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Keivan Borna; Abolfazl Mohajer

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ON THE REGULARITY AND DEFECT SEQUENCE OF MONOMIAL AND BINOMIAL IDEALS

Keivan Borna, Tehran, Abolfazl Mohajer, Mainz

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Abstract. When S is a polynomial ring or more generally a standard graded algebra over a field K, with homogeneous maximal ideal $\mathfrak{m}$, it is known that for an ideal I of S, the regularity of powers of I becomes eventually a linear function, i.e., $\operatorname{reg}(I^m) = dm + e$ for $m \gg 0$ and some integers d, e. This motivates writing $\operatorname{reg}(I^m) = dm + e_m$ for every $m \geqslant 0$. The sequence e_m, called the *defect sequence* of the ideal I, is the subject of much research and its nature is still widely unexplored. We know that e_m is eventually constant. In this article, after proving various results about the regularity of monomial ideals and their powers, we give several bounds and restrictions on e_m and its first differences when I is a primary monomial ideal. Our theorems extend the previous results about $\mathfrak{m}$-primary ideals in the monomial case. We also use our results to obtatin information about the regularity of powers of a monomial ideal using its primary decomposition. Finally, we study another interesting phenomenon related to the defect sequence, namely that of regularity jump, where we give an infinite family of ideals with regularity jumps at the second power.

Keywords: Castelnuovo-Mumford regularity; powers of ideal; defect sequence

MSC 2010: 13D02, 13P10

1. Introduction

Let I be a homogeneous ideal in a polynomial ring $S = K[x_1, \ldots, x_n]$, K a field of characteristic zero. Castelnuovo-Mumford regularity, or simply regularity, together with the projective dimension are the most important invariants of a homogeneous ideal in a polynomial ring $K[x_1, \ldots, x_n]$ (or a closed subscheme of $\mathbb{P}^n$). It measures the extent of cohomological complexity of such an ideal. More generally, let M be a finitely generated graded S-module. Consider a minimal graded free resolution of M as follows:

$$\mathbb{F}\colon \ \ldots \to F_i \xrightarrow{\delta_i} F_{i-1} \xrightarrow{\delta_{i-1}} \ldots \to F_0 \xrightarrow{\delta_0} M.$$

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There exist integers a_{ij} such that $F_i = \sum S(-a_{ij})$. The regularity of M, denoted $\mathrm{reg}(M)$, is then defined to be the supremum of the numbers $a_{ij} - i$.

Another way of defining the regularity is through graded local cohomology modules $H^i_{\mathfrak{m}}(M)$, where $\mathfrak{m} = (x_1, \ldots, x_n)$ denotes the irrelevant maximal ideal of S. As this module is Artinian, one can define $\mathrm{end}(H^i_{\mathfrak{m}}(M))$ as the maximum integer k such that $H^i_{\mathfrak{m}}(M)_k \neq 0$. Then one can equivalently define

$$\mathrm{reg}(M) = \max\{\mathrm{end}(H^i_m(M)) + i\}.$$

For equivalent definitions and various algebro-geometric properties of the regularity we refer to [3].

In this paper we are mostly concerned with the case where $M = I$ is a homogeneous ideal in S. One interesting problem in this setting is to determine the regularity of powers I^m. In Section 2, we prove several results and relations about the regularity of primary monomial ideals and their powers. Using these results, we then prove theorems about the regularity of an ideal by using its primary decomposition. Moreover, the main results of [6] and [8] show that the regularity of powers of ideals behaves linearly, i.e., $\mathrm{reg}(I^m) = dm + e$ for $m \gg 0$. See also [2]. The coefficient d is the *asymptotic generating degree* of I, i.e. the minimal number such that I is integral over $I_{\leqslant d}$, with $I_{\leqslant d}$ denoting the ideal generated by the forms in I of degree at most d. This prompts writing $\mathrm{reg}(I^m) = dm + e_m$ for every $m \geqslant 0$. The coefficients e_m are called the *regularity defect sequence* and is more interesting and the above results can be taken to mean that e_m is eventually constant, i.e., $e_{m+1} = e_m$ for $m \gg 0$. It is therefore also interesting to study the first difference sequence $e_{m+1} - e_m$. In particular it is interesting to give bounds for these differences. Another objective of Section 2 is to prove several bounds and constraints for the defect sequence and the above differences in the case that I is a monomial ideal. Here we again use the primary decomposition in order to deduce results for more general monomial ideals. Some of our results are generalizations of that of [1] to the case of primary ideals. For example we prove a generalization of the stabilization of the defect sequence (e.g. Theorems 2.13 and 2.14) which was proven in [1] for $\mathfrak{m}$-primary ideals.

