Haiying Li; Taotao Liu

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CONVEXITIES OF GAUSSIAN INTEGRAL MEANS AND WEIGHTED INTEGRAL MEANS FOR ANALYTIC FUNCTIONS

Haiying Li, Taotao Liu, Xinxiang

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Abstract. We first show that the Gaussian integral means of $f\colon \mathbb{C}\to\mathbb{C}$ (with respect to the area measure $\mathrm{e}^{-\alpha|z|^2}\,\mathrm{d}A(z)$) is a convex function of r on $(0,\infty)$ when $\alpha\leqslant 0$. We then prove that the weighted integral means $A_{\alpha,\beta}(f,r)$ and $L_{\alpha,\beta}(f,r)$ of the mixed area and the mixed length of $f(r\mathbb{D})$ and $\partial f(r\mathbb{D})$, respectively, also have the property of convexity in the case of $\alpha\leqslant 0$. Finally, we show with examples that the range $\alpha\leqslant 0$ is the best possible.

Keywords: Gaussian integral means; weighted integral means; analytic function; convexity

MSC 2010: 30H10, 30H20

1. Introduction

Let $\mathbb{D}$ represent a unit disk and $\mathrm{d}A$ denote the Euclidean area measure in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ stands for the space of holomorphic mappings $f\colon \mathbb{D}\to\mathbb{C}$, and $U(\mathbb{D})$ denotes univalent functions in $H(\mathbb{D})$. Recall that for any real number α and $0<r<1$, the weighted area measure is defined by

$$\mathrm{d}A_\alpha(z)=(1-|z|^2)^\alpha\,\mathrm{d}A(z),$$

where $\mathrm{d}A$ is the area measure of $\mathbb{D}$. Moreover, we already know that

$$r\mathbb{D}=\{z\in\mathbb{D}\colon\ |z|<r\},\quad r\mathbb{T}=\{z\in\mathbb{D}\colon\ |z|=r\}.$$

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For any real number α and $0 < p < \infty$ we define the Gaussian integral means of an analytic function $f\colon \mathbb{C} \to \mathbb{C}$ as

$$M_{p,\alpha}(f,r) = \frac{\int_{\{z\in\mathbb{C}\colon |z|\leqslant r\}} |f(z)|^p \mathrm{e}^{-\alpha|z|}\,\mathrm{d}A(z)}{\int_{\{z\in\mathbb{C}\colon |z|\leqslant r\}} \mathrm{e}^{-\alpha|z|^2}\,\mathrm{d}A(z)}, \quad r \in (0,\infty).$$

The above concept can be found in the theory of Fock spaces, e.g. see [2] and [12]. It is not hard to verify that the function $r \mapsto M_{p,\alpha}(f,r)$ strictly increases as $r \in (0,\infty)$ unless f is a constant. Readers can refer to [7] for more details.

In [11], Xiao and Zhu first introduced the notion of the integral means of an analytic function and discussed the area integral means of $f \in H(\mathbb{D})$:

$$\mathbb{M}_{p,\alpha}(f,r) = \frac{\int_{r\mathbb{D}} |f(z)|^p\,\mathrm{d}A_\alpha(z)}{\int_{r\mathbb{D}}\mathrm{d}A_\alpha(z)}, \quad 0 < p < \infty.$$

They proved that while $r \mapsto \mathbb{M}_{p,\alpha}(f,r)$ strictly increases unless f is a constant, it is different to the classical case in the sense that $\log \mathbb{M}_{p,\alpha}(f,r)$ is not always convex in $\log r$. Additionally, they proposed a conjecture where $\log r \mapsto \log \mathbb{M}_{p,\alpha}(f,r)$ is convex when $\alpha \leqslant 0$ and concave when $\alpha > 0$. In [9], Wang and Zhu obtained the result when $-3 \leqslant \alpha \leqslant 0$ and chose $p = 2$, $\alpha = 1$, $f(z) = 1 + z$ to verify that the conjecture is untrue. Subsequently, Wang, Xiao and Zhu got the conclusion when $-2 \leqslant \alpha \leqslant 0$ and $0 < p < \infty$ in [8]. Unfortunately, it is still unknown whether the conjecture is always true when $p \neq 2$. Inspired by previous research, Xiao and Xu discussed the fundamental case of $p = 1$ from a differential geometric viewpoint in their manuscript, see [10]. They also discussed monotonic growths and logarithmic convexities of the weighted integral means $A_{\alpha,\beta}(f,r)$ and $L_{\alpha,\beta}(f,r)$ of the mixed area and the mixed length of $f(r\mathbb{D})$ and $\partial f(r\mathbb{D})$ for the range $r \in [0,1)$.

At exactly the same time, the problem of Gaussian integral means was also studied. In [7], Wang and Xiao showed that the logarithmic convexity of function $M_{p,\alpha}(f,r)$ under the case of $f(z) = z^k$ is a monomial. Subsequently, the conclusions were improved. In [7], the case of an arbitrary analytic function f was considered.

Recently, Peng, Wang and Zhu investigated the (ordinary but not logarithmic) convexity of the area integral means of analytic functions in [6]. They claimed that for every $r \in [0,1)$ and when $p = 2$, the optimal range of $\mathbb{M}_{p,\alpha}(f,r)$ which is convex, is $\alpha \leqslant 0$.

Naturally, we can ask a fundamental question: When $p = 2$, are $M_{p,\alpha}(f,r)$, $A_{\alpha,\beta}(f,r)$ and $L_{\alpha,\beta}(f,r)$ convex functions? Indeed, we obtained the answer to the above question, which is the main result of this paper.

Theorem A. *Let $f\colon \mathbb{C}\to\mathbb{C}$ be an analytic function.*

(i) *If $\alpha\leqslant 0$, then $r\mapsto M_{2,\alpha}(f,r)$ is a convex function of r in the interval $(0,\infty)$.*

(ii) *If $\alpha>0$ and $k\geqslant 1$, then there exists some λ (depending on k and α) in the range $(0,\infty)$ such that $M_{2,\alpha}(z^k,r)$ is a convex function of r in the range $(0,\lambda)$ and a concave function of r in the interval (λ,∞).*

Furthermore, if we take $\lambda=\lambda(k,\alpha)$, the inflection point above, we have the following statements: for any fixed $\alpha>0$, $\lambda(k,\alpha)$ increases as k $(k\geqslant 1)$; for any fixed $k\geqslant 1$, $\lambda(k,\alpha)$ decreases as α $(\alpha>0)$. Based on Theorem A, we can easily see that the range $\alpha\leqslant 0$ is the best possible.

