0|} = \frac{\delta^{n+1} \prod_{k=0}^{n} (1 - \alpha_k \beta_k (1-\delta))}{\prod_{k=0}^{n} (1 - \beta_k(1-\delta))} \quad \forall\, n \in \mathbb{N},$$

which implies

$$\frac{\theta_{n+1}^{(1)}}{\theta_n^{(1)}} = \frac{\delta(1 - \alpha_{n+1}\beta_{n+1}(1-\delta))}{1 - \beta_{n+1}(1-\delta)} \quad \forall\, n \in \mathbb{N}.$$

By the assumption $\lim_{n\to\infty} \beta_n = 0$, we obtain

$$l = \lim_{n\to\infty} \frac{\theta_{n+1}^{(1)}}{\theta_n^{(1)}} = \delta < 1.$$

Since $l = \delta < 1$, the ratio test tells us that the series $\sum_{n=0}^{\infty} \theta_n^{(1)}$ converges. This allows us to conclude that

$$\lim_{n\to\infty} \theta_n^{(1)} = \lim_{n\to\infty} \frac{|\tau_n^{(1)} - 0|}{|\tau_n^{(2)} - 0|} = 0.$$

Hence, from Definition 1.2, we conclude that $\{\tau_n^{(1)}\}_{n=0}^{\infty}$ converges faster than $\{\tau_n^{(2)}\}_{n=0}^{\infty}$, which implies that $\{x_n\}_{n=0}^{\infty}$ converges faster than $\{w_n^{(3)}\}_{n=0}^{\infty}$.

Now, using (2.11)–(2.14) and the assumption $x_0 = w_0 = w_0^{(1)} = w_0^{(2)}$, we have

$$\text{(2.16)} \quad 0 \leqslant \frac{\|x_{n+1} - x^*\|}{\|w_{n+1} - x^*\|} \leqslant \frac{\delta^{2(n+1)} \prod_{k=0}^{n} (1 - \alpha_k \beta_k (1-\delta))}{\prod_{k=0}^{n} (1 - \alpha_k(1 + \delta(1-\beta_k) + \beta_k \delta^2 (1 - \gamma_k(1-\delta))))},$$

$$\text{(2.17)} \quad 0 \leqslant \frac{\|x_{n+1} - x^*\|}{\|w_{n+1}^{(1)} - x^*\|} \leqslant \frac{\delta^{2(n+1)} \prod_{k=0}^{n} (1 - \alpha_k \beta_k (1-\delta))}{\prod_{k=0}^{n} (1 - \alpha_k(1 + \delta(1 - \beta_k(1-\delta))))},$$

$$\text{(2.18)} \quad 0 \leqslant \frac{\|x_{n+1} - x^*\|}{\|w_{n+1}^{(2)} - x^*\|} \leqslant \frac{\delta^{2(n+1)} \prod_{k=0}^{n} (1 - \alpha_k \beta_k (1-\delta))}{\prod_{k=0}^{n} (1 - \alpha_k(1+\delta))} \quad \forall\, n \in \mathbb{N}.$$

Define

$$\theta_n^{(2)} = \frac{\delta^{2(n+1)} \prod_{k=0}^{n} (1 - \alpha_k \beta_k (1-\delta))}{\prod_{k=0}^{n} (1 - \alpha_k (1 + \delta(1-\beta_k) + \beta_k \delta^2 (1 - \gamma_k (1-\delta))))},$$

$$\theta_n^{(3)} = \frac{\delta^{2(n+1)} \prod_{k=0}^{n} (1 - \alpha_k \beta_k (1-\delta))}{\prod_{k=0}^{n} (1 - \alpha_k (1 + \delta(1-\beta_k(1-\delta))))},$$

$$\theta_n^{(4)} = \frac{\delta^{2(n+1)} \prod_{k=0}^{n} (1 - \alpha_k \beta_k (1-\delta))}{\prod_{k=0}^{n} (1 - \alpha_k (1+\delta))} \quad \forall\, n \in \mathbb{N}.$$