In [5] an interesting notion, namely that of *regularity jumps* has been defined. An ideal has regularity jump at the kth power if $\mathrm{reg}(I^k) - \mathrm{reg}(I^{k-1}) > d$. By definition of the defect sequence, this is equivalent to $e_k - e_{k-1} > 0$. In the same article the author mentions many new and known examples of ideals with this property. In Section 3 we consider the problem of ideals with regularity jumps and show that an infinite family of binomial ideals I_n for $n \geqslant 3$ have regularity jump at $k = 2$. The ideals I_n define Cohen-Macaulay rings of minimal multiplicity indicating that even among such ideals one can find examples whose squares do not have linear resolution.

The ideal I_3 has been shown in [5] to have such a regularity jump by declaring the existence of a nonlinear second syzygy. Our contribution here is to show that for all $n \geqslant 3$ the ideal I_n^2 has regularity strictly greater than 4. We achieve this by local cohomological methods.

2. Regularity of monomial ideals and their powers

In this section, we first prove some results about regularity of primary monomial ideals in $S = K[x_1, \ldots, x_n]$ and their powers and then apply them to get some information about the defect sequence. Let us introduce some notation that we are going to use throughout the whole paper.

Some notation. As in the introduction, $S = K[x_1, \ldots, x_n]$ with K a field of characteristic zero. If I is a monomial ideal, we denote by $\Lambda(I)$ the set of indeterminates that appear in the minimal set of monomials generating I. We also denote the highest degree of a minimal monomial generator of I by $\mu(I)$ and the maximum power of pure generators by $\nu(I)$. The set of minimal generators of I is denoted by $\mathrm{min.gen}(I)$.

Lemma 2.1. *Let $\mathfrak{p}$ be a prime monomial ideal in $S = K[x_1, \ldots, x_n]$.*

(i) *$\mathfrak{p}$ is generated by a subset of variables, i.e., it is of the form $(x_{i_1}, \ldots, x_{i_l})$ for $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, n\}$.*

(ii) *If I is a $\mathfrak{p}$-primary monomial ideal, then $I = (x_{i_1}^{a_1}, \ldots, x_{i_l}^{a_l}, h_1, \ldots, h_r)$, where h_i are monomials which contain only variables x_{i_j} for $1 \leqslant j \leqslant l$.*

Proof. (i) This is well-known and easy to verify.

(ii) If I is $\mathfrak{p}$-primary then $\sqrt{I} = \mathfrak{p} = (x_{i_1}, \ldots, x_{i_l})$ by (i). So $x_{i_j}^{a_j}$ lies in I for some power of x_{i_j}. The assumption that I is $\mathfrak{p}$-primary then implies that the nonpure minimal generators of I can only contain indeterminates among x_{i_j} for $1 \leqslant j \leqslant l$. $\square$

Remark 2.2. (i) Let J be an $\mathfrak{m}$-primary ideal so that S/J is artinian. Then $\operatorname{reg}(S/J) = \max\{i\colon (S/J)_i \neq 0\}$. In other words, $\operatorname{reg}(S/J)$ is the maximal degree of homogeneous elements of S/J. In this case, we denote by $\eta(J)$ the set of homogeneous polynomials of maximal degree of S/J. The set of nonzero elements of the socle of S/J will be denoted by $s(J)$ (see [1]). Note also that $\operatorname{reg}(J) = \operatorname{reg}(S/J) + 1$.

(ii) Let I be a homogeneous ideal and let $u \in S$ be a nonzero divisor on S/I. Then $\operatorname{reg}(S/(I + uS)) = \operatorname{reg}(S/I) + \deg(u) - 1$. In particular, if u is a linear nonzero divisor of S/I, then $\operatorname{reg}(S/(I + uS)) = \operatorname{reg}(S/I)$.