Theorem B. *Let $0\leqslant\beta\leqslant 1$ and $0<r<1$.*

(i) *If $\alpha\leqslant 0$, then $A_{\alpha,\beta}(f,r)$ is a convex function for all $f\in U(\mathbb{D})$. Furthermore, the range $\alpha\leqslant 0$ is the best possible.*

(ii) *If $\alpha\leqslant 0$, then $L_{\alpha,\beta}(f,r)$ is a convex function for all $f\in U(\mathbb{D})$. Furthermore, the range $\alpha\leqslant 0$ is the best possible.*

2. Preliminaries

For $f\in H(\mathbb{D})$ and $0<r<1$, we respectively define the integral means of the mixed area and the mixed length for $f(r\mathbb{D})$ and $\partial f(r\mathbb{D})$ as:

$$\Phi_A(f,r)=\frac{A(f,r)}{\pi r^2},\quad \Phi_L(f,r)=\frac{L(f,r)}{2\pi r},$$

where $A(f,t)$ and $L(f,t)$ denote the area of $f(r\mathbb{D})$ and the length of $\partial f(r\mathbb{D})$ with respect to the standard Euclidean metric on $\mathbb{C}$. Next, in the sense of isoperimetry, the mathematical expression

$$\Phi_A(f,t)=(\pi t^2)^{-1}\int_{t\mathbb{D}}|f'(z)|^2\,\mathrm{d}A(z)\leqslant\left[(2\pi t)^{-1}\int_{t\mathbb{T}}|f'(z)||\mathrm{d}z|\right]^2=[\Phi_L(f,t)]^2$$

holds. See [10].

Furthermore, we will use the following convention in the rest of this paper:

$$\mathrm{d}\mu_\alpha(t)=(1-t^2)^\alpha\,\mathrm{d}t^2,\quad v_\alpha(t)=\mu_\alpha([0,t])\quad\forall\, t\in(0,1),$$

and for $0\leqslant\beta\leqslant 1$ we define

$$\Phi_{A,\beta}(f,t)=\frac{A(f,t)}{(\pi t^2)^\beta},\quad \Phi_{L,\beta}(f,t)=\frac{L(f,t)}{(2\pi t)^\beta},$$

and

$$A_{\alpha,\beta}(f,r) = \frac{\int_0^r \Phi_{A,\beta}(f,t)\,\mathrm{d}\mu_\alpha(t)}{\int_0^r \mathrm{d}\mu_\alpha(t)}, \quad L_{\alpha,\beta}(f,r) = \frac{\int_0^r \Phi_{L,\beta}(f,t)\,\mathrm{d}\mu_\alpha(t)}{\int_0^r \mathrm{d}\mu_\alpha(t)},$$

which are called the weighted integral means of the mixed area and mixed length of $f(r\mathbb{D})$ and $\partial f(r\mathbb{D})$, respectively.

Recall that $M_p(f,r) = (2\pi)^{-1}\int_0^{2\pi} |f(\sqrt{r}\mathrm{e}^{\mathrm{i}\theta})|^p\,\mathrm{d}\theta$. If we write every analytic function $f\colon \mathbb{C}\to\mathbb{C}$ in the form of a power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

then we can immediately obtain that

$$M_2(f,r) = \sum_{k=0}^{\infty} |a_k|^2 r^k.$$

To simplify the notation, we will write

$$M = M(r) = M_2(f,r), \quad \varphi = \varphi(x) = \int_0^x \mathrm{e}^{-\alpha t}\,\mathrm{d}t, \quad H = H(x) = \int_0^x M(t)\mathrm{e}^{-\alpha t}\,\mathrm{d}t.$$

Note that φ and H depend on the parameter α, thus here and throughout the paper we will let $\partial\varphi/\partial\alpha$ and $\partial H/\partial\alpha$ denote the derivatives of φ and H with respect to α, respectively. In what follows, unspecified derivatives are taken with respect to the main variable x.

A calculation with polar coordinates gives

$$M_{2,\alpha}(f,r) = \frac{\int_0^{r^2} M_2(f,t)\mathrm{e}^{-\alpha t}\,\mathrm{d}t}{\int_0^{r^2} \mathrm{e}^{-\alpha t}\,\mathrm{d}t} = \frac{H(r^2)}{\varphi(r^2)}.$$

Using an elementary computation, we get the following formula:

$$\varphi(x) = \begin{cases} \dfrac{1-\mathrm{e}^{-\alpha x}}{\alpha}, & \alpha \neq 0, \\ x, & \alpha = 0. \end{cases}$$

Next, we also have:

$$\begin{cases} \varphi'(x) = \mathrm{e}^{-\alpha x}, \\ H'(x) = M(x)\varphi'(x), \\ M'(r) = \sum\limits_{k=0}^{\infty} (k+1)|a_{k+1}|^2 r^k \geqslant 0, \quad r \in (0,\infty), \\ M''(r) = \sum\limits_{k=0}^{\infty} (k+2)(k+1)|a_{k+2}|^2 r^k \geqslant 0, \quad r \in (0,\infty). \end{cases}$$

Throughout the paper, we use the notation $U \sim V$ to denote that U and V have the same sign, and employ the symbol $\equiv$ when a new notation is introduced. Finally, $\mathbb{N}$ is the set of all natural numbers.

3. Convexity for $M_{p,\alpha}(f,\cdot)$

3.1. The case $\alpha \leqslant 0$. In what follows, we investigate conditions for the function $M_{2,\alpha}(f,r)$ to be a convex function of r in the interval $(0,\infty)$. It is not hard to see that the convexity of the function $M_{2,\alpha}(f,\sqrt{r})$ depends on the sign of the weight parameter α, so we will first discuss the case of $\alpha \leqslant 0$. The following basic lemma is needed; it comes directly from [12] with $(0,1)$ being replaced by $(0,\infty)$.

Lemma 3.1. *Suppose $f(x)$ is twice differentiable on $(0,\infty)$. Then $f(x^2)$ is convex in the range $(0,\infty)$ if and only if $f'(x)+2xf''(x)$ is nonnegative on $(0,\infty)$. In particular, if $f(x)$ is nondecreasing and convex in the interval $(0,\infty)$, then $f(x^2)$ is convex on $(0,\infty)$.*

Proof. Let $g(x)=f(x^2)$, we easily have

$$g''(x) = 2[f'(x^2)+2x^2f''(x^2)].$$

Then the desired result follows. □

Lemma 3.2. *Suppose $\alpha > 0$, then the function*

$$E(x) = 4x\varphi'(x) - (1-2\alpha x)\varphi(x)$$

is strictly positive on $(0,\infty)$.

Proof. Take $x_0 = 1/2\alpha$, we can easily obtain that

$$1-2\alpha x \leqslant 0, \quad x\in[x_0,\infty),$$

which implies that $E(x)>0$ in the range $[x_0,\infty)$. For $x\in(0,x_0)$ we get

$$1-2\alpha x > 0$$

and

$$E(x) \sim \frac{4x\varphi'(x)}{1-2\alpha x} - \varphi(x) \equiv E_1(x).$$

It follows from direct computations that:

$$E_1'(x) = \frac{4(\varphi' + x\varphi'')(1-2\alpha x) + 8\alpha x\varphi'}{(1-2\alpha x)^2} - \varphi'$$
$$= \frac{4(\varphi' - \alpha x\varphi')(1-2\alpha x) + 8\alpha x\varphi' - (1-2\alpha x)^2\varphi'}{(1-2\alpha x)^2} = \frac{(4\alpha^2x^2+3)\varphi'}{(1-2\alpha x)^2} > 0.$$

Thus, $E(x) \sim E_1(x) > E_1(0) = 0$ on $(0, x_0)$. This completes the proof of the lemma. □

Lemma 3.3. *If* $\alpha > 0$, $k \geqslant 1$, $x \in (0,\infty)$ *and*

$$h = h(x) = \int_0^x t^k \mathrm{e}^{-\alpha t}\,\mathrm{d}t,$$

then the following statements hold:

(I) $g_1(x) := x^k\varphi(x) - h(x) > 0$,
(II) $g_2(x) := (\partial\varphi/\partial\alpha)(x) + x\varphi(x) > 0$,
(III) $g_3(x) := h(x)(\partial\varphi/\partial\alpha)(x) - (\partial h/\partial\alpha)(x)\varphi(x) > 0$,
(IV) $g_4(x) := -2\mathrm{e}^{-\alpha x}\big((\partial\varphi/\partial\alpha)(x) + x\varphi(x)\big) + \varphi^2(x) > 0$.

