Kybernetika

Ya-Ming Wang; Hua-Wen Liu

On the distributivity equation for uni-nullnorms

Kybernetika, Vol. 55 (2019), No. 1, 24–43

Persistent URL: http://dml.cz/dmlcz/147704

KYBERNETIKA — VOLUME 55 (2019), NUMBER 1, PAGES 24–43

ON THE DISTRIBUTIVITY EQUATION FOR UNI-NULLNORMS

Ya-Ming Wang and Hua-Wen Liu

A uni-nullnorm is a special case of 2-uninorms obtained by letting a uninorm and a nullnorm share the same underlying t-conorm. This paper is mainly devoted to solving the distributivity equation between uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms and some binary operators, such as, continuous t-norms, continuous t-conorms, uninorms, and nullnorms. The new results differ from the previous ones about the distributivity in the class of 2-uninorms, which have not yet been fully characterized.

Keywords: fuzzy connectives, uni-nullnorms, T-norms, T-conorms, nullnorms, uninorms, distributivity equation

Classification: 46F10, 62E86

1. INTRODUCTION

Uninorms as aggregation operators generalizing and unifying the concepts of t-norms and t-conorms were introduced by Yager and Rybalov [30], and have been proved to be useful in many fields like fuzzy system modeling, decision making and so on [31]. Fodor et al. [7] studied the structure of uninorms and gave the characterization of uninorms, which can be built up from t-norms and t-conorms by using an ordinal sum structure. And Fodor and Li et al. gave a single-point characterization of uninorms in recent years [8, 10, 11]. Nullnorms and t-operators as another generalizations of t-norms and t-conorms were respectively introduced by Calvo [3] and Mas et al. [12], and they have been proved to be equivalent [13]. 2-uninorms as a generalization of uninorms and nullnorms, and such generalization further extended to n-uninorms, were introduced by Akella in [2]. However, the structure of 2-uninorms has not yet been fully characterized. Recently, Sun et al. [26] developed the concept of a uni-nullnorm (null-uninorm) by letting a uninorm and a nullnorm share the same underlying t-conorm (t-norm). Uni-nullnorms and null-uninorms as special cases of 2-uninorms generalize uninorms and nullnorms and have been proved that they are a pair of dual binary operations. After that, Sun et al. showed the full characterizations of uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms [27]. Although a uni-nullnorm with continuous Archimedean underlying t-norms and a t-conorm is a special

DOI: 10.14736/kyb-2019-1-0024

case of 2-uninorms, it is not included in the structures of 2-uninorms given in [5] that just have five special structures of 2-uninorms.

The role of functional equations involving aggregation operators is very important in theories of fuzzy sets and fuzzy logic. According to the literature we can find, there are a lot of studies on distributivity equation about uninorms, nullnorms, semi-uninorms, semi-nullnorms and semi-t-operators [11, 13, 16-20, 22-25, 29]. But for 2-uninorms, there are only a few studies on distributivity equation [5, 15, 28]. The reason for the lack of study on the distributivity equation about 2-uninorms is that there is no complete characterization of 2-uninorms. Now Sun et al. have given the full description of uni-nullnorms, which is a special case of 2-uninorms. Obviously, the study of distributivity about uni-nullnorms is the preparation for the discussion of distributive 2-uninorms, so it is necessary to characterize the distributivity equation of uni-nullnorms.

This paper is organized as follows. In Section 2, we present the definitions and structures of some concerning binary operators used later in the paper. In Section 3, we investigate the distributivity between uni-nullnorms and continuous t-norms (t-conorms). After that, we characterize the distributivity equation between uni-nullnorms and uninorms in Section 4, and the distributivity equation between uni-nullnorms and nullnorms in Section 5. Section 6 is conclusion and further work.

2. PRELIMINARIES

We only recall some facts on t-norms, t-conorms, uninorms, nullnorms and uni-nullnorms, which will be used throughout the paper. More details about them can be found in [3, 4, 7, 9, 22, 27].

Definition 2.1. (Klement et al. [9]) A *t-norm* is a binary function $T : [0,1]^2 \to [0,1]$ which is a commutative, associative, increasing binary operator with a neutral element 1. A *t-conorm* is a binary function $S : [0,1]^2 \to [0,1]$ which is a commutative, associative, increasing binary operator with a neutral element 0. A t-norm T (t-conorm S) is called *Archimedean* if for each $(x, y) \in (0,1)^2$ there is an $n \in \mathbb{N}$ such that $T(x, \cdots, x) = x_T^{(n)} < y$ ($x_S^{(n)} > y$).

A t-norm and a t-conorm are a pair of dual operators, we can obtain the corresponding results for t-conorms by interchanging the words t-norms and t-conorms and the roles of 0 and 1, respectively. Thus we only give some definitions and lemmas about t-norms.

For a t-norm T, an element $c \in (0,1)$ is called a *nilpotent element* of T if there exists some $n \in \mathbb{N}$ such that $c_T^{(n)} = 0$. T is said to be *strictly monotone* (SM) if $T(x,y) < T(x,z)$ whenever $x > 0$ and $y < z$. T satisfies the *cancellation law* (CL) if $T(x,y) = T(x,z)$ implies $x = 0$ or $y = z$. T satisfies the *conditional cancellation law* (CCL) if $T(x,y) = T(x,z) > 0$ implies $y = z$. For a continuous t-norm T, T is called *strict* if it is strictly monotone, and T is called *nilpotent* if every $c \in (0,1)$ is a nilpotent element of T. Any continuous Archimedean t-norms is either strict or nilpotent.

Lemma 2.2. (Klement et al. [9]) A continuous t-norm T is Archimedean if and only if it satisfies CCL.

Lemma 2.3. (Klement et al. [9]) A continuous t-norm T is Archimedean if and only if $T(x,x) < x$ for all $x \in (0,1)$.

Definition 2.4. (Fodor et al. [7]) A *uninorm* is a binary function $U : [0,1]^2 \to [0,1]$ which is commutative, associative, increasing in each variable and there is a neutral element $f \in [0,1]$ such that $U(f,x) = x$ for all $x \in [0,1]$.

Evidently, a uninorm with a neutral element $f = 1$ is a t-norm and a uninorm with a neutral element $f = 0$ is a t-conorm. For any uninorm U, we have $U(0,1) \in \{0,1\}$, and U is called *conjunctive* when $U(0,1) = 0$ and *disjunctive* when $U(0,1) = 1$. By $\mathcal{U}_f$ we denote the family of all uninorms with neutral element $f \in [0,1]$. In what follows we give the description of uninorms by using the notation $A(f) = [0,f) \times (f,1] \cup (f,1] \times [0,f)$.

Theorem 2.5. (Fodor et al. [7]) Let $f \in (0,1)$. $U \in \mathcal{U}_f$ if and only if

$$U(x,y) = \begin{cases} fT_U(\frac{x}{f}, \frac{y}{f}) & \text{if } (x,y) \in [0,f]^2, \\ f + (1-f)S_U(\frac{x-f}{1-f}, \frac{y-f}{1-f}) & \text{if } (x,y) \in [f,1]^2, \\ C(x,y) & \text{if } (x,y) \in A(f), \end{cases} \tag{1}$$

where T_U and S_U are respectively a t-norm and a t-conorm, the associative and increasing operation $C : A(f) \to [0,1]$ fulfills $\min(x,y) \leqslant C(x,y) \leqslant \max(x,y)$ for $(x,y) \in A(f)$.

Definition 2.6. (Fodor et al. [7]) Consider $f \in (0,1)$. A binary operator $U : [0,1]^2 \to [0,1]$ is a *representable uninorm* if and only if there exists a continuous strictly increasing function $h : [0,1] \to [-\infty, +\infty]$ with $h(0) = -\infty$, $h(f) = 0$ and $h(1) = +\infty$ such that

$$U(x,y) = h^{-1}(h(x) + h(y))$$

for all $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ and $U(0,1) = U(1,0) \in \{0,1\}$. The function h is usually called an additive generator of U.