In Section 3, we will need the following definition. If m is a monomial, then we will denote the highest index of a variable dividing m by $t(m)$ and the highest power of $x_{t(m)}$ dividing m by $l(m)$.

Definition 2.3. A monomial ideal is called weakly stable, if for every monomial $m \in I$, and for every $j < t(m)$, there exists an integer power r such that $x_j^r m / x_{l(m)}^{t(m)} \in I$.

The following remark describes the behavior of regularity in exact sequences and can be proven by taking the associated long exact local cohomology sequence.

Remark 2.4. Consider the following exact sequence of finitely generated graded S-modules $0 \to M_1 \to M_2 \to M_3 \to 0$. Then

(i) $\operatorname{reg} M_2 \leqslant \max\{\operatorname{reg} M_1, \operatorname{reg} M_3\}$.

(ii) $\operatorname{reg} M_1 \leqslant \max\{\operatorname{reg} M_2, \operatorname{reg} M_3 + 1\}$.

(iii) $\operatorname{reg} M_3 \leqslant \max\{\operatorname{reg} M_1 - 1, \operatorname{reg} M_2\}$.

iv) If $\operatorname{reg} M_1 > \operatorname{reg} M_2$, then $\operatorname{reg} M_3 = \operatorname{reg} M_1 - 1$.

(v) If $\operatorname{reg} M_2 > \operatorname{reg} M_1$, then $\operatorname{reg} M_3 = \operatorname{reg} M_2$.

(vi) If $\operatorname{reg} M_3 > \operatorname{reg} M_2$, then $\operatorname{reg} M_1 = \operatorname{reg} M_3 + 1$.

With the above notation and remarks, we begin with our first result.

Proposition 2.5. (i) *Let $J \subseteq I$ be monomial ideals such that I is $\mathfrak{p}$-primary and J is $\mathfrak{q}$-primary. Then $\operatorname{reg}(I^m) \leqslant \operatorname{reg}(J^m) + t$ for every $m \in \mathbb{N}$, for $t = m \sum\limits_{j=1}^{k} a_{i_j} - k$, where $x_{i_1}^{a_{i_1}}, \ldots, x_{i_k}^{a_{i_k}}$ are the pure powers in $\operatorname{min.gen}(I)$ of variables in $\Lambda(I) \setminus \Lambda(J)$.*

(ii) *Let $J \subseteq I$ be $\mathfrak{p}$-primary monomial ideals. Then $\operatorname{reg}(I^m) \leqslant \operatorname{reg}(J^m)$ for every $m \in \mathbb{N}$.*

P r o o f. (i) Since $J \subseteq I$ it follows that $\Lambda(J) \subseteq \Lambda(I)$. We may assume, without loss of generality, that $\Lambda(J) = \{x_1, \ldots, x_l\}$ and $\Lambda(I) = \{x_1, \ldots, x_l, x_{l+1}, \ldots, x_s\}$. Let $x_{l+1}^{a_{l+1}}, \ldots, x_s^{a_s}$ be the pure power generators of I that are not in J, i.e., powers of variables in $\Lambda(I) \setminus \Lambda(J)$ as indicated above. Then

$$(x_{l+1}^{ma_{l+1}}, \ldots, x_s^{ma_s}, x_{s+1}, \ldots, x_n)$$

is a regular sequence for S/J^m and $(x_{s+1}, \ldots, x_n)$ is a regular sequence for S/I^m. Set $J_m = J^m + (x_{l+1}^{ma_{l+1}}, \ldots, x_s^{ma_s}, x_{s+1}, \ldots, x_n)$ and $I_m = I^m + (x_{s+1}, \ldots, x_n)$. Then J_m and I_m are $\mathfrak{m}$-primary and it follows from Remark 2.2 (ii) that $\operatorname{reg}(I_m) = \operatorname{reg}(I^m)$ and $\operatorname{reg}(S/J_m) = \operatorname{reg}(S/J^m) + \sum\limits_{l+1}^{s} (ma_i - 1)$. Moreover, $J_m \subseteq I_m$ and hence $\eta(I_m) \cap J_m = \emptyset$, where $\eta(I_m)$ is as in Remark 2.2 (i). It follows that each element of $\eta(I_m)$ is nonzero in S/J_m and therefore $\operatorname{reg}(I_m) \leqslant \operatorname{reg}(J_m)$. Alternatively, one can argue (notationally easier) that $S/J_m \twoheadrightarrow S/I_m$ and hence $\operatorname{end}(S/I_m) \leqslant \operatorname{end}(S/J_m)$ and the claim follows.