Proof. (I) Obviously,

$$h(x) = \int_0^x t^k \mathrm{e}^{-\alpha t}\,\mathrm{d}t \leqslant x^k \int_0^x \mathrm{e}^{-\alpha t}\,\mathrm{d}t = x^k\varphi(x),$$

which means $g_1(x) > 0$.

(II) It is not difficult to get

$$\frac{\partial^2\varphi}{\partial\alpha\partial x} = -x\mathrm{e}^{-\alpha x} = -x\varphi'(x), \qquad \frac{\partial^2 h}{\partial\alpha\partial x} = -x^{k+1}\mathrm{e}^{-\alpha x} = -x^{k+1}\varphi'(x),$$

and

$$g_2'(x) = \frac{\partial^2\varphi}{\partial\alpha\partial x} + \varphi(x) + x\varphi'(x) = -x\varphi'(x) + \varphi(x) + x\varphi'(x) = \varphi(x) > 0.$$

Thus $g_2(x) > g_2(0) = 0$, for which (II) holds.

(III) Based on the definition of $g_3(x)$ and several calculations we have:

$$g_3'(x) = h'(x)\frac{\partial\varphi}{\partial\alpha}(x) + h(x)\frac{\partial^2\varphi}{\partial\alpha\partial x} - \frac{\partial^2 h}{\partial\alpha\partial x}\varphi(x) - \frac{\partial h}{\partial\alpha}(x)\varphi'(x)$$
$$= \mathrm{e}^{-\alpha x}\Big[x^k\frac{\partial\varphi}{\partial\alpha}(x) - xh(x) + x^{k+1}\varphi(x) - \frac{\partial h}{\partial\alpha}(x)\Big]$$
$$= \mathrm{e}^{-\alpha x}\Big[x^k\Big(\frac{\partial\varphi}{\partial\alpha}(x) + x\varphi(x)\Big) - xh(x) - \frac{\partial h}{\partial\alpha}(x)\Big]$$
$$\sim x^k\Big(\frac{\partial\varphi}{\partial\alpha}(x) + x\varphi(x)\Big) - xh(x) - \frac{\partial h}{\partial\alpha}(x) \equiv g(x).$$

Note that

$$g'(x) = kx^{k-1}\Big(\frac{\partial\varphi}{\partial\alpha}(x) + x\varphi(x)\Big) + x^k\alpha(x) - h(x) - x^{k+1}\mathrm{e}^{-\alpha x} + x^{k+1}\mathrm{e}^{-\alpha x}$$

$$= kx^{k-1}\Big(\frac{\partial\varphi}{\partial\alpha}(x) + x\varphi(x)\Big) + x^k\alpha(x) - h(x).$$

Then $g'(x) > 0$ follows from (I) and (II). Hence $g_3'(x) \sim g(x) > g(0) = 0$, then $g_3(x) > g_3(0) = 0$, which proves (III).

(IV) It is easy to check that

$$g_4'(x) = 2\alpha\mathrm{e}^{-\alpha x}\Big(\frac{\partial\varphi}{\partial\alpha}(x) + x\varphi(x)\Big) > 0.$$

Thus $g_4(x) > g_4(0) = 0$, which means (IV) holds. □

Lemma 3.4. *Let $k \geqslant 1$ and $x \in (0, \infty)$. Then the function*

$$v(\alpha) = \Big(x^k - \frac{h(x)}{\varphi(x)}\Big)\Big(\frac{4x\varphi'(x)}{\varphi(x)} + 2\alpha x - 1\Big)$$

increases for $\alpha \in (0, \infty)$, where $h(x)$ is defined above.

Proof. In order to simplify the above formulae, we will represent $h(x)$ as h and $\varphi(x)$ as φ. It follows from direct computations that

$$\begin{aligned} v'(\alpha) &= \frac{1}{\varphi^2}\Big[h\frac{\partial\varphi}{\partial\alpha} - \frac{\partial h}{\partial\alpha}\varphi\Big]\Big[\frac{4x\varphi'}{\varphi} + 2\alpha x - 1\Big] \\ &\quad + \Big(x^k - \frac{h}{\varphi}\Big)\Big[\frac{-4x^2\mathrm{e}^{-\alpha x}\varphi - 4x\varphi'\partial\varphi/\partial\alpha}{\varphi^2} + 2x\Big] \\ &= \frac{x}{\varphi^3}\Big\{\frac{1}{x}\Big[h\frac{\partial\varphi}{\partial\alpha} - \frac{\partial h}{\partial\alpha}\varphi\Big]\Big[4x\varphi' - (1-2\alpha x)\varphi\Big] \\ &\quad + 2(x^k\varphi - h)\Big[-2\mathrm{e}^{-\alpha x}\Big(\frac{\partial\varphi}{\partial\alpha} + x\varphi\Big) + \varphi^2\Big]\Big\} \\ &\sim \frac{1}{x}\Big[h\frac{\partial\varphi}{\partial\alpha} - \frac{\partial h}{\partial\alpha}\varphi\Big][4x\varphi' - (1-2\alpha x)\varphi] \\ &\quad + 2(x^k\varphi - h)\Big[-2\mathrm{e}^{-\alpha x}\Big(\frac{\partial\varphi}{\partial\alpha} + x\varphi\Big) + \varphi^2\Big] \\ &= \frac{1}{x}g_3(x)E(x) + 2g_1(x)g_4(x) > 0. \end{aligned}$$

Then the desired result follows. □

Theorem 3.5. *Let $f\colon \mathbb{C} \to \mathbb{C}$ be an analytic function. If $\alpha \leqslant 0$, then both $M_{2,\alpha}(f,\sqrt{r})$ and $M_{2,\alpha}(f,r)$ are convex functions of r on $(0,\infty)$.*