Recall that there are no continuous uninorms with neutral element $f \in (0,1)$. In fact, representable uninorms were characterized as those uninorms that are continuous in $[0,1]^2 \setminus \{(0,1),(1,0)\}$ (see [22]) as well as those that are strictly increasing in the open unit square (see [8, 22]).

Definition 2.7. (Calvo et al. [3]) A *nullnorm* is a binary function $V : [0,1]^2 \to [0,1]$ which is commutative, associative, increasing and has a zero element $k \in [0,1]$ such that

(i) $V(0,x) = V(x,0) = x$ for all $x \leqslant k$,

(ii) $V(1,x) = V(x,1) = x$ for all $x \geqslant k$.

By Definition 2.7, the case $k = 0$ leads back to a t-norm, while the case $k = 1$ leads back to a t-conorm. By $\mathcal{V}_k$ we denote the family of all nullnorms with the zero element $k \in [0,1]$.

Theorem 2.8. (Calvo et al. [3]) Let $k \in (0,1)$. $V \in \mathcal{V}_k$ if and only if there exists a t-norm T_V and a t-conorm S_V such that

$$V(x,y) = \begin{cases} kS_V(\frac{x}{k}, \frac{y}{k}) & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)T_V(\frac{x-k}{1-k}, \frac{y-k}{1-k}) & \text{if } (x,y) \in [k,1]^2, \\ k & \text{otherwise.} \end{cases} \tag{2}$$

Definition 2.9. (De Baets [4]) An element $s \in [0,1]$ is called an *idempotent element* of operation $F : [0,1]^2 \to [0,1]$ if $F(s,s) = s$. Operation F is called *idempotent* if all elements from $[0,1]$ are idempotent.

Definition 2.10. (Akella [2], Fechner et al. [6]) Let G be a binary operator on $[0,1]$ which is commutative. Then $\{e_1, e_2\}_a$ is called the *2-neutral element* of G if $G(e_1, x) = x$ for all $x \leqslant a$ and $G(e_2, x) = x$ for all $x \geqslant a$ where $0 < a < 1$ and $e_1 \in [0,a]$, $e_2 \in [a,1]$.

Definition 2.11. (Akella [2], Fechner et al. [6]) A binary operator G on $[0,1]$ is a *2-uninorm* if it is commutative, associative, increasing in both the variable and has a 2-neutral element $\{e_1, e_2\}_a$.

Definition 2.12. (Sun et al. [26]) A *uni-nullnorm* G is a 2-uninorm having a 2-neutral element $\{e, 1\}_a$ with a being the zero element over $[e, 1]$.

Note that a uni-nullnorm is a special case of a 2-uninorm, where the underlying upper uninorm of the 2-uninorm is a t-norm. Obviously, uni-nullnorms generalize both uninorms and nullnorms. For a uni-nullnorm G with a 2-neutral element $\{e,1\}_a$, G is a uninorm if $a = 1$ and a nullnorm if $e = 0$. A uni-nullnorm with a 2-neutral element $\{e,1\}_a$ is proper if $0 < e < 1$.

Theorem 2.13. (Sun et al. [26]) Let $G : [0,1]^2 \to [0,1]$ be a uni-nullnorm such that the underlying t-norms T_G^{ll}, T_G^{ur} and t-conorm S_G are continuous Archimedean. Then G has one of the following structures:

(i)

$$G(x,y) = \begin{cases} eT_G^{ll}(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (a-e)S_G(\frac{x-e}{a-e}, \frac{y-e}{a-e}) & \text{if } (x,y) \in [e,a]^2, \\ a + (1-a)T_G^{ur}(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{if } (x,y) \in [e,a] \times [a,1] \cup [a,1] \times [e,a], \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{3}$$

(ii)

$$G(x,y) = \begin{cases} eT_G^{ll}(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (a-e)S_G(\frac{x-e}{a-e}, \frac{y-e}{a-e}) & \text{if } (x,y) \in [e,a]^2, \\ a + (1-a)T_G^{ur}(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a], \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{4}$$

(iii)

$$G(x,y)=\begin{cases} eT_G^{ll}(\frac{x}{e},\frac{y}{e}) & \text{if } (x,y)\in[0,e]^2,\\ e+(a-e)S_G(\frac{x-e}{a-e},\frac{y-e}{a-e}) & \text{if } (x,y)\in[e,a]^2,\\ a+(1-a)T_G^{ur}(\frac{x-a}{1-a},\frac{y-a}{1-a}) & \text{if } (x,y)\in[a,1]^2,\\ a & \text{if } (x,y)\in(0,a]\times[a,1]\cup[a,1]\times(0,a],\\ \min(x,y) & \text{otherwise.}\end{cases} \tag{5}$$

(iv)

$$G(x,y)=\begin{cases} eT_G^{ll}(\frac{x}{e},\frac{y}{e}) & \text{if } (x,y)\in[0,e]^2,\\ e+(a-e)S_G(\frac{x-e}{a-e},\frac{y-e}{a-e}) & \text{if } (x,y)\in[e,a]^2,\\ a+(1-a)T_G^{ur}(\frac{x-a}{1-a},\frac{y-a}{1-a}) & \text{if } (x,y)\in[a,1]^2,\\ a & \text{if } (x,y)\in[0,a]\times[a,1]\cup[a,1]\times[0,a],\\ \max(x,y) & \text{otherwise.}\end{cases} \tag{6}$$

(v)

$$G(x,y)=\begin{cases} eT_G^{ll}(\frac{x}{e},\frac{y}{e}) & \text{if } (x,y)\in[0,e]^2,\\ e+(a-e)S_G(\frac{x-e}{a-e},\frac{y-e}{a-e}) & \text{if } (x,y)\in[e,a]^2,\\ a+(1-a)T_G^{ur}(\frac{x-a}{1-a},\frac{y-a}{1-a}) & \text{if } (x,y)\in[a,1]^2,\\ a & \text{if } (x,y)\in[0,a]\times[a,1]\cup[a,1]\times[0,a],\\ 0 & \text{if } (x,y)\in\{0\}\times(e,a)\cup(e,a)\times\{0\},\\ \max(x,y) & \text{otherwise.}\end{cases} \tag{7}$$

(vi)

$$G(x,y)=\begin{cases} eT_G^{ll}(\frac{x}{e},\frac{y}{e}) & \text{if } (x,y)\in[0,e]^2,\\ e+(a-e)S_G(\frac{x-e}{a-e},\frac{y-e}{a-e}) & \text{if } (x,y)\in[e,a]^2,\\ a+(1-a)T_G^{ur}(\frac{x-a}{1-a},\frac{y-a}{1-a}) & \text{if } (x,y)\in[a,1]^2,\\ a & \text{if } (x,y)\in(0,a]\times[a,1]\cup[a,1]\times(0,a],\\ 0 & \text{if } (x,y)\in\{0\}\times[0,1]\cup[0,1]\times\{0\},\\ \max(x,y) & \text{otherwise.}\end{cases} \tag{8}$$

(vii)

$$G(x,y)=\begin{cases} aU_G(\frac{x}{a},\frac{y}{a}) & \text{if } (x,y)\in[0,a]^2,\\ a+(1-a)T_G^{ur}(\frac{x-a}{1-a},\frac{y-a}{1-a}) & \text{if } (x,y)\in[a,1]^2,\\ a & \text{if } (x,y)\in(0,a]\times[a,1]\cup[a,1]\times(0,a],\\ 0 & \text{otherwise,}\end{cases} \tag{9}$$

where U_G is a conjunctive representable uninorm.

(viii)

$$G(x,y)=\begin{cases} aU_G(\frac{x}{a},\frac{y}{a}) & \text{if } (x,y)\in[0,a]^2,\\ a+(1-a)T_G^{ur}(\frac{x-a}{1-a},\frac{y-a}{1-a}) & \text{if } (x,y)\in[a,1]^2,\\ a & \text{if } (x,y)\in[0,a]\times[a,1]\cup[a,1]\times[0,a], \end{cases} \tag{10}$$

where U_G is a disjunctive representable uninorm.

Now, we recall the distributivity equations.