(ii) Since both I and J are $\mathfrak{p}$-primary, it follows that $\Lambda(I) = \Lambda(J)$. So (ii) is a special case of (i) where $s = l$ and $t = 0$. $\square$

Proposition 2.6. *Let I be a $\mathfrak{p}$-primary monomial ideal. Then $\operatorname{reg}(I^{m+1}) > \operatorname{reg}(I^m)$ for every $m > 0$. In other words, $e_{m+1} - e_m > -d$.*

P r o o f. Applying Proposition 2.5 (ii) to the pair $I^{m+1} \subset I^m$, we see that $\operatorname{reg}(I^{m+1}) \geqslant \operatorname{reg}(I^m)$. We may assume that $\Lambda(I) = \{x_1, \ldots, x_s\}$ and define $I_m = I^m + (x_{s+1}, \ldots, x_n)$. It is $\mathfrak{m}$-primary and $\operatorname{reg}(I_m) = \operatorname{reg}(I^m)$ by Remark 2.2 (i). Let $f \in \eta(I_m)$. Since $x_{s+1} = \ldots = x_n = 0$ in S/I_m, we may assume that $f \in K[x_1, \ldots, x_s]$. It follows that $x_j f \in I_m$ for $1 \leqslant j \leqslant s$ or equivalently that $x_j f \in I^m$. This implies that $f \notin \eta(I_{m+1})$ and since by assumption $f \notin I_{m+1}$, we conslude that $\operatorname{reg}(I_m) < \operatorname{reg}(I_{m+1})$. $\square$

Example (Conca, [7]). The assumption that the monomial ideal I is $\mathfrak{p}$-primary is necessary in Proposition 2.6. Indeed, in $S = K[x_1, x_2, x_3]$, consider the monomial ideal $I = (x_1^4, x_2^4, x_1^3x_2, x_1x_2^3, x_1^2x_2^2x_3^5)$. Then $7 = \operatorname{reg}(I^2) < \operatorname{reg}(I) = 8$. In this example even $\sqrt{I} = (x_1, x_2)$. However, I is not (x_1, x_2)-primary.

Corollary 2.7. *Let I be a primary monomial ideal. Then $\operatorname{reg}(I^{n-1}/I^n) = \operatorname{reg}(I^n) - 1$ for every $n > 0$.*

P r o o f. Consider the exact sequence

$$0 \to I^n \to I^{n-1} \to I^{n-1}/I^n \to 0.$$

By Proposition 2.6, $\operatorname{reg}(I^n) > \operatorname{reg}(I^{n-1})$ for every $n > 0$. The claim follows from Remark 2.4 (iv) applied to the above exact sequence. $\square$

The following corollary, which is interesting in its own right, will be useful later on in the paper.

Corollary 2.8. *Let I and J be primary monomial ideals such that $\Lambda(I) \cap \Lambda(J) = \emptyset$. Then $\operatorname{reg}(I+J)^n = \max\limits_{i\in[0,n-1]}\{\operatorname{reg}(I^{i+1}) + \operatorname{reg}(J^{n-i})\} - 1$ for every $n > 0$.*

P r o o f. If $n = 1$, then one checks that $I + J$ is also primary and if $f \in \eta(I)$ and $g \in \eta(J)$, then $fg \in \eta(I + J)$. Consequently, $\operatorname{reg}(I + J) = \operatorname{reg}(I) + \operatorname{reg}(J) - 1$. If $n > 1$, By [7], Theorem 3.3,

$$\operatorname{reg}((I+J)^{n-1}/(I+J)^n) = \max_{i\in[0,n-1]}\{\operatorname{reg}(I^i/I^{i+1}) + \operatorname{reg}(J^{n-i-1}/J^{n-i})\}.$$

By Corollary 2.7, $\operatorname{reg}(I^i/I^{i+1}) = \operatorname{reg}(I^{i+1}) - 1$ and similarly, $\operatorname{reg}(J^{n-i-1}/J^{n-i}) = \operatorname{reg}(J^{n-i}) - 1$. Finally, note that the ideal $(I+J)^n$ is also primary and hence $\operatorname{reg}((I+J)^{n-1}/(I+J)^n) = \operatorname{reg}((I+J)^n) - 1$, which completes the proof. $\square$

Next we prove a result on the regularity of powers of a monomial ideal using its primary decomposition. Before this, we need a lemma.