Proof. Note that $M_{2,\alpha}(f,\sqrt{r}) = H(r)/\varphi(r)$, hence in order to prove the convexity of $M_{2,\alpha}(f,\sqrt{r})$ we just need to show that the function $H(x)/\varphi(x)$ is convex in the range $(0,\infty)$. In the following section, we also write h for $h(x)$, φ for $\varphi(x)$ and M for $M(x)$. These functions were defined previously. A basic calculation gives

$$\begin{aligned}
\Big(\frac{H}{\varphi}\Big)' &= \frac{H'\varphi - H\varphi'}{\varphi^2} = \frac{H'}{\varphi} - \frac{H\varphi'}{\varphi^2}.\\
\Big(\frac{H}{\varphi}\Big)'' &= \frac{H''\varphi - H'\varphi'}{\varphi^2} - \frac{(H'\varphi' + H\varphi'')\varphi^2 - H\varphi'(2\varphi\varphi')}{\varphi^4}\\
&= \frac{H''}{\varphi} - \frac{2H'\varphi'}{\varphi^2} - \frac{H\varphi''}{\varphi^2} + \frac{2H(\varphi')^2}{\varphi^3} = \frac{H''}{\varphi} - 2\frac{H'}{\varphi}\frac{\varphi'}{\varphi} + 2\frac{H}{\varphi}\Big(\frac{\varphi'}{\varphi}\Big)^2 - \frac{H}{\varphi}\frac{\varphi''}{\varphi}\\
&= \frac{M'\varphi' + M\varphi''}{\varphi} - 2\frac{M(\varphi')^2}{\varphi^2} + \Big(\frac{2(\varphi')^2}{\varphi^3} - \frac{\varphi''}{\varphi^2}\Big)H\\
&\sim M'\varphi'\varphi^2 + M\varphi''\varphi^2 - 2M(\varphi')^2\varphi + 2(\varphi')^2H - \varphi''\varphi H\\
&= M'\varphi'\varphi^2 + M(-\alpha\varphi')\varphi^2 - 2M(\varphi')^2\varphi + 2(\varphi')^2H - (-\alpha\varphi')\varphi H\\
&= \varphi'[M'\varphi^2 + (1+\varphi')(H - M\varphi)] \sim M'\varphi^2 + (1+\varphi')(H - M\varphi)\\
&= (1+\varphi')\Big[\frac{M'\varphi^2}{1+\varphi'} + H - M\varphi\Big] \sim \frac{M'\varphi^2}{1+\varphi'} + H - M\varphi \equiv \sigma(x).
\end{aligned}$$

Here we used the identity

$$\alpha\varphi = 1 - \varphi',$$

which is valid for all α including $\alpha = 0$.

Next, we will proceed to determine the sign of $\sigma(x)$ for the interval $(0,\infty)$. By a direct calculation we have:

$$\begin{aligned}
\sigma'(x) &= M''\frac{\varphi^2}{1+\varphi'} + M'\Big(\frac{\varphi^2}{1+\varphi'}\Big)' + M\varphi' - M\varphi' - M'\varphi\\
&= M''\frac{\varphi^2}{1+\varphi'} + M'\Big(\frac{\varphi^2}{1+\varphi'}\Big)' - M'\varphi \geqslant M'\Big[\Big(\frac{\varphi^2}{1+\varphi'}\Big)' - \varphi\Big]\\
&= \frac{M'\varphi}{(1+\varphi')^2}[2\varphi'(1+\varphi') - \varphi\varphi'' - (1+\varphi')^2]\\
&= \frac{M'\varphi}{(1+\varphi')^2}[(\varphi')^2 + \alpha\varphi\varphi' - 1] = \frac{-\alpha M'\varphi^2}{(1+\varphi')^2} \geqslant 0.
\end{aligned}$$

Thus $\sigma(x) \geqslant \sigma(0) = 0$, which means that $(H/\varphi)'' \geqslant 0$ holds for $\alpha \leqslant 0$ and $x \in (0,\infty)$. This proves that the function $M_{2,\alpha}(f,\sqrt{r})$ is convex for $r \in (0,\infty)$. Note that $M_{2,\alpha}(f,r)$ is increasing, then by Lemma 1 we can easily get that $M_{2,\alpha}(f,r)$ is also convex for $r \in (0,\infty)$. This completes the proof of Theorem 3.5. □

3.2. The case $\alpha > 0$. In the following section we use examples to show that $M_{2,\alpha}(f,r)$ is generally not a convex function of r for positive α. These examples actually reveal more delicate behaviour of $M_{2,\alpha}(f,r)$ when $\alpha > 0$.

Theorem 3.6. *Suppose $k \geqslant 1$ and $\alpha \geqslant 0$. Then there exists some $\lambda = \lambda(k,\alpha) \in (0,\infty)$ such that $M_{2,\alpha}(z^k,r)$ is a convex function of r on $(0,\lambda)$ and a concave function of r on (λ,∞). Furthermore, for any fixed α, $\lambda(k,\alpha)$ is increasing in k; and for any fixed k, $\lambda(k,\alpha)$ is decreasing in α.*

P r o o f. When $f(z) = z^k$, it follows that

$$M(t) = M_2(f,t) = \frac{1}{2\pi}\int_0^{2\pi} |(\sqrt{t}\mathrm{e}^{\mathrm{i}\theta})^k|^2 \,\mathrm{d}\theta = t^k,$$

thus

$$H(r) = \int_0^r M(t)\mathrm{e}^{-\alpha t}\,\mathrm{d}t = \int_0^r t^k\mathrm{e}^{-\alpha t}\,\mathrm{d}t = h(r).$$

Consequently,

$$M_{2,\alpha}(z^k,r) = \frac{H(r^2)}{\varphi(r^2)} = \frac{h(r^2)}{\varphi(r^2)}.$$

By Lemma 3.1, in order to prove the theorem, we only need to determine the sign of the function

$$\Delta(x) = \Big(\frac{h(x)}{\varphi(x)}\Big)' + 2x\Big(\frac{h(x)}{\varphi(x)}\Big)''$$

on $(0,\infty)$. Via a rewrite,

$$h = h(x,\alpha,k) = \int_0^x t^k\mathrm{e}^{-\alpha t}\,\mathrm{d}t$$

and

$$\varphi = \varphi(x) = \int_0^x \mathrm{e}^{-\alpha t}\,\mathrm{d}t.$$

By direct computations we have

$$\begin{aligned}
\Delta(x) &= \frac{h'}{\varphi} - \frac{h\varphi'}{\varphi^2} + 2x\Big[\frac{h''}{\varphi} - 2\frac{h'\varphi'}{\varphi^2} + 2\frac{h(\varphi')^2}{\varphi^3} - \frac{h\varphi''}{\varphi^2}\Big]\\
&= \frac{1}{\varphi^3}\big[h'\varphi^2 - h\varphi'\varphi + 2xh''\varphi^2 - 4xh'\varphi'\varphi + 4xh(\varphi')^2 - 2xh\varphi''\varphi\big]\\
&= \frac{1}{\varphi^3}\big[\varphi(h'\varphi + 2xh''\varphi - 4xh'\varphi') + h(4x(\varphi')^2 - 2x\varphi''\varphi - \varphi'\varphi)\big]\\
&\sim \varphi\mathrm{e}^{\alpha x}(h'\varphi + 2xh''\varphi - 4xh'\varphi') + h\mathrm{e}^{\alpha x}(4x(\varphi')^2 - 2x\varphi''\varphi - \varphi'\varphi)
\end{aligned}$$

$$
\begin{aligned}
&= \varphi e^{\alpha x}(\varphi x^k e^{-\alpha x} + 2\varphi k x^k e^{-\alpha x} - 2\alpha\varphi x^{k+1} e^{-\alpha x} - 4x^{k+1} e^{-2\alpha x}) \\
&\quad + h e^{\alpha x}(4x e^{-2\alpha x} + 2\alpha\varphi x e^{-\alpha x} - e^{-\alpha x}\varphi) \\
&= 2\varphi^2 k x^k + (4x e^{-\alpha x} + 2x\alpha\varphi - \varphi)(h - \varphi x^k) \\
&= 2kx^k\varphi^2 + [4x\varphi' - (1 - 2\alpha x)\varphi](h - \varphi x^k) \equiv \omega(x, \alpha, k).
\end{aligned}
$$