Then, we have

$$\frac{\theta_{n+1}^{(2)}}{\theta_n^{(2)}} = \frac{\delta^2 (1 - \alpha_{n+1} \beta_{n+1} (1-\delta))}{1 - \alpha_{n+1} (1 + \delta(1-\beta_{n+1}) + \beta_{n+1} \delta^2 (1 - \gamma_{n+1}(1-\delta)))},$$

$$\frac{\theta_{n+1}^{(3)}}{\theta_n^{(3)}} = \frac{\delta^2 (1 - \alpha_{n+1} \beta_{n+1} (1-\delta))}{1 - \alpha_{n+1} (1 + \delta(1-\beta_{n+1}(1-\delta)))},$$

$$\frac{\theta_{n+1}^{(4)}}{\theta_n^{(4)}} = \frac{\delta^2 (1 - \alpha_{n+1} \beta_{n+1} (1-\delta))}{1 - \alpha_{n+1}(1+\delta)} \quad \forall\, n \in \mathbb{N}.$$

By the assumption $\lim_{n\to\infty} \alpha_n = 0$, we obtain

$$l_i = \lim_{n\to\infty} \frac{\theta_{n+1}^{(i)}}{\theta_n^{(i)}} = \delta^2 < 1 \quad \text{for } i = 2, 3, 4.$$

Since $l_i = \delta^2 < 1$ for $i = 2, 3, 4$, the ratio test tells us that the series $\sum_{n=0}^{\infty} \theta_n^{(i)}$ converges for $i = 2, 3, 4$. This allows us to conclude that

$$\lim_{n\to\infty} \theta_n^{(i)} = 0 \quad \text{for } i = 2, 3, 4.$$

It follows from (2.16)–(2.18) that

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x^*\|}{\|w_{n+1} - x^*\|} = \lim_{n\to\infty} \frac{\|x_{n+1} - x^*\|}{\|w_{n+1}^{(1)} - x^*\|} = \lim_{n\to\infty} \frac{\|x_{n+1} - x^*\|}{\|w_{n+1}^{(2)} - x^*\|} = 0.$$

Hence, from Definition 1.3, we can say that $\{x_n\}_{n=0}^{\infty}$ converges faster than $\{w_n\}_{n=0}^{\infty}$, $\{w_n^{(1)}\}_{n=0}^{\infty}$ and $\{w_n^{(2)}\}_{n=0}^{\infty}$ to the fixed point $x^*$. □

E x a m p l e 2.1. Let $X=[0,1]$ and $T\colon X\to X$ be an operator defined by

$$Tx=\frac{\cos(x^2-5)}{8}+\frac{\mathrm{e}^{-x^3}}{2}.$$

It is clear that $T$ satisfies (1.4) with $\delta=0.677865$ and $x^*=0.462220$. Take $\alpha_n=\beta_n=\gamma_n=(n^3+50)^{-1}$ and $x_0=\frac{7}{10}$. Table 1 and Figure 1 show that the Picard-S iterative scheme (1.2) converges to $x^*=0.462220$ faster than normal-S, Mann, Ishikawa and Noor iteration methods.

| # of Iter. | Picard-S | Normal-S | Noor | Ishikawa | Mann |
|---|---|---|---|---|---|
| 1 | 0.504620 | 0.334800 | 0.692695 | 0.692696 | 0.692594 |
| 2 | 0.469452 | 0.502597 | 0.685769 | 0.685771 | 0.685579 |
| 3 | 0.463417 | 0.445218 | 0.679867 | 0.679869 | 0.679614 |
| 4 | 0.462417 | 0.468807 | 0.675539 | 0.675539 | 0.675251 |
| 5 | 0.462254 | 0.459554 | 0.672665 | 0.672668 | 0.672368 |
| 6 | 0.462226 | 0.463289 | 0.669613 | 0.670821 | 0.670517 |
| 7 | 0.462221 | 0.461789 | 0.668802 | 0.669616 | 0.669311 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 14 | | 0.462221 | 0.666871 | 0.666987 | 0.666683 |
| 15 | | ⋮ | 0.666779 | 0.666874 | 0.66657 |
| ⋮ | | | ⋮ | ⋮ | ⋮ |

Table 1. Comparison of the rate of convergence among various iteration methods for Example 2.1.