Lemma 2.9. *Let I and J be monomial ideals such that $\Lambda(I) \cap \Lambda(J) = \emptyset$. Then*
(i) *$IJ = I \cap J$.*
(ii) *If I and J are primary ideals, then $\operatorname{reg}(IJ) = \operatorname{reg}(I) + \operatorname{reg}(J)$.*

Proof. (i) Note that $I \cap J$ is generated by $c_{ij} = \operatorname{lcm}(f_i, g_j)$ where the f_i and g_j are the monomial generators of I and J. The assumption $\Lambda(I) \cap \Lambda(J) = \emptyset$ implies that $c_{ij} = f_i g_j \in IJ$, respectively. So every generator of $I \cap J$ is contained in IJ. This implies that $I \cap J \subseteq IJ$. As the other inclusion is automatic, one concludes (i).
(ii) If I and J are primary ideals, then consider the short exact sequence

$$0 \to I \cap J \to I \oplus J \to I + J \to 0.$$

By Corollary 2.8, it follows that $\operatorname{reg}(I+J) > \max\{\operatorname{reg}(I), \operatorname{reg}(J)\} = \operatorname{reg}(I \oplus J)$. Now Remark 2.4 (vi) together with Corollary 2.8 gives that $\operatorname{reg}(I \cap J) = \operatorname{reg}(I+J) + 1 = \operatorname{reg}(I) + \operatorname{reg}(J)$. Now use part (i). $\square$

Recall from the introduction that if I is a homogeneous ideal in a polynomial ring S, then we set $e_m = \operatorname{reg}(I^m) - dm$. It is known by [6], and independently [8], that e_m is eventually constant. Of particular interest is to determine when this sequence becomes stationary, i.e., to determine $l \in \mathbb{N}$ such that $e_{m+1} = e_m$ for $m \geqslant l$. Therefore it is interesting to get bounds for the differences $e_{m+1} - e_m$.

Theorem 2.10. *Let I be a monomial ideal with primary decomposition $I = \bigcap\limits_1^l Q_j$ where Q_j is $\mathfrak{p}_j$-primary and such that $\Lambda(Q_i) \cap \Lambda(Q_j) = \emptyset$ for $i \neq j$. Also, let f_m^j be the first difference of the defect sequence of Q_j, i.e., $f_m^j = e_{j,m} - e_{j,m-1}$, where $e_{j,m}$ is the defect sequence of the ideal Q_j. Then*
(i) *$\operatorname{reg}(I^m) > \operatorname{reg}(I^{m-1})$ for every $m > 0$,*
(ii) *$e_m - e_{m-1} = \sum\limits_{j=1}^{l} f_m^j + \sum\limits_{j=1}^{l} \nu(Q_j) - d$.*

Proof. (i) By our assumptions, one deduces from a generalization of Lemma 2.9 to arbitrary finite number of ideals that

$$I^m = (Q_1 \ldots Q_l)^m = Q_1^m \cap \ldots \cap Q_l^m \quad \text{for every } m > 0.$$

Note that Q_j^m is also a primary ideal by Lemma 2.1. By Proposition 2.6 we have that $\operatorname{reg}(Q_j^m) > \operatorname{reg}(Q_j^{m-1})$ for every j. Now, Lemma 2.9 (ii) gives that $\operatorname{reg}\Bigl(\bigcap\limits_{j=1}^{l} Q_j^m\Bigr) = \sum\limits_{j=1}^{l} \operatorname{reg}(Q_j^m)$ for $m > 0$. This implies in particular the above claim.