With the help of Lemma 3.2 we get

$$
\omega(x, \alpha, k) \sim \frac{2kx^k\varphi^2}{4x\varphi' - (1 - 2\alpha x)\varphi} + h - \varphi x^k \equiv \Delta_1(x).
$$

It is not hard to obtain that

$$
\begin{cases}
(2kx^k\varphi^2)' = 2k^2 x^{k-1}\varphi^2 + 4kx^k\varphi - 4\alpha k x^k \varphi^2, \\
4x\varphi' - (1 - 2\alpha x)\varphi = 4x - \varphi - 2\alpha x\varphi, \\
(4x\varphi' - (1 - 2\alpha x)\varphi)' = 3 - \alpha\varphi - 2\alpha x + 2\alpha^2 x\varphi, \\
(h - \varphi x^k)' = -kx^{k-1}\varphi.
\end{cases}
$$

Then

$$
\begin{aligned}
\Delta_1'(x) &= \frac{(2k^2 x^{k-1}\varphi^2 + 4kx^k\varphi - 4\alpha k x^k\varphi^2)(4x - \varphi - 2\alpha x\varphi)}{(4x - \varphi - 2\alpha x\varphi)^2} \\
&\quad - \frac{2kx^k\varphi^2(3 - \alpha\varphi - 2\alpha x + 2\alpha^2 x\varphi)}{(4x - \varphi - 2\alpha x\varphi)^2} - kx^{k-1}\varphi \\
&= \frac{kx^{k-1}\varphi^2[8kx - 2k\varphi - 4k\alpha x\varphi - 2x - 4\alpha x^2 + 2\alpha x\varphi - \varphi]}{(4x - \varphi - 2\alpha x\varphi)^2} \\
&\sim 8kx - 2k\varphi - 4k\alpha x\varphi - 2x - 4\alpha x^2 + 2\alpha x\varphi - \varphi \\
&= 4kx - 4\alpha x^2 - \frac{2k+1}{\alpha} + \Big(4kx - 2x + \frac{2k+1}{\alpha}\Big)e^{-\alpha x} \equiv \delta(x).
\end{aligned}
$$

To continue the calculation we have

$$
\delta'(x) = 4k - 8\alpha x + (2k - 3 - 4\alpha k x + 2\alpha x)e^{-\alpha x}.
$$

Note that

$$
\delta'(0) = 3(2k - 1) > 0, \quad \delta'(\infty) < 0,
$$

so there exists some $\lambda_1 \in (0, \infty)$ such that $\delta'(x) > 0$ on $(0, \lambda_1)$ and $\delta'(x) < 0$ on (λ_1, ∞). Since $\delta(0) = 0, \delta(\infty) < 0$, it follows that there exists a point $\lambda_2 \in (0, \infty)$ such $\delta(x) > 0$ for $x \in (0, \lambda_2)$ and $\delta(x) < 0$ for $x \in (\lambda_2, \infty)$. It is easy to see that

$$
\lim_{x\to 0^+} \Delta_1(x) = 0, \quad \lim_{x\to\infty} \Delta_1(x) < 0,
$$

with details deferred to after the proof. So there exists $\lambda \in (0,\infty)$ such $\Delta(x) > 0$ for $x \in (0,\lambda)$ and $\Delta(x) < 0$ for $x \in (\lambda,\infty)$. That is to say there exists some $\lambda = \lambda(k,\alpha) \in (0,\infty)$ such that $M_{2,\alpha}(z^k, r)$ is a convex function of r on $(0,\lambda)$ and a concave function of r on (λ,∞).

Take $\lambda = \lambda(\alpha, k)$ as a solution of equation

$$\omega(x,\alpha,k) = 0,$$

or equivalently, $\Delta(x) = 0$. For any $l > k$ we will proceed to determine the sign of

$$\omega(\lambda(\alpha,k),\alpha,l) = \omega(\lambda,\alpha,l).$$

Since

$$\omega(\lambda,\alpha,k) = 2k\lambda^k\varphi^2(\lambda,\alpha) + [4\lambda\varphi'(\lambda,\alpha) - (1-2\alpha\lambda)\varphi(\lambda,\alpha)](h(\lambda,\alpha,k) - \lambda^k\varphi(\lambda,\alpha)) = 0,$$

it follows that

$$4\lambda\varphi'(\lambda,\alpha) - (1-2\alpha\lambda)\varphi(\lambda,\alpha) = \frac{2k\lambda^k\varphi^2(\lambda,\alpha)}{\lambda^k\varphi(\lambda,\alpha) - h(\lambda,\alpha,k)}.$$

Thus, we can get

$$\begin{aligned}
\omega(\lambda,\alpha,l) &= 2l\lambda^l\varphi^2(\lambda,\alpha) + [4\lambda\varphi'(\lambda,\alpha) - (1-2\alpha\lambda)\varphi(\lambda,\alpha)](h(\lambda,\alpha,l) - \lambda^l\varphi(\lambda,\alpha)) \\
&= 2l\lambda^l\varphi^2(\lambda,\alpha) + \frac{2k\lambda^k\varphi^2(\lambda,\alpha)}{\lambda^k\varphi(\lambda,\alpha) - h(\lambda,\alpha,k)}(h(\lambda,\alpha,l) - \lambda^l\varphi(\lambda,\alpha)) \\
&= \frac{2k\lambda^k\varphi^2(\lambda,\alpha)}{\lambda^k\varphi(\lambda,\alpha) - h(\lambda,\alpha,k)}[l\lambda^{l-k}(\lambda^k\varphi(\lambda,\alpha) - h(\lambda,\alpha,k)) + k(h(\lambda,\alpha,l) - \lambda^l\varphi(\lambda,\alpha))] \\
&\sim l\lambda^{l-k}(\lambda^k\varphi(\lambda,\alpha) - h(\lambda,\alpha,k)) + k(h(\lambda,\alpha,l) - \lambda^l\varphi(\lambda,\alpha)) \\
&\equiv \omega_1(\lambda,\alpha,k,l).
\end{aligned}$$

Since

$$\frac{\partial\omega_1(\lambda,\alpha,k,l)}{\partial\lambda} = l(l-k)\lambda^{l-k-1}(\lambda^k\varphi(\lambda,\alpha) - h(\lambda,\alpha,k)) > 0,$$

we obtain

$$\omega(\lambda,\alpha,l) \sim \omega_1(\lambda,\alpha,k,l) > \omega_1(0,\alpha,k,l) = 0,$$

which implies that for any fixed α, λ is increasing in k.