Definition 2.14. (Aczél [1]) Let $G,F:[0,1]^2\to[0,1]$. We say that

(i) G is *left distributive* over F, if for all $x,y,z\in[0,1]$,

$$G(x,F(y,z))=F(G(x,y),G(x,z)). \tag{11}$$

(ii) G is *right distributive* over F, if for all $x,y,z\in[0,1]$,

$$G(F(x,y),z)=F(G(x,z),G(y,z)). \tag{12}$$

If Eqs. (11) and (12) are fulfilled simultaneously, for example, G is commutative, we say that G is *distributive* over F.

Lemma 2.15. (Rak [18]) Every increasing operation $F:[0,1]^2\to[0,1]$ is distributive over max and min.

Theorem 2.16. (Ruiz and Torrens [22]) Let U be a representable uninorm with neutral element $f\in(0,1)$ and T be a continuous t-norm. The following conditions are equivalent.

(i) U is distributive over T.

(ii) U is conditionally distributive over T.

(iii) We have either one of the following cases:

(a) $T=T_M$.

(b) T is strict and if t is the additive generator of T satisfying $t(f)=1$, then $\frac{1}{t}$ is also a multiplicative generator of U.

Theorem 2.17. (Ruiz and Torrens [22]) Let U be a representable uninorm with neutral element $f\in(0,1)$ and S be a continuous t-conorm. The following conditions are equivalent.

(i) U is distributive over S.

(ii) U is conditionally distributive over S.

(iii) We have either one of the following cases:

(a) $S=S_M$.

(b) S is strict and if s is the additive generator of S satisfying $s(f)=1$, then $\frac{1}{s}$ is also a multiplicative generator of U.

3. DISTRIBUTIVITY BETWEEN UNI-NULLNORMS AND CONTINUOUS T-NORMS (T-CONORMS)

In this section, we will discuss distributivity between continuous t-norms (t-conorms) and proper uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms. Here we investigate the distributivity equation between continuous t-norms and uni-nullnorms in detail, and only list the results of this equation for uni-nullnorms and continuous t-conorms as corollaries in this section because t-norms and t-conorms are a pair of dual operators. First of all, let us discuss the distributivity for uni-nullnorms over continuous t-norms.

Theorem 3.1. Let uni-nullnorm G be one of Eqs. (3) – (8) with $0 < e < a < 1$ and T be a continuous t-norm. Then G is distributive over T if and only if $T = \min$.

Proof. Suppose uni-nullnorm G is one of Eqs. (3),(4),(5). Let $x = a$ and $y = z = 1$ in the distributivity equation, then we can easily obtain $a = G(a, 1) = G(a, T(1, 1)) = T(G(a, 1), G(a, 1)) = T(a, a)$.

Now let us prove $T(e, e) = e$. It follows from the continuity of T that there exists $t_1 \in [e, a)$ such that $T(t_1, t_1) = e$. Then we have $x = G(x, e) = G(x, T(t_1, t_1)) = T(G(x, t_1), G(x, t_1)) = T(x \wedge t_1, x \wedge t_1) = T(x, x)$ for any $x \in (0, e)$. Thus we obtain $T(e, e) = e$ by the continuity of T.

Next we will prove T is idempotent. In fact, we have $x = G(x, e) = G(x, T(e, e)) = T(G(x, e), G(x, e)) = T(x, x)$ for all $x \in [0, a]$, and $x = G(x, 1) = G(x, T(1, 1)) = T(G(x, 1), G(x, 1)) = T(x, x)$ for all $x \in [a, 1]$. That is, $T(x, x) = x$ for all $x \in [0, 1]$.

The converse statement is obvious from Lemma 2.15.

Suppose uni-nullnorm G is one of Eqs. (6),(7),(8). Based on the above proof, here we only need to prove $T(e, e) = e$. Assume $T(e, e) < e$, then it follows from the continuity of T that there exists $t_2 \in (e, a)$ such that $T(t_2, t_2) = e$. Then for any $x \in (0, e)$ we have $x = G(x, e) = G(x, T(t_2, t_2)) = T(G(x, t_2), G(x, t_2)) = T(x \vee t_2, x \vee t_2) = T(t_2, t_2) = e$, which is a contradiction. So we have $T(e, e) = e$. □

Theorem 3.2. Let uni-nullnorm G be one of Eqs. (9),(10) with $0 < e < a < 1$ and T be a continuous t-norm. Then G is distributive over T if and only if one of the following cases holds:

(i) $T = T_M$.

(ii) The structure of T is as follows:

$$T(x, y) = \begin{cases} aT_1(\frac{x}{a}, \frac{y}{a}) & \text{if } (x, y) \in [0, a]^2, \\ \min(x, y) & \text{otherwise}, \end{cases} \tag{13}$$

where T_1 is a strict t-norm and if t_1 is the additive generator of T_1 satisfying $t_1(\frac{e}{a}) = 1$, then $\frac{1}{t_1}$ is a multiplicative generator of the underlying representable uninorm U_G of uni-nullnorm G.

Proof. Firstly, it is obvious that $x = G(x, 1) = G(x, T(1, 1)) = T(G(x, 1), G(x, 1)) = T(x, x)$ for all $x \in [a, 1]$. So we have $T(x, y) = \min(x, y)$ for $(x, y) \in [0, a] \times [a, 1] \cup [a, 1] \times$

$[0, 1]$ by the continuity of T. Then it follows from Theorem 2.16 that one of the cases (i) and (ii) holds.

Conversely, it is easy to verify that G is distributive over T. □

Next we will discuss the distributivity for continuous t-norms over proper uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms.

Theorem 3.3. Let G be a proper uni-nullnorm with continuous Archimedean underlying t-norms T_G^{ll}, T_G^{ur} and t-conorm S_G, and T be a continuous t-norm. Then T is not distributive over G.

Proof. Let $y = z = 1$ in the distributivity equation, then for any $x \in [0, 1]$ we have $x = T(x, 1) = T(x, G(1, 1)) = G(T(x, 1), T(x, 1)) = G(x, x)$, which contradicts with $G(x, x) < x$ for $x \in (0, e) \cup (a, 1)$ and $G(x, x) > x$ for $x \in (e, a)$. □

Corollary 3.4. Let uni-nullnorm G be one of Eqs. (3) – (8) with $0 < e < a < 1$, and S be a continuous t-conorm. Then G is distributive over S if and only if $S = \max$.

Corollary 3.5. Let uni-nullnorm G be one of Eqs. (9),(10) with $0 < e < a < 1$ and S be a continuous t-conorm. Then G is distributive over S if and only if one of the following cases holds:

(i) $S = S_M$.

(ii) The structure of S is as follows:

$$S(x, y) = \begin{cases} aS_1(\frac{x}{a}, \frac{y}{a}) & \text{if } (x, y) \in [0, a]^2, \\ \max(x, y) & \text{otherwise}, \end{cases} \tag{14}$$

where S_1 is a strict t-conorm and if s_1 is the additive generator of S_1 satisfying $s_1(\frac{e}{a}) = 1$, then s_1 is also a multiplicative generator of the underlying representable uninorm U_G of uni-nullnorm G.

Corollary 3.6. Let G be a proper uni-nullnorm with continuous Archimedean underlying t-norms T_G^{ll}, T_G^{ur} and t-conorm S_G, and S be a continuous t-conorm. Then S is not distributive over G.

4. DISTRIBUTIVITY BETWEEN UNI-NULLNORMS AND UNINORMS

From now on, G will denote a uni-nullnorm with a 2-neutral element $\{e, 1\}_a$, where $0 < e < a < 1$ and the underlying t-norms and t-conorm are continuous Archimedean, and $U \in \mathcal{U}_f$ will denote a uninorm with a neutral element $0 < f < 1$. Depending on whether the neutral elements of G and U are same or not, there are two cases: distributivity between G and U with $f = e$, and distributivity between G and U with $f \neq e$.

4.1. Distributivity between G and U with $f = e$

In this section, we will investigate distributivity for G over U and U over G with $f = e$, respectively. Firstly, let us discuss the distributivity equation for uni-nullnorms over uninorms.