(ii) By the arguments of part (i), $\operatorname{reg}(I^m) = \operatorname{reg}\Bigl(\bigcap\limits_{j=1}^{l} Q_j^m\Bigr) = \sum\limits_{j=1}^{l} \operatorname{reg}(Q_j^m)$. It then follows that

$$(e_m - e_{m-1}) + d = \operatorname{reg}(I^m) - \operatorname{reg}(I^{m-1}) = \sum(\operatorname{reg}(Q_j^m) - \operatorname{reg}(Q_j^{m-1})) = \sum f_m^j + \sum d_j$$

where d_j is the asymptotic generating degree of Q_j. Now, by the description of the ideals Q_j in Lemma 2.1, the asymptotic generating degree of Q_j is equal to the greatest degree of the minimal pure generators of Q_j, i.e., $\nu(Q_j)$. $\square$

The following proposition gives bounds for the defect sequence of I in terms of the defect sequence of I_1 defined in the proof of Proposition 2.5, which is $\mathfrak{m}$-primary.

Proposition 2.11. *Let $\{e_m\}$ and $\{e'_m\}$ be the defect sequences of the $\mathfrak{p}$-primary ideal I and $\mathfrak{m}$-primary ideal I_1 respectively. We have $e_m \leqslant e'_m < e_m + m$.*

Proof. Set $I_1 = I + (x_{s+1}, \ldots, x_n)$ as before. By [7], Proposition 2.9 and Theorem 2.4 respectively, it holds that

$$\operatorname{reg}(I^m) \leqslant \operatorname{reg}(I_1^m) \leqslant \max_{\substack{i \in [1, m-1] \\ j \in [1, m]}} \{\operatorname{reg}(I^{m-i} + i), \operatorname{reg}(I^{m-j+1} + j)\}.$$

By Proposition 2.6, this maximum is strictly less than $\operatorname{reg}(I^m) + m$. Now bearing in mind that the asymptotic generating degrees of I and I_1 are equal, we set $\operatorname{reg}(I^m) = dm + e_m$ and $\operatorname{reg}(I_1^m) = dm + e'_m$ to get the claimed inequalities. $\square$

The following results generalize the analogous results for $\mathfrak{m}$-primary ideals proved in [1] for primary ideals in the monomial case.

Theorem 2.12. *Let $J \subseteq I$ be such that J is a homogeneous ideal and I is a primary monomial ideal with asymptotic generating degree d. Let $I_m = I^m + (x_i : x_i \notin \Lambda(I))$. Write $\operatorname{reg}(I^m) = dm + e_m$ and let c be the maximal degree of a minimal generator of J. If $J \cap \eta(I_m) \neq \emptyset$, then $e_m - e_{m-1} \leqslant c - d$.*

Proof. We may assume that $\Lambda(I) = \{x_1, \ldots, x_s\}$. Take $f \in J \cap \eta(I_m)$, then $\deg(f) = \operatorname{reg}(S/I_m) = \operatorname{reg}(S/I^m)$. Since $f \in J$, we have $f = \sum t_i g_i$, where the g_i are minimal generators of J. Since $f \in \eta(I_m)$, it follows that $f \notin I_m$. Observe that $II_{m-1} \subseteq I_m$. Indeed, write $t = \alpha + \beta \in I_{m-1}$, where $\alpha \in I^{m-1}$ and $\beta \in$

$(x_{s+1},\dots,x_n)$. If $g\in I$, then $tg=\alpha g+\beta g\in I^m+(x_{s+1},\dots,x_n)=I_m$. Since $J\subseteq I$, we have that $g_i\in I$. As $f\notin I_m$, the above observation shows that not all of the t_i can lie in I_{m-1}. Suppose that $t_1\notin I_{m-1}$. It follows that

$$\begin{aligned}\operatorname{reg}(I^m)=\operatorname{reg}(I_m)&=\deg(f)+1=\deg(t_1)+\deg(g_1)+1\\&\leqslant \operatorname{reg}(I_{m-1})+c=\operatorname{reg}(I^{m-1})+c.\end{aligned}$$

Hence $\operatorname{reg}(I^m)-\operatorname{reg}(I^{m-1})\leqslant c$ which amounts to saying that $e_m-e_{m-1}\leqslant c-d$. $\square$

Proposition 2.13. *Let $J\subseteq I$ be monomial ideals such that I is $\mathfrak{p}$-primary and J is $\mathfrak{q}$-primary. Write $\operatorname{reg}(I^m)=dm+e_m$ and let $c=\max\{\mu(J),b_i\}$ where b_i are the powers of pure minimal generators of I in variables in $\Lambda(I)\setminus\Lambda(J)$. If $\operatorname{reg}(I^m)>\operatorname{reg}(J)+\sum(b_i-1)$, then $e_m-e_{m-1}\leqslant c-d$.*