**Lemma 2.1.** *Let $X$ be a real normed linear space and $T\colon X\to X$ be a quasi-strictly contractive operator satisfying* (1.4) *with a fixed point $x^*$. Assume that $\widetilde{T}\colon X\to X$ is an approximate operator of $T$ for given $\varepsilon$. Then*

$$\|Tx-\widetilde{T}y\|\leqslant 2\delta\|x-x^*\|+\delta\|y-x\|+\varepsilon. \tag{2.19}$$

P r o o f. Using triangle inequality, Definition 1.1 and condition (1.4), we get

$$\begin{aligned}\|Tx-\widetilde{T}y\| &\leqslant \|Tx-Ty\|+\|Ty-\widetilde{T}y\|\\ &\leqslant \|Tx-x^*\|+\|Ty-x^*\|+\varepsilon\\ &\leqslant \delta\|x-x^*\|+\delta\|y-x^*\|+\varepsilon\\ &\leqslant \delta\|x-x^*\|+\delta\|y-x\|+\delta\|x-x^*\|+\varepsilon\\ &= 2\delta\|x-x^*\|+\delta\|y-x\|+\varepsilon.\end{aligned}$$

□

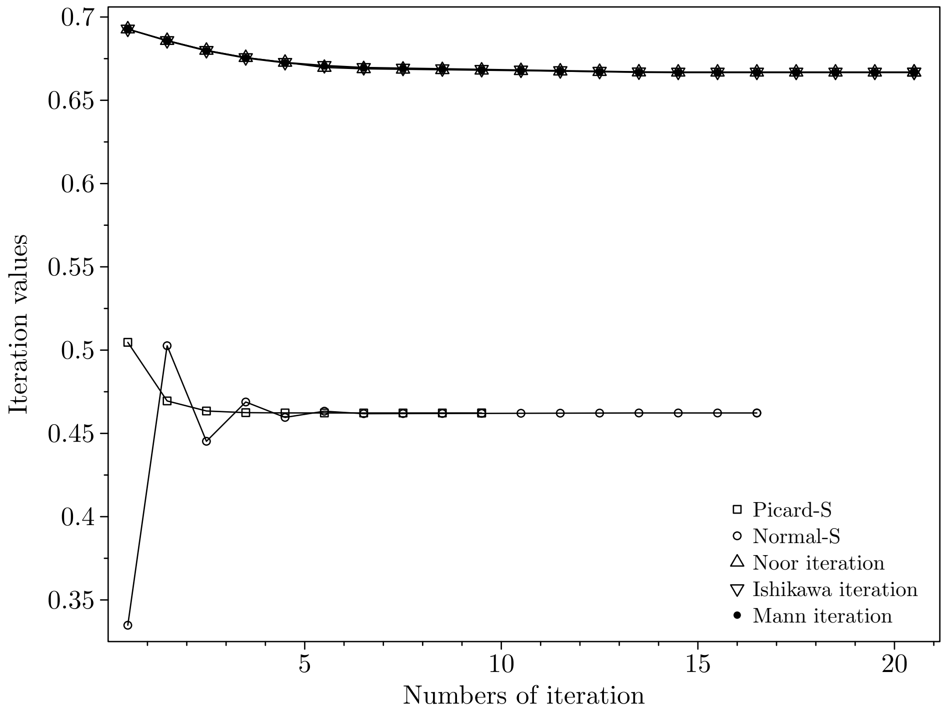


Figure 1. Convergence behavior of various iteration methods.