Next, we are going to determine the sign of $\omega(\lambda(\alpha,k),\beta,k) = \omega(\lambda,\beta,k)$ for $\beta > \alpha$. Since

$$\begin{aligned}
\omega(\lambda,\alpha,k) &= 2k\lambda^k\varphi^2(\lambda,\alpha) \\
&\quad + [4\lambda\varphi'(\lambda,\alpha) - (1-2\alpha\lambda)\varphi(\lambda,\alpha)](h(\lambda,\alpha,k) - \lambda^k\varphi(\lambda,\alpha)) = 0,
\end{aligned}$$

it follows that

$$2k\lambda^k = \frac{1}{\varphi^2(\lambda,\alpha)}(\lambda^k\varphi(\lambda,\alpha) - h(\lambda,\alpha,k))[4\lambda\varphi'(\lambda,\alpha) - (1-2\alpha\lambda)\varphi(\lambda,\alpha)].$$

With the help of Lemma 3.4 and direct calculations, we have

$$\begin{aligned}
\omega(\lambda,\beta,k) &= 2k\lambda^k\varphi^2(\lambda,\beta) + [4\lambda\varphi'(\lambda,\beta) - (1-2\beta\lambda)\varphi(\lambda,\beta)](h(\lambda,\beta,k) - \lambda^k\varphi(\lambda,\beta)) \\
&= \varphi^2(\lambda,\beta)\Big\{\Big[\lambda^k - \frac{h(\lambda,\alpha,k)}{\varphi(\lambda,\alpha)}\Big]\Big[\frac{4\lambda\varphi'(\lambda,\alpha)}{\varphi(\lambda,\alpha)} + 2\alpha\lambda - 1\Big] \\
&\quad - \Big[\lambda^k - \frac{h(\lambda,\beta,k)}{\varphi(\lambda,\beta)}\Big]\Big[\frac{4\lambda\varphi'(\lambda,\beta)}{\varphi(\lambda,\beta)} + 2\beta\lambda - 1\Big]\Big\} \\
&\sim \Big[\lambda^k - \frac{h(\lambda,\alpha,k)}{\varphi(\lambda,\alpha)}\Big]\Big[\frac{4\lambda\varphi'(\lambda,\alpha)}{\varphi(\lambda,\alpha)} + 2\alpha\lambda - 1\Big] \\
&\quad - \Big[\lambda^k - \frac{h(\lambda,\beta,k)}{\varphi(\lambda,\beta)}\Big]\Big[\frac{4\lambda\varphi'(\lambda,\beta)}{\varphi(\lambda,\beta)} + 2\beta\lambda - 1\Big] < 0,
\end{aligned}$$

which implies that for any fixed k, λ is decreasing in α. This completes the proof of Theorem 3.6. □

Remark 3.7. In the proof of Theorem 3.6 we claimed that

$$\lim_{x\to 0^+}\Delta_1(x) = 0, \quad \lim_{x\to\infty}\Delta_1(x) < 0.$$

This is elementary but cumbersome, so we deferred the details here. Recall that

$$\Delta_1(x) = \frac{2kx^k\varphi^2}{4x\varphi' - (1-2\alpha x)\varphi} + h - \varphi x^k.$$

Then L'Hopital's rule gives us

$$\lim_{x\to 0^+}\frac{2kx^k\varphi^2}{4x\varphi' - (1-2\alpha x)\varphi} = \lim_{x\to 0^+}\frac{2k^2x^{k-1}\varphi^2 + 4kx^k\varphi - 4\alpha kx^k\varphi^2}{3 - \alpha\varphi - 2\alpha x + 2\alpha^2 x\varphi} = 0.$$

From the explicit formulae for h and φ we deduce that

$$\lim_{x\to 0^+} h = 0, \quad \lim_{x\to 0^+} x^k\varphi = 0.$$

Thus $\lim_{x\to 0^+}\Delta_1(x) = 0$.

Again with the help of L'Hopital's rule we obtain

$$\begin{aligned}
\lim_{x\to\infty}\frac{2kx^k\varphi^2}{4x\varphi' - (1-2\alpha x)\varphi} &= \lim_{x\to\infty}\frac{2k^2x^{k-1}\varphi^2 + 4kx^k\varphi - 4\alpha kx^k\varphi^2}{3 - \alpha\varphi - 2\alpha x + 2\alpha^2 x\varphi} \\
&= \lim_{x\to\infty}\frac{2k^2x^{k-2}\varphi[(k-1)\varphi + 2x - 2\alpha x\varphi] + 4kx^{k-1}(k\varphi - x - 2x\alpha)\varphi'}{-2\alpha + 2\alpha^2\varphi + (2\alpha^2 x - \alpha)\varphi'} \\
&= \lim_{x\to\infty}\frac{2k^2x^{k-2}\varphi[(k-1)\varphi + 2x\varphi'] + 4kx^{k-1}(k\varphi - x - 2x\alpha)\varphi'}{-2\alpha + 2\alpha^2\varphi + (2\alpha^2 x - \alpha)\varphi'} < 0.
\end{aligned}$$

The last inequality holds due to the fact that

$$\lim_{x\to\infty}\varphi=\frac{1}{\alpha},\quad \lim_{x\to\infty}\varphi'=0.$$

Moreover, Lemma 3.3 (I) states that $h-x^k\varphi<0$, hence $\lim\limits_{x\to\infty}\Delta_1(x)<0$.

4. Convexity for $A_{\alpha,\beta}(f,\cdot)$

In this section, we deal with the convexity of $A_{\alpha,\beta}(f,r)$. First, we consider the case when $f(z)=z^n$ is a monomial. For our purpose we need the following preliminary results, which come directly from [10].

Lemma 4.1. *Let* $-\infty<\alpha<\infty$, $0\leqslant\beta\leqslant 1$ *and* $f\in H(\mathbb{D})$. *Then* $r\mapsto A_{\alpha,\beta}(f,r)$ *strictly increases on* $(0,1)$ *unless*

$$f=\begin{cases}\text{constant} & \text{when } \beta<1,\\ \text{linear map} & \text{when } \beta=1.\end{cases}$$

Proposition 4.2. *Let* $0\leqslant\beta\leqslant 1$ *and* $0<r<1$. *If* $\alpha\leqslant 0$ *and* $n\in\mathbb{N}$, *then both* $A_{\alpha,\beta}(z^n,\sqrt{r})$ *and* $A_{\alpha,\beta}(z^n,r)$ *are convex functions on* $(0,1)$. *Consequently,* $A_{\alpha,\beta}(f,r)$ *is convex for all* $f\in U(\mathbb{D})$.

Proof. From [9] we know that $f_\lambda(x)=\int_0^x t^\lambda(1-t)^\alpha\,\mathrm{d}t$. Given $n\in\mathbb{N}$, a direct calculation gives $\Phi_{A,\beta}(z^n,t)=n\pi^{1-\beta}t^{2(n-\beta)}$, and by a change of variable we have

$$\begin{aligned}A_{\alpha,\beta}(z^n,r)&=\frac{\int_0^r\Phi_{A,\beta}(z^n,t)\,\mathrm{d}\mu_\alpha(t)}{\int_0^r\mathrm{d}\mu_\alpha(t)}=\frac{n\pi^{1-\beta}\int_0^{r^2}t^{n-\beta}(1-t)^\alpha\,\mathrm{d}t}{\int_0^{r^2}(1-t)^\alpha\,\mathrm{d}t}\\&=\frac{n\pi^{1-\beta}f_{(n-\beta)}(r^2)}{f_0(r^2)}.\end{aligned}$$

To prove the convexity of $A_{\alpha,\beta}(z^n,\sqrt{r})$ we just need to show that the function $F(x)/\psi(x)$ is convex on $(0,1)$. Here

$$F(x)=\int_0^x t^{n-\beta}(1-t)^\alpha\,\mathrm{d}t,\quad \psi(x)=\int_0^x(1-t)^\alpha\,\mathrm{d}t.$$

To simplify the displayed formulae we will write F for $F(x)$ and ψ for $\psi(x)$. Next, let $N=N(x):=x^{n-\beta}$, then $F'=N\psi'$. Obviously, both N' and N'' are nonnegative.