Proof. As $\Lambda(J)\subseteq\Lambda(I)$ by assumption, we may assume that $\Lambda(J)=\{x_1,\dots,x_l\}$ and $\Lambda(I)=\{x_1,\dots,x_l,x_{l+1},\dots,x_s\}$. Let $I_m=I^m+(x_{s+1},\dots,x_n)$ for $m>0$. The assumption $\operatorname{reg}(I^m)>\operatorname{reg}(J)+\sum(b_i-1)$ implies that

$$\eta(I_m)\subseteq J_1=J+(x_{l+1}^{b_{l+1}},\dots,x_s^{b_s},x_{s+1},\dots,x_n)\subseteq I_1=I+(x_{s+1},\dots,x_n).$$

Note that I_1 and J_1 are $\mathfrak{m}$-primary ideals. Let $f\in\eta(I_m)$ and write $f=\sum t_ig_i$ where the g_i are minimal generators of J_1. Next observe that $J_1I_{m-1}\subseteq I_m$. This can be proved by using the above inclusions, and the arguments similar to the ones in the proof of Theorem 2.12. This shows that there exists a t_i, say t_1, such that $t_1\notin I_{m-1}$. The last step of the proof is exactly that of Theorem 2.12. $\square$

In [1], Theorem 2.7 a bound for the defect sequence of $\mathfrak{m}$-primary monomial ideals has been proven. In what follows we generalize this result to the case of arbitrary primary monomial ideals.

Theorem 2.14. *Let $J\subseteq I$ be monomial ideals such that I is $\mathfrak{p}$-primary and J is $\mathfrak{q}$-primary. Let c,c' be respectively the maximum and minimum degree of a minimal generator of J. Let d' be the minimum degree of an element of $\operatorname{min.gen}(I)\setminus J$. If $m>\min\{(\operatorname{reg}(J)+t)/d,\operatorname{reg}(J)/d'+\max\{1-c'/d',0\}\}$, where $t=\sum(b_j-1)$ and b_j are the powers of the pure generators of I in variables in $\Lambda(I)\setminus\Lambda(J)$, then $e_m-e_{m-1}\leqslant c-d$.*

This holds for each m such that $\eta(J)\not\subseteq s(I'_m)+\eta(I'^{m-1}J)$, where I' is the ideal generated by $\operatorname{min.gen}(I)\setminus J$ and $s(I'_m)$ is as in Remark 2.2.

P r o o f. If $m > (\operatorname{reg}(J) + t)/d$, then $\operatorname{reg}(I^m) = dm + e_m > \operatorname{reg}(J) + t$ and consequently, $e_m - e_{m-1} \leqslant c - d$ by Proposition 2.13. On the other hand, if $d'm > \operatorname{reg}(J) + t$, then $I^m \subseteq J_1 = J + (x_{l+1}, \ldots, x_s, x_{s+1}, \ldots, x_n)$ and hence also $I_m \subseteq J_1$ with the notation of Proposition 2.13. Thus we have $\operatorname{reg}(I^m) \geqslant \operatorname{reg}(J)$. Now $I_m = (J + I')^m + (x_{s+1}, \ldots, x_n) = \sum_{i=0}^{m} J^i I'^{m-i} + (x_{s+1}, \ldots, x_n) \subseteq \sum_{i=0}^{m} J_i$ and since all of the summands but the last two are contained in J_2 and $\operatorname{reg}(J_2) > \operatorname{reg}(J)$ by Theorem 2.6, one concludes that in order for $\operatorname{reg}(I^m) = \operatorname{reg}(J)$ to have a chance, we must have $\eta(J_1) \subseteq s(I'_m) + \eta(I'^m J_1)$. The minimum degree of an element of $s(I'_m)$ is $d'm - 1$, and the minimum degree of an element of $s((I'^{m-1} J_1))$ is $d'(m-1) + c' - 1$. Consequently, if $d'm > \operatorname{reg}(J) + d' - c'$ then the degrees of the above elements are larger than $\operatorname{reg}(J) - 1 = \operatorname{reg}(S/J)$, which is exactly the degree of an element of $\eta(J_1)$. □