Theorem 4.1. Let uni-nullnorm G be Eq.(3) with $0 < e < a < 1$ and U be a uninorm with neutral element $e \in (0, 1)$. Then G is distributive over U if and only if the structure of U is as follows:

$$U(x, y) = \begin{cases} \max(x, y) & \text{if } (x, y) \in [e, 1]^2, \\ \min(x, y) & \text{otherwise.} \end{cases} \tag{15}$$

Proof. It is easy for us to prove that U is idempotent. In fact, we have $x = G(x, e) = G(x, U(e, e)) = U(G(x, e), G(x, e)) = U(x, x)$ for any $x \in [0, a]$, and $x = G(x, 1) = G(x, U(1, 1)) = U(G(x, 1), G(x, 1)) = U(x, x)$ for any $x \in [a, 1]$. That is, $U(x, x) = x$ for all $x \in [0, 1]$.

Now let us prove $U(x, y) = \min(x, y)$ for $(x, y) \in [0, e) \times [e, 1]$. Let $x \in (0, e)$, $y \in (0, e)$ and $z \in [e, 1]$, then it follows from the structures of G and U that $G(x, U(y, z)) = U(G(x, y), G(x, z)) = U(G(x, y), x \wedge z) = U(G(x, y), x) = G(x, y) \wedge x = G(x, y)$. Assume there exists some $y_0 \in (0, e)$ and $z_0 \in [e, 1]$ such that $U(y_0, z_0) \geqslant e$, then for any $x \in (0, e)$ we have $G(x, y_0) = G(x, U(y_0, z_0)) = x \wedge U(y_0, z_0) = x$, which contradicts with the structure of G. So we have $U(y, z) < e$ for all $(y, z) \in (0, e) \times [e, 1]$. Then from the continuity of the underlying t-norm T_G^{ll} of G, it follows that for every $y \in (0, e)$ there must exist $x_0 \in (0, e)$ such that $G(x_0, y) > 0$, that is, $G(x_0, U(y, z)) = G(x_0, y) > 0$. Thus we have $U(y, z) = y = \min(y, z)$ for $(y, z) \in (0, e) \times [e, 1]$ by the conditional cancellation law. From above, we know $U(0, 1) = 0$ by the monotonicity of U, so $U(0, z) = 0$ for $z \in [e, 1]$. Therefore, $U(y, z) = y = \min(y, z)$ for $(y, z) \in [0, e) \times [e, 1]$.

The converse statement is obvious. □

Theorem 4.2. Let uni-nullnorm G be Eq.(4) with $0 < e < a < 1$ and U be a uninorm with neutral element $e \in (0, 1)$. Then G is distributive over U if and only if the structure of U is as follows:

$$U(x, y) = \begin{cases} \max(x, y) & \text{if } (x, y) \in [0, e] \times [a, 1] \cup [a, 1] \times [0, e] \cup [e, 1]^2, \\ \min(x, y) & \text{otherwise.} \end{cases} \tag{16}$$

Proof. Based on the proof of Theorem 4.1, we only need to prove $U(x, y) = \max(x, y)$ for $(x, y) \in [0, e] \times [a, 1]$. Firstly, let $x \in [0, e]$, $y = e$ and $z \in [a, 1]$ in distributivity equation, then $a = G(x, z) = G(x, U(e, z)) = U(G(x, e), G(x, z)) = U(x, a)$ for any $x \in [0, e]$.

Now let $x \in (a, 1]$, $y \in [0, e]$ and $z \in (a, 1]$, then it follows from the structures of G and U that $G(x, U(y, z)) = U(G(x, y), G(x, z)) = U(a, G(x, z)) = a \vee G(x, z) = G(x, z)$. So we have $U(y, z) = z = \max(y, z)$ for $(y, z) \in [0, e] \times (a, 1]$ since the underlying operator T_G^{ur} of G is a continuous Archimedean t-norm, which satisfies the conditional cancellation law.

Conversely, it is easy to verify that G is distributive over U. □

Theorem 4.3. Let uni-nullnorm G be Eq.(5) with $0 < e < a < 1$ and U be a uninorm with neutral element $e \in (0,1)$. Then G is distributive over U if and only if the structure of U is as follows:

$$U(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in (0,e] \times [a,1] \cup [a,1] \times (0,e] \cup [e,1]^2, \\ 0 & \text{if } (x,y) \in \{0\} \times [a,1] \cup [a,1] \times \{0\}, \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{17}$$

Proof. Based on the proof of Theorem 4.2, here we only prove $U(0,a) = U(0,1) = 0$. Firstly, take $x = a$, $y = 0$ and $z = e$ in distributivity equation, then we have $0 = G(a,0) = G(a,U(0,e)) = U(G(a,0),G(a,e)) = U(0,a)$. And take $x = e$, $y = 0$ and $z = 1$ in distributivity equation, then we have $G(e,U(0,1)) = U(G(e,0),G(e,1)) = U(0,a) = 0$. Suppose $U(0,1) = 1$, then it follows from the structure of G that $0 = G(e,U(0,1)) = G(e,1) = a$, which is a contradiction. So we have $U(0,1) = 0$. $\square$

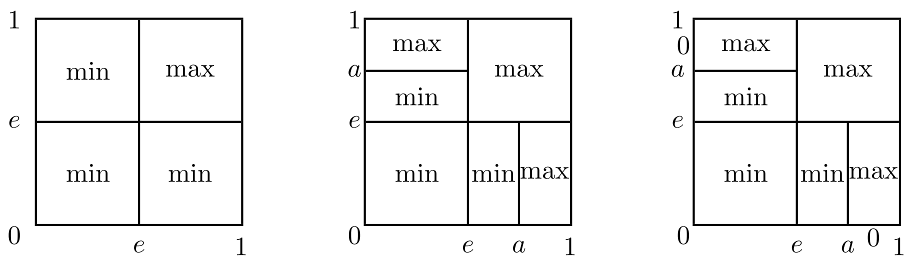

Fig. 1. (Left:) U in Theorem 4.1; (middle:) U in Theorem 4.2; (right:) U in Theorem 4.3.

Theorem 4.4. Let uni-nullnorm G be Eq.(6) with $0 < e < a < 1$ and U be a uninorm with neutral element $e \in (0,1)$. Then G is distributive over U if and only if the structure of U is as follows:

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,e]^2, \\ \max(x,y) & \text{otherwise.} \end{cases} \tag{18}$$

Proof. Firstly, we know that U is idempotent from the proof of Theorem 4.1. Let $x \in [0,e]$, $y = e$ and $z \in [a,1]$ in distributivity equation, then we have $a = G(x,z) = G(x,U(e,z)) = U(G(x,e),G(x,z)) = U(x,a)$. That is, $U(x,a) = a$ for all $x \in [0,e]$.

Now let us prove $U(x,y) = \max(x,y)$ for $(x,y) \in [0,e] \times (e,1]$. Let $x \in (e,a)$, $y \in [0,e]$ and $z \in (e,a)$, then it follows from the structures of G and U that $G(x,U(y,z)) = U(G(x,y),G(x,z)) = U(x \vee y, G(x,z)) = U(x,G(x,z)) = x \vee G(x,z) = G(x,z)$. Assume there exists some $y' \in [0,e]$ and $z' \in (e,a)$ such that $U(y',z') \leqslant e$, then for any $x \in (e,a)$ we have $x = x \vee U(y',z') = G(x,U(y',z')) = G(x,z')$, which contradicts with the structure of G. So $e < U(y,z) \leqslant a$ for all $(y,z) \in [0,e] \times (e,a)$. Thus we obtain

$U(y,z) = z = \max(y,z)$ for $(y,z) \in [0,e] \times (e,a)$ since the underlying operator S_G is a continuous Archimedean t-conorm, which satisfies the conditional cancellation law. And let $x \in (a,1]$, $y \in [0,e]$ and $z \in (a,1]$, it follows from the structures of G and U that $G(x,U(y,z)) = U(G(x,y),G(x,z)) = U(a,G(x,z)) = a \vee G(x,z) = G(x,z)$, then $U(y,z) = z = \max(y,z)$ for $(y,z) \in [0,e] \times (a,1]$ since the underlying operator T_G^{ur} is a continuous Archimedean t-norm, which satisfies the conditional cancellation law.