A basic calculation gives

$$\begin{aligned}\Big(\frac{F}{\psi}\Big)'' &= \frac{F''}{\psi} - \frac{2F'\psi'}{\psi^2} - \frac{F\psi''}{\psi^2} + \frac{2F(\psi')^2}{\psi^3}\\ &\sim \psi[N'\psi'\psi + N(\psi''\psi - 2(\psi')^2)] + (2(\psi')^2 - \psi''\psi)F\\ &\sim (1-x)N'\psi^2 + [2-(\alpha+2)\psi](F - N\psi).\end{aligned}$$

Then from the proof of Theorem 6 in [6] we find $A_{\alpha,\beta}(z^n, \sqrt{r})$ is convex for $r \in (0,1)$. Since $A_{\alpha,\beta}(z^n, r)$ is nondecreasing (see Lemma 4.1), which we combine with Lemma 3.1, we see that $A_{\alpha,\beta}(z^n, r)$ is also convex on $(0,1)$.

For $f \in U(\mathbb{D})$, writing $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ we can easily get that

$$\Phi_{A,\beta}(f(z), t) = (\pi t^2)^{-\beta} A(f,t) = \pi^{1-\beta} \sum_{n=0}^{\infty} n|a_n|^2 t^{2(n-\beta)},$$

whence

$$\begin{aligned}A_{\alpha,\beta}(f,r) &= \frac{\int_0^r \pi^{1-\beta} \sum_{n=0}^{\infty} n|a_n|^2 t^{2(n-\beta)}\,\mathrm{d}\mu_\alpha(t)}{\int_0^r \mathrm{d}\mu_\alpha(t)}\\ &= \frac{\sum_{n=0}^{\infty} |a_n|^2 \int_0^r (\pi t^2)^{-\beta} A(z^n, t)\,\mathrm{d}\mu_\alpha(t)}{\int_0^r \mathrm{d}\mu_\alpha(t)} = \sum_{n=0}^{\infty} |a_n|^2 A_{\alpha,\beta}(z^n, r).\end{aligned}$$

Therefore $A_{\alpha,\beta}(f,r)$ is also convex function on $(0,1)$ for all $f \in U(\mathbb{D})$. □

Proposition 4.3. *Let* $0 \leqslant \beta \leqslant 1$ *and* $0 < r < 1$, *and suppose* $\alpha > 0$. *Then there exists a positive integer* n *such that the function* $A_{\alpha,\beta}(z^n, r)$ *is not convex in the interval* $(0,1)$.

P r o o f. Note that

$$A_{\alpha,\beta}(z^n, r) = n\pi^{1-\beta} \frac{f_{(n-\beta)}(r^2)}{f_0(r^2)},$$

so by Lemma 3.1, in order to prove the conclusion, we need to determine the sign of the function

$$\Delta_2(x) = \Big(\frac{F(x)}{\psi(x)}\Big)' + 2x\Big(\frac{F(x)}{\psi(x)}\Big)''$$

for the range $(0,1)$. Let $\lambda = n - \beta \geqslant 1$, then $n \geqslant \beta + 1$. A rewrite results in

$$F = F(x,\alpha,\lambda) = \int_0^x t^\lambda (1-t)^\alpha\,\mathrm{d}t$$

and

$$\psi = \psi(x,\alpha) = \int_0^x (1-t)^\alpha \, \mathrm{d}t.$$

From the proof of Theorem 7 in [6] we know that there exists a unique point $x_0 \in (0,1)$ such that $\Delta_2(x) > 0$ for $x \in (0,x_0)$ and $\Delta_2(x) < 0$ for $x \in (x_0,1)$. We therefore find that $A_{\alpha,\beta}(z^n,r)$ is not convex on $(0,1)$ when $n \geqslant \beta+1$. □

Theorem 4.4. *Let $0 \leqslant \beta \leqslant 1$ and $0 < r < 1$. If $\alpha \leqslant 0$, then $A_{\alpha,\beta}(f,r)$ is a convex function for all $f \in U(\mathbb{D})$. Furthermore, the range $\alpha \leqslant 0$ is the best possible.*

P r o o f. The result directly follows from Proposition 4.2 and Proposition 4.3. □

We have proved that when $\alpha \leqslant 0$, $A_{\alpha,\beta}(f,r)$ is a convex function. Naturally, when $\alpha > 0$, is the function $A_{\alpha,\beta}(f,r)$ concave for the interval $(0,1)$? In fact, it is not. In what follows, for $\alpha > 0$ we give an example such that the function $A_{\alpha,\beta}(f,r)$ is neither convex nor concave on $(0,1)$. First, we need the following lemma:

Lemma 4.5. *Let $f \in H(\mathbb{D})$. Then f belongs to $U(\mathbb{D})$ provided that one of the following two conditions is valid:*

$$f(0) = f'(0) - 1 = 0 \quad \textit{and} \quad \left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1$$

(see [5] or [1]),

$$\left|\left[\frac{f''(z)}{f'(z)}\right]' - \frac{1}{2}\left[\frac{f''(z)}{f'(z)}\right]^2\right| \leqslant 2(1-|z|^2)^{-2} \quad \forall z \in \mathbb{D}$$

(see [4] or [3]).

Example 4.6. Let $\alpha = 1$, $\beta = 1$ and $f(z) = z + z^2/2$. Then function $A_{\alpha,\beta}(f,r)$ is neither convex nor concave for $r \in (0,1)$.

P r o o f. Since

$$|z| < 1 < 2 - |z| \leqslant |z+2| \quad \forall z \in \mathbb{D},$$
$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| = \left|\frac{z^2(1+z)}{(z+z^2/2)^2} - 1\right| = \frac{|z|^2}{|z+2|^2} < 1,$$

therefore $f \in U(\mathbb{D})$ due to Lemma 4.5. As $f'(z) = z+1$, we have

$$A(f,t) = \int_{t\mathbb{D}} |z+1|^2 \, \mathrm{d}A(z) = \pi\Big(t^2 + \frac{t^4}{2}\Big)$$

and

$$\int_0^r \Phi_{A,1}(f,t)\,\mathrm{d}\mu_1(t) = r^2 - \frac{r^4}{4} - \frac{r^6}{6}.$$

Meanwhile,

$$v_1(r) = \int_0^r (1-t^2)\,\mathrm{d}t^2 = r^2 - \frac{r^4}{2},$$

thus we get

$$A_{1,1}(f,r) = \frac{12 - 3r^2 - 2r^4}{6(2-r^2)} := P(r^2).$$

Hence we just need to consider the convexity of $P(x^2)$ on $(0,1)$. Note that

$$\Delta_2(x) = P'(x) + 2xP''(x) = \frac{Q(x)}{3(2-x)^3},$$

where $Q(x) = 6 - 15x + 6x^2 - x^3$.