**Theorem 2.4.** *Let $X$ be a real normed linear space, $T\colon X \to X$ be a quasi-strictly contractive operator satisfying* (1.4) *with a fixed point $x^*$, $\widetilde{T}\colon X \to X$ be an approximate operator of $T$ for given $\varepsilon$, and $\widetilde{x}$ be the fixed point of $\widetilde{T}$. Assume that $\{x_n\}_{n=0}^{\infty}$ is the sequence in* (1.2) *and $\{\widetilde{x}_n\}_{n=0}^{\infty}$ is a sequence defined by*

$$\begin{cases} \widetilde{x}_{n+1} = \widetilde{T}\widetilde{y}_n, \\ y_n = (1-\alpha_n)\widetilde{T}\widetilde{x}_n + \alpha_n\widetilde{T}\widetilde{z}_n, \\ z_n = (1-\beta_n)\widetilde{x}_n + \beta_n\widetilde{T}\widetilde{x}_n \quad \forall\, n \in \mathbb{N}, \end{cases} \tag{2.20}$$

*where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying $\lim\limits_{n\to\infty} \alpha_n = 0$ or $\lim\limits_{n\to\infty} \beta_n = 0$. If $\{\widetilde{x}_n\}_{n=0}^{\infty}$ converges to $\widetilde{x}$, then we have*

$$\|x^* - \widetilde{x}\| \leqslant \frac{\varepsilon(\delta+1)}{1-\delta}.$$

Proof. From (1.2), (1.4), (2.19) and (2.20), we have

$$\|x_{n+1} - \widetilde{x}_{n+1}\| = \|Ty_n - \widetilde{T}\widetilde{y}_n\| \leqslant 2\delta\|y_n - x^*\| + \delta\|\widetilde{y}_n - y_n\| + \varepsilon, \tag{2.21}$$

$$
\begin{aligned}
(2.22)\qquad \|y_n - \widetilde{y}_n\| &= \|(1-\alpha_n)Tx_n + \alpha_n Tz_n - (1-\alpha_n)\widetilde{T}\widetilde{x}_n - \alpha_n\widetilde{T}\widetilde{z}_n\| \\
&\leqslant (1-\alpha_n)\|Tx_n - \widetilde{T}\widetilde{x}_n\| + \alpha_n\|Tz_n - \widetilde{T}\widetilde{z}_n\| \\
&\leqslant (1-\alpha_n)(2\delta\|x_n - x^*\| + \delta\|\widetilde{x}_n - x_n\| + \varepsilon) \\
&\quad + \alpha_n(2\delta\|z_n - x^*\| + \delta\|\widetilde{z}_n - z_n\| + \varepsilon) \\
&= (1-\alpha_n)(2\delta\|x_n - x^*\| + \delta\|\widetilde{x}_n - x_n\|) \\
&\quad + \alpha_n(2\delta\|z_n - x^*\| + \delta\|\widetilde{z}_n - z_n\|) + \varepsilon \\
&= 2(1-\alpha_n)\delta\|x_n - x^*\| + 2\alpha_n\delta\|z_n - x^*\| \\
&\quad + (1-\alpha_n)\delta\|\widetilde{x}_n - x_n\| + \alpha_n\delta\|\widetilde{z}_n - z_n\| + \varepsilon, \\
(2.23)\qquad \|z_n - \widetilde{z}_n\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - (1-\beta_n)\widetilde{x}_n - \beta_n\widetilde{T}\widetilde{x}_n\| \\
&\leqslant (1-\beta_n)\|x_n - \widetilde{x}_n\| + \beta_n\|Tx_n - \widetilde{T}\widetilde{x}_n\| \\
&\leqslant (1-\beta_n)\|x_n - \widetilde{x}_n\| \\
&\quad + \beta_n(2\delta\|x_n - x^*\| + \delta\|\widetilde{x}_n - x_n\| + \varepsilon) \\
&= (1-\beta_n(1-\delta))\|x_n - \widetilde{x}_n\| + 2\beta_n\delta\|x_n - x^*\| + \beta_n\varepsilon.
\end{aligned}
$$

Combining (2.21)–(2.23) and using the fact $1-\alpha_n\beta_n(1-\delta) < 1$ for all $n \in \mathbb{N}$ in the resulting inequality, we get

$$
\begin{aligned}
(2.24)\quad \|x_{n+1} - \widetilde{x}_{n+1}\| \leqslant (1-\varrho)\|x_n - \widetilde{x}_n\| + 2\delta\|y_n - x^*\| + 2(1-\alpha_n)\delta^2\|x_n - x^*\| \\
+ 2\alpha_n\delta^2\|z_n - x^*\| + \alpha_n\delta^2\beta_n\varepsilon + \varepsilon\delta + \varepsilon,
\end{aligned}
$$

where $\varrho = 1-\delta^2 \in (0,1)$.