The converse statement is obvious. □

Theorem 4.5. Let uni-nullnorm G be Eq.(7) with $0 < e < a < 1$ and U be a uninorm with neutral element $e \in (0,1)$. Then G is distributive over U if and only if the structure of U is as follows:

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,e]^2, \\ 0 & \text{if } (x,y) \in \{0\} \times (e,a) \cup (e,a) \times \{0\}, \\ \max(x,y) & \text{otherwise.} \end{cases} \tag{19}$$

Proof. Based on the proof of Theorem 4.4, we just prove $U(x,0) = 0$ for $x \in (e,a)$. Let $x \in (e,a)$, $y \in (0,e]$ and $z = 0$ in distributivity equation, then we have $0 = G(x,0) = G(x,U(y,0)) = U(G(x,y),G(x,0)) = U(x \vee y, 0) = U(x,0)$. □

Theorem 4.6. Let uni-nullnorm G be Eq.(8) with $0 < e < a < 1$ and U be a uninorm with neutral element $e \in (0,1)$. Then G is distributive over U if and only if the structure of U is as follows:

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,e]^2, \\ 0 & \text{if } (x,y) \in \{0\} \times [e,1] \cup [e,1] \times \{0\}, \\ \max(x,y) & \text{otherwise.} \end{cases} \tag{20}$$

Proof. Based on the proof of Theorem 4.5, we just prove $U(0,a) = U(0,1) = 0$. Take $x = a$, $y = 0$ and $z \in (0,e]$ in distributivity equation, we have $0 = G(a,0) = G(a,U(0,z)) = U(G(a,0),G(a,z)) = U(0,a)$. And take $x = a$, $y = 0$ and $z = 1$ in distributivity equation, we have $G(a,U(0,1)) = U(G(a,0),G(a,1)) = U(0,a) = 0$. Suppose $U(0,1) = 1$, then $a = G(a,1) = G(a,U(0,1)) = 0$, which is a contradiction. So we have $U(0,1) = 0$. □

The distributivity for representable uninorms over uninorms was already studied in [21], and it was proved that there are no solutions in this case. That is, representable uninorms are not distributive over uninorms. Thus, we have the following conclusion.

Theorem 4.7. Let uni-nullnorm G be one of Eqs. (9),(10) with $0 < e < a < 1$ and U be a uninorm with neutral element $e \in (0,1)$. Then G is not distributive over U.

Proof. First of all, we know that U is idempotent from the proof of Theorem 4.1. Let $x \in (0,e)$, $y \in (e,a)$ and $z = e$ in distributivity equation, then we have $G(x,y) = G(x,U(y,e)) = U(G(x,y),G(x,e)) = U(G(x,y),x)$.

Now let us prove $U(x, y) \neq e$ for $(x, y) \in (0, e) \times (e, a)$. Assume there exists some $x_0 \in (0, e)$ and $y_0 \in (e, a)$ such that $G(x_0, y_0) = e$, then $e = G(x_0, y_0) = U(G(x_0, y_0), x_0) = U(e, x_0) = x_0$, which is a contradiction. So we have $G(x, y) \neq e$ and $x = G(x, e) < G(x, y) < G(e, y) = y$ for $(x, y) \in (0, e) \times (e, a)$ from the strict monotonicity of the underlying representable uninorm U_G of G. Thus, there are two possibilities: $0 < x < G(x, y) < e < y < a$ and $0 < x < e < G(x, y) < y < a$. When $0 < x < G(x, y) < e < y < a$, then from the structure of U it follows that $G(x, y) = U(G(x, y), x) = G(x, y) \wedge x = x$, which is a contradiction. When $0 < x < e < G(x, y) < y < a$, then from the structure of U it follows that $G(x, y) = G(y, x) = U(G(y, x), y) = G(x, y) \vee y = y$, which is a contradiction. Therefore, G is not distributive over U. □

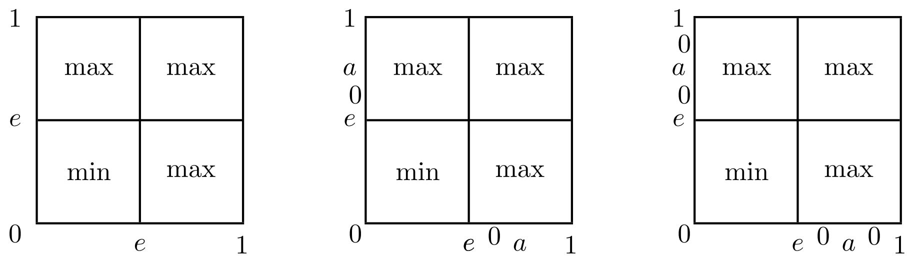

Fig. 2. (Left:) U in Theorem 4.4; (middle:) U in Theorem 4.5; (right:) U in Theorem 4.6.

Next we will discuss the distributivity for uninorms over proper uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms.

Theorem 4.8. Let G be a proper uni-nullnorm with continuous Archimedean underlying t-norms T_G^{ll}, T_G^{ur} and t-conorm S_G, and U be a uninorm with neutral element $e \in (0, 1)$. Then U is not distributive over G.

Proof. The proof is similar to the one of Theorem 3.3. □

4.2. Distributivity between G and U with $f \neq e$

In this section, we will investigate distributivity for G over U and U over G with $f \neq e$, respectively. Firstly, let us discuss the distributivity equation for uni-nullnorms over uninorms.

Theorem 4.9. Let uni-nullnorm G be Eq.(3) with $0 < e < a < 1$ and U be a uninorm with neutral element $f \neq e$, where the underlying t-norm T_U is continuous. Then G is distributive over U if and only if $f = a$ and U has the form (15).

Proof. First of all, let us prove $f = a$. Assume $f \in (0, e)$, then we know $G(f, f) < f$ from Lemma 2.3. Taking $x = z = f$ and $y = e$ in distributivity equation, then we have

$f = G(f,e) = G(f,U(e,f)) = U(G(f,e),G(f,f)) = U(f,G(f,f)) = G(f,f)$, which is a contradiction. Assume $f \in (e,a)$, then we know $G(f,f) > f$. Taking $x = z = f$ and $y = e$ in distributivity equation, then we have $f = G(f,e) = G(f,U(e,f)) = U(G(f,e),G(f,f)) = U(f,G(f,f)) = G(f,f)$, which is a contradiction. Assume $f \in (a,1)$, then we know $G(f,f) < f$ from Lemma 2.3. Taking $x = z = f$ and $y = 1$ in distributivity equation, then we have $f = G(f,1) = G(f,U(1,f)) = U(G(f,1),G(f,f)) = U(f,G(f,f)) = G(f,f)$, which is a contradiction. So we have $f = a$.

Next, let us prove that U is idempotent. From the structure of U, it follows that $U(e,e) \leqslant e$, then there must exist $t_1 \in [e,a)$ such that $U(t_1,t_1) = e$ by the continuity of T_U. So for any $x \in (0,e)$, we have $x = G(x,e) = G(x,U(t_1,t_1)) = U(G(x,t_1),G(x,t_1)) = U(x \wedge t_1, x \wedge t_1) = U(x,x)$. Thus, we obtain $U(e,e) = e$ by the continuity of T_U. Therefore, we have $x = G(x,e) = G(x,U(e,e)) = U(G(x,e),G(x,e)) = U(x,x)$ for any $x \in [0,a]$ and $x = G(x,1) = G(x,U(1,1)) = U(G(x,1),G(x,1)) = U(x,x)$ for any $x \in [a,1]$. That is, U is idempotent. And similar to the proof of Theorem 4.1, we have $U(x,y) = \min(x,y)$ for $(x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a]$. The converse statement is obvious from Lemma 2.15. □

Theorem 4.10. Let uni-nullnorm G be one of Eqs. (4) − (10) with $0 < e < a < 1$ and U be a uninorm with neutral element $f \neq e$. Then G is distributive over U if and only if $f = a$ and U has the form (15).