Note that $Q'(x) = -15 + 12x - 3x^2$ is an open-downward parabola with its axis of symmetry about $x = 2 > 1$, so $Q'(x)$ increases on $(0,1)$ and thus $Q'(x) < Q'(1) = -6 < 0$, hence $Q(x)$ decreases on $(0,1)$. Since $Q(0) = 6 > 0$, $Q(1) = -4 < 0$, then there exists $x_0 \in (0,1)$ such that $Q(x) > 0$ for $x \in (0,x_0)$ and $Q(x) < 0$ for $x \in (x_0,1)$. Consequently, function $A_{\alpha,\beta}(f,r)$ is neither convex nor concave for $r \in (0,1)$. □

5. Convexity of $L_{\alpha,\beta}(f,\cdot)$

Analogously, we can obtain the following results for the mixed lengths, but in this section we need the following lemma from [10].

Lemma 5.1. *Let $-\infty < \alpha < \infty$, $0 \leqslant \beta \leqslant 1$ and $f \in U(\mathbb{D})$ or $f(z) = a_0 + a_n z^n$ with $n \in \mathbb{N}$. Then $r \mapsto L_{\alpha,\beta}(f,r)$ strictly increases in the interval $(0,1)$ unless*

$$f = \begin{cases} \text{constant} & \text{when } \beta < 1, \\ \text{linear map} & \text{when } \beta = 1. \end{cases}$$

Proposition 5.2. *Let $0 \leqslant \beta \leqslant 1$ and $0 < r < 1$. If $\alpha \leqslant 0$ and $n \in \mathbb{N}$, then both $L_{\alpha,\beta}(z^n, \sqrt{r})$ and $L_{\alpha,\beta}(z^n, r)$ are convex functions on $(0,1)$. Consequently, $L_{\alpha,\beta}(f,r)$ is convex for all $f \in U(\mathbb{D})$.*

P r o o f. The proof is similar to that of Proposition 4.2, except for the following statement: If $f \in U(\mathbb{D})$, then there exists $g(z) = \sum\limits_{n=0}^{\infty} b_n z^n$ such that g is the square root of the zero-free derivative f' on $\mathbb{D}$ and $f'(0) = g^2(0)$, and hence

$$\Phi_{L,\beta}(f,t) = (2\pi t)^{-\beta} \int_{t\mathbb{T}} |f'(z)||\mathrm{d}z| = (2\pi t)^{-\beta} \int_{t\mathbb{T}} |g(z)|^2 |\mathrm{d}z| = (2\pi t)^{1-\beta} \sum_{n=0}^{\infty} |b_n|^2 t^{2n}.$$

Thus, we have completed the proof. □

Similar to Proposition 4.3 and Theorem 4.4 we then have the following two results.

Proposition 5.3. *Let $0 \leqslant \beta \leqslant 1$ and $0 < r < 1$, and suppose $\alpha > 0$. Then there exists a positive integer n such that function $L_{\alpha,\beta}(z^n, r)$ is not convex on $(0,1)$.*

Theorem 5.4. *Let $0 \leqslant \beta \leqslant 1$ and $0 < r < 1$. If $\alpha \leqslant 0$, then $L_{\alpha,\beta}(f,r)$ is a convex function for all $f \in U(\mathbb{D})$. Furthermore, the range $\alpha \leqslant 0$ is the best possible.*

Next, we give an example to verify that when $\alpha > 0$, $L_{\alpha,\beta}(f,r)$ is neither convex nor concave for $r \in (0,1)$.

Example 5.5. Let $\alpha = 1$, $\beta = 0$ and $f(z) = (z+2)^3$. Then function $L_{\alpha,\beta}(f,r)$ is neither convex nor concave for $r \in (0,1)$.

P r o o f. Obviously, we can obtain that $f'(z) = 3(z+2)^2$ and $f''(z) = 6(z+2)$, thus

$$\Big[\frac{f''(z)}{f'(z)}\Big]' - \frac{1}{2}\Big[\frac{f''(z)}{f'(z)}\Big]^2 = -\frac{4}{(z+2)^2}.$$

It is not hard to see that

$$\sqrt{2}(1-|z|^2) \leqslant 2 - |z| \quad \forall z \in \mathbb{D}.$$

So,

$$\bigg|\Big[\frac{f''(z)}{f'(z)}\Big]' - \frac{1}{2}\Big[\frac{f''(z)}{f'(z)}\Big]^2\bigg| = \frac{4}{|z+2|^2} \leqslant \frac{4}{(2-|z|)^2} \leqslant \frac{2}{(1-|z|^2)^2}.$$

Then we know $f \in U(\mathbb{D})$ via Lemma 4.5. If we continue the computation, we have

$$L(f,t) = \int_0^{2\pi} |f'(t\mathrm{e}^{\mathrm{i}\theta})| t \,\mathrm{d}\theta = 6\pi t(t^2+4)$$

and

$$\int_0^r \Phi_{L,\beta}(f,t)\,\mathrm{d}\mu_1(t) = 12\pi\Big(\frac{4}{3}r^3 - \frac{3}{5}r^5 - \frac{1}{7}r^7\Big).$$

Combining this with $v_1(r) = r^2 - r^4/2$ we get

$$L_{1,\beta}(f,r) = \frac{24\pi(140r - 63r^3 - 15r^5)}{105(2-r^2)}.$$

For our purpose we just need to determine the convexity of the function

$$R(x) = \frac{140x - 63x^3 - 15x^5}{2-x^2}.$$

Note that

$$R'(x) = \frac{280 - 238x^2 - 77x^4 + 45x^6}{(2-x^2)^2}$$

and

$$R''(x) = \frac{6x(28 - 182x^2 + 90x^4 - 15x^6)}{(2-x^2)^3} = \frac{6xT(x)}{(2-x^2)^3},$$

where $T(x) = 28 - 182x^2 + 90x^4 - 15x^6$.

If we let $s = x^2$, then we get

$$T(x) = U(s) = 28 - 182s + 90s^2 - 15s^3.$$

Since $U'(s) = -182 + 180s - 45s^2$ is an open-downward parabola with its axis of symmetry being $s = 2 > 1$, we get $U'(s)$ increases in the interval $(0,1)$, whence $U'(s) < U'(1) = -47 < 0$. Therefore $U(s)$ decreases in the range $(0,1)$. Obviously, we also have the equalities

$$U(0) = 28, \quad U(1) = -79.$$

Summing up, we therefore find that there exists $s_0 \in (0,1)$ such that $U(s) > 0$ for $s \in (0, s_0)$ and $U(s) < 0$ for $s \in (s_0, 1)$. Then there exists $x_0 \in (0,1)$ such that $R''(x) > 0$ for $x \in (0, x_0)$ and $R''(x) < 0$ for $x \in (x_0, 1)$. Consequently, $L_{\alpha,\beta}(f,r)$ is neither convex nor concave on $(0,1)$. □

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Authors' address: Haiying Li, Taotao Liu, School of Mathematics and Information Science, Henan Normal University, 46# East of Construction Road, Xinxiang, Henan, 453007 P. R. China, e-mail: `haiyingli2012@yahoo.com`.