Denote

$$
\mu_n = \|x_{n+1} - \widetilde{x}_{n+1}\| \geqslant 0, \quad \nu_n = \varrho \in (0,1),
$$

$$
\xi_n = \frac{2\delta(\|y_n - x^*\| + (1-\alpha_n)\delta\|x_n - x^*\| + \alpha_n\delta\|z_n - x^*\|) + (\alpha_n\delta^2\beta_n + \delta + 1)\varepsilon}{\varrho} \geqslant 0.
$$

It is now easy to check that (2.24) fulfills all the requirements of Lemma 1.2 and so by its conclusion, we obtain

$$
\|x^* - \widetilde{x}\| \leqslant \frac{\varepsilon(\delta+1)}{1-\delta^2}.
$$

□

Example 2.2. Let $X = [0,1]$ and $T\colon X \to X$ be defined by

$$
Tx = \frac{1}{2}\sqrt{x^2 + \tfrac{1}{10}}\,\mathrm{e}^{-(\cosh x)/4}.
$$

It is clear that $T$ satisfies (1.4) with $\delta = 0.265205$ and $x^* = 0.133342$. Define an operator $\widetilde{T}\colon X \to X$ by

$$\widetilde{T}x = 0.05 + \frac{0.0644541}{(0.1 + 0.5x)(3 + \cos(2\pi x))} + 0.875619x^2 \sin\left(\tfrac{1}{2}\mathrm{e}^{-2x}\pi\right).$$

The fixed point of $\widetilde{T}$ is $\widetilde{x} = 0.173261$. If we put $\alpha_n = \beta_n = (5n^3 + 1000)^{-1}$ for all $n \in \mathbb{N}$ in (2.20), then the resulting iteration method converges to $\widetilde{x} = 0.173261$ as shown in Table 2.

| # of Iter. | Iteration (2.20) |
|---|---|
| 0 | 0.600000 |
| 1 | 0.195153 |
| 2 | 0.173779 |
| 3 | 0.173270 |
| 4 | 0.173261 |
| ⋮ | ⋮ |

Table 2. Convergence behavior of iteration method (2.20).

By using Wolfram Mathematica 9 software package, we get

$$|Tx - \widetilde{T}x| < 0.0949569$$

for all $x \in X$ and for a fixed $\varepsilon = 0.0949569 > 0$. That is, $\widetilde{T}$ is an approximate operator of $T$. Now, we have $|x^* - \widetilde{x}| = 0.039919$. Actually, without knowing and computing the fixed point $\widetilde{x}$, we can find the following estimate via Theorem 2.4:

$$|x^* - \widetilde{x}| \leqslant \frac{\varepsilon(\delta + 1)}{1 - \delta^2} = \frac{0.0949569(0.265205 + 1)}{1 - 0.265205^2} = 0.129229.$$

## 3. Conclusion

Pertaining to the iteration methods employed in [1], [6], [8], [9], [12], [17], [21], [20], [19], [23], [24], [25], [29] and [31], it is the usual practice to impose some conditions like $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$ (or $\sum\limits_{n=0}^{\infty} \beta_n = \infty$, $\sum\limits_{n=0}^{\infty} \gamma_n = \infty$), $\sum\limits_{n=0}^{\infty} \alpha_n\beta_n = \infty$, $\sum\limits_{n=0}^{\infty} (\alpha_n + \beta_n) = \infty$, $0 < \alpha \leqslant \alpha_n$, $0 < \beta \leqslant \beta_n$, $\beta_n \leqslant \alpha_n$, $\frac{1}{2} \leqslant \alpha_n$, $\frac{1}{2} \leqslant \alpha_n(1 - \delta)$ for all $n \in \mathbb{N}$ on the parametric sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty} \subseteq [0, 1]$ for the type of convergency, stability and data dependency problems considered in those papers. However, none of these conditions has been used in our corresponding results. Therefore, our results are improvements over the corresponding results in all the above mentioned references and some other previous results in the literature.

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