Proof. The proof is similar to the one of Theorem 4.9. □

Theorem 4.11. Let uni-nullnorm G be Eq.(3) with $0 < e < a < 1$ and U be a uninorm with neutral element $f \neq e$. Then U is not distributive over G.

Proof. Suppose U is distributive over G. If $f \in (0,e)$, then we have $x = U(x,f) = U(x,G(e,f)) = G(U(x,e),U(x,f)) = G(U(x,e),x)$ for $x \in (e,a)$. Assume there exists $x_0 \in (e,a)$ such that $U(x_0,e) \geqslant a$, then from the structure of G we have $x_0 = G(U(x_0,e),x_0) = a$, which is a contradiction. That is, $e \leqslant U(x,e) < a$ for $x \in (e,a)$. Thus, from the structure of the underlying t-conorm S_G of G, it follows that $G(x,x) > x = G(U(x,e),x)$ for any $x \in (e,a)$. So we obtain $U(x,e) < x$ for $x \in (e,a)$, which contradicts with the fact that $U(x,e) \geqslant x$ for $x \in (e,a)$ by the structure of U. If $f \in (e,a]$, then we have $x = U(x,f) = U(x,G(e,f)) = G(U(x,e),U(x,f)) = G(U(x,e),x)$ for $x \in (0,e)$. From the structure of the underlying t-norm T_G^{ll} of G, it follows that $G(x,x) < x = G(U(x,e),x)$ for any $x \in (0,e)$. So we obtain $U(x,e) > x$ for $x \in (0,e)$, which contradicts with the fact that $U(x,e) \leqslant x$ for $x \in (0,e)$ by the structure of U. If $f \in (a,1)$, then we have $x = U(x,f) = U(x,G(1,f)) = G(U(x,1),U(x,f)) = G(U(x,1),x)$ for $x \in (e,a)$. Assume there exists $x_0 \in (e,a)$ such that $U(x_0,1) \geqslant a$, then from the structure of G we have $x_0 = G(U(x_0,1),x_0) = a$, which is a contradiction. That is, $e < x \leqslant U(x,1) < a$ for $x \in (e,a)$. Thus, from the structure of the underlying t-conorm S_G of G, it follows that $G(x,x) > x = G(U(x,1),x)$ for any $x \in (e,a)$. So we obtain $U(x,1) < x$ for $x \in (e,a)$, which contradicts with the fact that $U(x,1) \geqslant x$ for $x \in (e,a)$ by the structure of U. □

Theorem 4.12. Let uni-nullnorm G be one of Eqs. (4) − (8) with $0 < e < a < 1$ and U be a uninorm with neutral element $f \neq e$. Then U is not distributive over G.

Proof. The proof is similar to the one of Theorem 4.11. □

Theorem 4.13. Let uni-nullnorm G be Eq.(9) with $0 < e < a < 1$ and U be a uninorm with neutral element $f \neq e$. Then U is not distributive over G.

Proof. Suppose U is distributive over G. If $f \in (0, e)$, then we have $x = U(x, f) = U(x, G(e, f)) = G(U(x, e), U(x, f)) = G(U(x, e), x)$ for $x \in (e, a)$. Assume there exists $x_0 \in (0, a)$ such that $U(x_0, e) \geqslant a$, then by the structure of G we have $x_0 = G(U(x_0, e), x_0) = a$, which is a contradiction. That is, $U(x, e) < a$ for $x \in (0, a)$. Thus, from the structure of the underlying representable uninorm U_G of G, it follows that $G(x, e) = x = G(U(x, e), x)$ for any $x \in (0, a)$. So we obtain $U(x, e) = e$ for $x \in (0, a)$, which contradicts with the fact that $U(x, e) \geqslant x > e$. And we can similarly obtain contradictions if $f \in (e, 1)$. □

Theorem 4.14. Let uni-nullnorm G be Eq.(10) with $0 < e < a < 1$ and U be a uninorm with neutral element $f \neq e$. Then U is not distributive over G.

Proof. The proof is similar to the one of Theorem 4.13. □

5. DISTRIBUTIVITY BETWEEN UNI-NULLNORMS AND NULLNORMS

From now on, $V \in \mathcal{V}_k$ will denote a nullnorm with a zero element $0 < k < 1$. Depending on whether the zero elements of G and V are same or not, there are two cases: distributivity between G and V with $k = a$, and distributivity between G and V with $k \neq a$.

5.1. Distributivity between G and V with $k = a$

In this section, we will investigate distributivity for G over V and V over G with $k = a$, respectively. Firstly, let us discuss the distributivity equation for uni-nullnorms over nullnorms.

Theorem 5.1. Let uni-nullnorm G be Eq.(3) with $0 < e < a < 1$ and V be a nullnorm with zero element $a \in (0, 1)$. If G is distributive over V, then $V(x, y) = \max(x, y)$ for $(x, y) \in [0, e)^2$ and $V(x, y) = \min(x, y)$ for $(x, y) \in [a, 1]^2$.

Proof. Taking $y = z = e$ and $x \in [0, e)$ in distributivity equation, then we have $x = x \wedge V(e, e) = G(x, V(e, e)) = V(G(x, e), G(x, e)) = V(x, x)$. And let $y = z = 1$ and $x \in [a, 1]$ in distributivity equation, we have $x = G(x, 1) = G(x, V(1, 1)) = V(G(x, 1), G(x, 1)) = V(x, x)$. That is, $V(x, y) = \max(x, y)$ for $(x, y) \in [0, e)^2$ and $V(x, y) = \min(x, y)$ for $(x, y) \in [a, 1]^2$. □

Theorem 5.2. Let uni-nullnorm G be Eq.(4) with $0 < e < a < 1$ and V be a nullnorm with zero element $a \in (0, 1)$. If G is distributive over V, then $e \leqslant V(e, e) < a$, and $V(x, y) = \max(x, y)$ for $(x, y) \in [0, e)^2$ and $V(x, y) = \min(x, y)$ for $(x, y) \in [a, 1]^2$. Especially, if $V(e, e) = e$, then G is distributive over V if and only if V is idempotent.

Proof. Based on the proof of Theorem 5.1, we only need to prove $V(e,e) < a$. Assume $V(e,e) = a$, then we have $a = G(0,a) = G(0,V(e,e)) = V(G(0,e),G(0,e)) = V(0,0) = 0$, which is a contradiction. □

Theorem 5.3. Let uni-nullnorm G be Eq.(5) with $0 < e < a < 1$ and V be a nullnorm with zero element $a \in (0,1)$. Then one of the following cases holds:

(i) If $V(e,e) = e$, then G is distributive over V if and only if V is idempotent.

(ii) If $e < V(e,e) < a$, and G is distributive over V, then $V(x,y) = \max(x,y)$ for $(x,y) \in [0,e)^2$ and $V(x,y) = \min(x,y)$ for $(x,y) \in [a,1]^2$.

(iii) If $V(e,e) = a$, then G is distributive over V if and only if the underlying t-norm T_G^{ll} is strict and the structure of V is as follows:

$$V(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [a,1]^2, \\ y & \text{if } (x,y) \in \{0\} \times [0,a], \\ x & \text{if } (x,y) \in [0,a] \times \{0\}, \\ a & \text{otherwise.} \end{cases} \tag{21}$$

Proof. We only prove (iii) here. For any $x \in (0,a]$, we can easily obtain $a = G(x,a) = G(x,V(e,e)) = V(G(x,e),G(x,e)) = V(x,x)$. So for any $x,y \in (0,a]$, we have $a = V(x \wedge y, x \wedge y) \leqslant V(x,y) \leqslant V(x \vee y, x \vee y) = a$. Now assume the underlying t-norm T_G^{ll} is nilpotent7, then there must exist $t \in (0,e)$ such that $G(t,t) = 0$. Thus, for any $x,y,z \in (0,t]$, we have $a = G(x,a) = G(x,V(y,z)) = V(G(x,y),G(x,z)) = V(0,0) = 0$, which is a contradiction.

Conversely, in order to verify that G is distributive over V, we need to consider the following cases by the commutativity of V.

1) When $y,z \in (0,a]$.

If $x > 0$, then $G(x,V(y,z)) = G(x,a) = a = V(G(x,y),G(x,z))$ by the structures of G and V.

If $x = 0$, then $G(x,V(y,z)) = G(0,a) = 0 = V(0,0) = V(G(0,y),G(0,z)) = V(G(x,y),G(x,z))$ by the structures of G and V.

2) When $y = 0$ and $z \in [0,a]$. It is obvious for any $x \in [0,1]$ that $G(x,V(0,z)) = G(x,z) = V(0,G(x,z)) = V(G(x,0),G(x,z))$ by the structures of G and V.

3) When $y \in [0,a]$ and $z \in [a,1]$.

If $x > 0$, then $G(x,V(y,z)) = G(x,a) = a = V(G(x,y),G(x,z))$ by the structures of G and V.

If $x = 0$, then $G(x,V(y,z)) = G(0,a) = 0 = V(0,0) = V(G(0,y),G(0,z)) = V(G(x,y),G(x,z))$ by the structures of G and V.

4) When $y,z \in [a,1]$. Without loss of generality, we assume $y \leqslant z$.

If $x > 0$, then $G(x,V(y,z)) = G(x,y \wedge z) = G(x,y) = G(x,y) \wedge G(x,z) = V(G(x,y), G(x,z))$ by the structures of G and V.

If $x = 0$, then $G(x,V(y,z)) = G(0,y) = 0 = V(0,0) = V(G(0,y),G(0,z)) = V(G(x,y),G(x,z))$ by the structures of G and V. □

Theorem 5.4. Let uni-nullnorm G be Eq.(6) with $0 < e < a < 1$ and V be a nullnorm with zero element $a \in (0, 1)$. Then G is distributive over V if and only if V is idempotent.

Proof. In order to prove that V is idempotent, we just need to prove $V(e, e) = e$. Firstly, we know that $e \leqslant V(e, e) \leqslant a$ by the structure of V. Assume $V(e, e) = a$, then we have $a = G(0, a) = G(0, V(e, e)) = V(G(0, e), G(0, e)) = V(0, 0) = 0$, which is a contradiction. Assume $e < V(e, e) < a$, then we have $V(e, e) = 0 \vee V(e, e) = G(0, V(e, e)) = V(G(0, e), G(0, e)) = V(0, 0) = 0$, which is a contradiction. So $V(e, e) = e$. □

Theorem 5.5. Let uni-nullnorm G be Eq.(7) with $0 < e < a < 1$ and V be a nullnorm with zero element $a \in (0, 1)$. If G is distributive over V, then $e \leqslant V(e, e) < a$. Especially,

(i) If $V(e, e) = e$, then G is distributive over V if and only if V is idempotent.

(ii) If $e < V(e, e) < a$, and G is distributive over V, then $V(x, y) = V(e, e)$ for $(x, y) \in (0, e]^2$ and $V(x, y) = \min(x, y)$ for $(x, y) \in [a, 1]^2$.

Proof. The proof is similar to the one of Theorem 5.1 and Theorem 5.2. □

Theorem 5.6. Let uni-nullnorm G be Eq.(8) with $0 < e < a < 1$ and V be a nullnorm with zero element $a \in (0, 1)$. Then one of the following cases holds:

(i) If $V(e, e) = e$, then G is distributive over V if and only if V is idempotent.

(ii) If $e < V(e, e) < a$, and G is distributive over V, then $V(x, y) = V(e, e)$ for $(x, y) \in (0, e]^2$ and $V(x, y) = \min(x, y)$ for $(x, y) \in [a, 1]^2$.

(iii) If $V(e, e) = a$, then G is distributive over V if and only if the underlying t-norm T_G^{ll} is strict and the structure of V is Eq. (21).

Proof. The proof is similar to the one of Theorem 5.3. □

Theorem 5.7. Let uni-nullnorm G be one of Eqs. (9),(10) with $0 < e < a < 1$ and V be a nullnorm with zero element $a \in (0, 1)$. If G is distributive over V, then $V(x, y) = \min(x, y)$ for $(x, y) \in [a, 1]^2$. Especially, if the underlying t-conorm S_V of V is continuous, then G is distributive over V if and only if one of the following cases holds:

(i) V is idempotent.

(ii) The structure of V is as follows:

$$V(x, y) = \begin{cases} aS_V(\frac{x}{a}, \frac{y}{a}) & \text{if } (x, y) \in [0, a]^2, \\ \min(x, y) & \text{if } (x, y) \in [a, 1]^2, \\ a & \text{otherwise}, \end{cases} \tag{22}$$

where the underlying t-conorm S_V of V is strict, and if s is the additive generator of S_V satisfying $s(\frac{e}{a}) = 1$, is also a multiplicative generator of U_G.

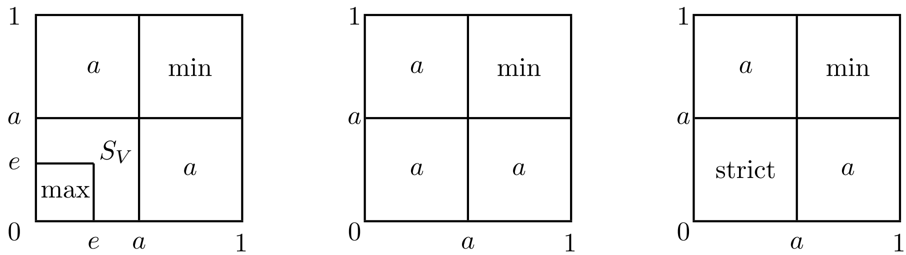

Fig. 3. (Left:) V in Theorem 5.3(ii); (middle:) V in Theorem 5.3(iii); (right:) V in Theorem 5.7(ii).

P r o o f. The proof is similar to the one of Theorem 3.2. □

Next we will discuss the distributivity for nullnorms over proper uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms.

Theorem 5.8. Let G be a proper uni-nullnorm with continuous Archimedean underlying t-norms T_G^{ll}, T_G^{ur} and t-conorm S_G, and V be a nullnorm. Then V is not distributive over G.

P r o o f. The proof is similar to the one of Theorem 3.3. □

5.2. Distributivity between G and V with $k \neq a$

In this section, we will investigate distributivity for G over V and V over G with $k \neq a$, respectively. Firstly, let us discuss the distributivity equation for uni-nullnorms over nullnorms.

Theorem 5.9. Let uni-nullnorm G be Eq.(3) with $0 < e < a < 1$ and V be a nullnorm with zero element $k \neq a$. Then G is not distributive over V.

P r o o f. Suppose G is distributive over V.

If $0 < k < e$, then we have $G(k,k) = G(k,V(0,e)) = V(G(k,0),G(k,e)) = V(0,k) = k$, which contradicts with $G(x,x) < x$ for all $x \in (0,e)$.

If $k = e$, then we have $a = G(1,e) = G(1,V(1,0)) = V(G(1,1),G(1,0)) = V(1,0) = e$, which is a contradiction.

If $e < k < a$, then we have $G(k,k) = G(k,V(a,e)) = V(G(k,a),G(k,e)) = V(a,k) = k$, which contradicts with $G(x,x) > x$ for all $x \in (e,a)$.

If $a < k < 1$, then we have $G(k,k) = G(k,V(a,1)) = V(G(k,a),G(k,1)) = V(a,k) = k$, which contradicts with $G(x,x) < x$ for all $x \in (a,1)$. □

Theorem 5.10. Let uni-nullnorm G be one of Eqs. (4)−(10) with $0 < e < a < 1$ and V be a nullnorm with zero element $k \neq a$. Then G is not distributive over V.

Proof. The proof is similar to the one of Theorem 5.9. □

Theorem 5.11. Let G be a proper uni-nullnorm with continuous Archimedean underlying t-norms T_G^{ll}, T_G^{ur} and t-conorm S_G, and V be a nullnorm with $k \neq a$. Then V is not distributive over G.

Proof. Suppose that V is distributive over G, then we have $x = V(x,1) = V(x, G(1,1)) = G(V(x,1), V(x,1)) = G(x,x)$ for all $x \in (k,1)$, which contradicts with the structure of the underlying t-norm T_G^{ur} of G. □

From the above Theorems, we can easily obtain the following conclusion.

Remark. Let G and F be two proper uni-nullnorms with continuous Archimedean underlying t-norms and t-conorm. Then G is not distributive over F.

6. CONCLUSION AND FURTHER WORK

In this paper, we investigated the distributivity between uni-nullnorms and some other binary operators, such as, continuous t-norms, continuous t-conorms, uninorms and nullnorms. From the above Theorems and corollaries, we know that there is no solution of distributivity equation for continuous t-norms (continuous t-conorms, uninorms or nullnorms) over uni-nullnorms, which have continuous Archimedean underlying t-norms and t-conorms. As for the distributivity for uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms over continuous t-norms, continuous t-conorms or uninorms, we have obtained full characterizations. But for the distributivity for uni-nullnorms over nullnorms, we obtained partial characterizations about nullnorms.

In the future work, we will consider characterizations of uni-nullnorms with continuous or left-continuous underlying t-norms and t-conorms. And we also want to investigate other functional equations about uni-nullnorms.

ACKNOWLEDGEMENT

The authors would like to thank the anonymous referees for their valuable comments and suggestions. This work was supported by the National Natural Science Foundation of China No. 61573211.

(Received September 25, 2017)

REFERENCES

[1] J. Aczél: Lectures on Functional Equations and Their Applications. Academia, New York 1966. DOI:10.1002/zamm.19670470321

[2] P. Akella: Structure of n-uninorms. Fuzzy Sets Syst. *158* (2007), 1631–1651. DOI:10.1016/j.fss.2007.02.015

[3] T. Calvo, B. De Baets, and J. C. Fodor: The functional equations of Frank and Alsina for uninorms and nullnorms. Fuzzy Sets Syst. *120* (2001), 385–394. DOI:10.1016/s0165-0114(99)00125-6

[4] B. De Baets: Idempotent uninorms. Eur. J. Oper. Res. *118* (1999), 631–642. DOI:10.1016/s0377-2217(98)00325-7

[5] P. Drygaś and E. Rak: Distributivity equation in the class of 2-uninorms. Fuzzy Sets Syst. *291* (2015), 82–97. DOI:10.1016/j.fss.2015.02.014

[6] W. Fechner, E. Rak, and L. Zedam: The modularity law in some classes of aggregation operators. Fuzzy Sets Syst. *332* (2018), 56–73. DOI:10.1016/j.fss.2017.03.010

[7] J. C. Fodor, R. R. Yager, and A. Rybalov: Structure of uninorms. Int. J. Uncertain. Fuzziness Knowl.-Based Syst. *5* (1997), 411–427. DOI:10.1142/s0218488597000312

[8] J. C. Fodor and B. De Baets: A single-point characterization of representable uninorms. Fuzzy Sets Syst. *202* (2012), 89–99. DOI:10.1016/j.fss.2011.12.001

[9] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Kluwer, Dordrecht 2000. DOI:10.1007/978-94-015-9540-7

[10] G. Li, H. W. Liu, and J. Fodor: Single-point characterization of uninorms with nilpotent underlying t-norm and t-conorm. Int. J. Uncertain. Fuzziness Knowl.-Based Syst. *22* (2014), 591–604. DOI:10.1142/s0218488514500299

[11] G. Li and H. W. Liu: Distributivity and conditional distributivity of a uninorm with continuous underlying operators over a continuous t-conorm. Fuzzy Sets Syst. *287* (2016), 154–171. DOI:10.1016/j.fss.2015.01.019

[12] M. Mas, G. Mayor, and J. Torrens: T-operators. Int. J. Uncertain. Fuzziness Knowl.-Based Syst. *7* (1999), 31–50. DOI:10.1142/s0218488599000039

[13] M. Mas, G. Mayor, and J. Torrens: The distributivity condition for uninorms and t-operators. Fuzzy Sets Syst. *128* (2002), 209–225. DOI:10.1016/s0165-0114(01)00123-3

[14] M. Mas, R. Mesiar, M. Monserat, and J. Torrens: Aggregation operations with annihilator. Internat, J. Gen. Syst. *34* (2015), 17–38. DOI:10.1080/03081070512331318347

[15] Y. M. Min and F. Qin: The distributivity for semi-nullnorms over 2-uninorms. (In Chinese.) J. Jiangxi Normal Univ. (Nature Science) *40* (2016), 3, 263267.

[16] F. Qin and B. Zhao: The distributive equations for idempotent uninorms and nullnorms. Fuzzy Sets Syst. *155* (2005), 446–458. DOI:10.1016/j.fss.2005.04.010

[17] F. Qin: Distributivity between semi-uninorms and semi-t-operators. Fuzzy Sets Syst. *299* (2015), 66–88. DOI:10.1016/j.fss.2015.10.012

[18] E. Rak: Distributivity equation for nullnorms. J. Electr. Eng. *56* (2005), 53–55. DOI:10.1109/t-aiee.1936.5057143

[19] E. Rak and P. Drygaś: Distributivity equation between uninorms. J. Electr. Engrg. *57* (2006), 35–38.

[20] E. Rak: Some remarks about distributivity equation between uninorms. J. Electr. Engrg. *58* (2007), 41–42.

[21] D. Ruiz-Aguilera and J. Torrens: Distributivity of strong implications over conjunctive and disjunctive uninorms. Kybernetika *42* (2006), 319–336.

[22] D. Ruiz and J. Torrens: Distributivity and conditional distributivity of a uninorm and a continuous t-conorm. IEEE Trans. Fuzzy Syst. *14* (2006), 180–190. DOI:10.1109/tfuzz.2005.864087

[23] Y. Su, W. Zong, and H. W. Liu: On distributivity equations for uninorms over semi-t-operators. Fuzzy Sets Syst. *287* (2015), 41–65. DOI:10.1016/j.fss.2015.08.001

[24] Y. Su, W. Zong, H. W. Liu, and P. Xue: On distributivity equations for semi-t-operators over uninorms. Fuzzy Sets Syst. *287* (2016), 172–183. DOI:10.1016/j.fss.2015.03.009

[25] Y. Su, H.W. Liu, D. Ruiz-Aguilera, J. Vicente Riera, and J. Torrens: On the distributivity property for uninorms. Fuzzy Sets Syst. *287* (2016), 184–202. DOI:10.1016/j.fss.2015.06.023

[26] F. Sun, X. P. Wang, and X. B. Qu: Uni-nullnorms and null-uninorms. J. Intell. Fuzzy Syst. *32* (2017), 1969–1981. DOI:10.3233/jifs-161495

[27] F. Sun, X. P. Wang, and X. B. Qu: Characterizations of uni-nullnorms with continuous Archimedean underlying t-norms and t-conorms. Fuzzy Sets Syst. *334* (2018), 24–35. DOI:10.1016/j.fss.2017.03.001

[28] Y. M. Wang and F. Qin: Distributivity for 2-uninorms over semi-uninorms. Int. J. Uncertain. Fuzziness Knowl.-Based Syst. *25* (2017), 317–345. DOI:10.1142/s0218488517500131

[29] A. F. Xie and H. W. Liu: On the distributivity of uninorms over nullnorms. Fuzzy Sets Syst. *211* (2013), 62–72. DOI:10.1016/j.fss.2012.05.008

[30] R. R. Yager and A. Rybalov: Uninorm aggregation operators. Fuzzy Sets Syst. *80* (1996), 111–120. DOI:10.1016/0165-0114(95)00133-6

[31] R. R. Yager: Uninorms in fuzzy system modeling. Fuzzy Sets Syst. *122* (2001), 167–175. DOI:10.1016/s0165-0114(00)00027-0

Ya-Ming Wang, School of Mathematics, Shandong University, Jinan, 250100. P. R. China.

e-mail: 623073044@qq.com

Hua-Wen Liu, School of Mathematics, Shandong University, Jinan, 250100. P. R. China.

e-mail: hw.liu@sdu.edu.cn