Corollary 2.15. *Let I be a primary monomial ideal. Let $d + b$ be the maximum degree of a minimal generator of I. Then $e_{m+1} \leqslant e_m + b$ for all $m > 1$.*

P r o o f. Since by Theorem 2.6, $\operatorname{reg}(I^m) > \operatorname{reg}(I)$ for all $m > 1$, the claim follows from Proposition 2.13. Note that by setting $J = I$ in this proposition, $\Lambda(I) \backslash \Lambda(J) = \emptyset$ and hence we may set $\sum(b_j - 1) = 0$. □

Remark 2.16. In the above generalizations from $\mathfrak{m}$-primary ideals to arbitrary primary monomial ideals we have restrict ourselves to the $\mathfrak{m}$-primary ideals by considering ideals I_m. An alternative way to view this reduction is the following approach: If $I \subset (x_{i_1}, \ldots, x_{i_r})$ is a homogeneous ideal, we know that $I = (I \cap K[x_{i_1}, \ldots, x_{i_r}])S$. As the extension $K[x_{i_1}, \ldots, x_{i_r}] \to S$ is faithfully flat, the Betti numbers of $I \cap K[x_{i_1}, \ldots, x_{i_r}]$ over $K[x_{i_1}, \ldots, x_{i_r}]$ are exactly the Betti numbers of I over S. Therefore we may assume, without loss of generality, that I is $(x_{i_1}, \ldots, x_{i_r})$-primary and replace S by $K[x_{i_1}, \ldots, x_{i_r}]$ and I by $I \cap K[x_{i_1}, \ldots, x_{i_r}]$ and therefore restrict ourselves to the case that I is $\mathfrak{m}$-primary (in $K[x_{i_1}, \ldots, x_{i_r}]$). However, note that in Theorems 2.13 and 2.14 unlike the analogous results in [1], in which both ideals are $\mathfrak{m}$-primary, the ideals are associated to different prime ideals in general. Therefore, some changes and correction terms are needed to be added in the generalizations that we have considered in the above results.

3. Regularity jumps of a binomial ideal

We begin with the definition of the notion of a regularity jump as given in [5].

Definition 3.1. Let I be an ideal generated in a single degree d. We say that the regularity of powers of I jumps at place k if $e_k > e_{k-1}$.

By definition of the defect sequence, the above condition is equivalent to $\operatorname{reg}(I^k) - \operatorname{reg}(I^{k-1}) > d$ and indeed this is the original condition stated in [5]. Before we proceed, let us introduce a remark.

Remark 3.2. Let I be a homogeneous ideal in the polynomial ring $S = K[x_1, \ldots, x_n]$ and $\mathfrak{m}$ be the irrelevant maximal ideal of S. If I is not $\mathfrak{m}$-primary, then $\operatorname{reg}(I) = \min\{\mu\colon\ H^i(S/I)_{\mu-i} = 0 \text{ for all } i\}$. See [4], Proposition 9.5.

We are now ready to state our result. The example is as follows:

Example. Let $n \geqslant 3$ and

$$\begin{aligned} I_n = (&x_1^2, \ldots, x_{n+1}^2, x_1x_2, \ldots, x_1x_{n+1},\\ &x_2x_3 - x_1x_{n+2}, x_2x_4 - x_1x_{n+3}, \ldots, x_2x_{n+1} - x_1x_{2n}, \ldots,\\ &x_3x_4 - x_1x_{2n+1}, \ldots, x_3x_{n+1} - x_1x_{3n-2}, \ldots, x_nx_{n+1} - x_1x_s), \end{aligned}$$

where $s = \frac{1}{2}n(n+1) + 1$.

In this description, a generator apart from $x_1^2, \ldots, x_{n+1}^2, x_1x_2, \ldots, x_1x_{n+1}$ is of the form $x_ix_j - x_1x_{t(n,i,j)}$, where $2 \leqslant i < j \leqslant n+1$ and $t(n,i,j) = (i-1)n - \frac{1}{2}(i-1)(i-2) + 1 + (j-i)$.

Another description of this ideal as given in [5] is as follows:

$$I_n = (x^2, y_1^2, y_2^2, \ldots, y_n^2, xy_1, \ldots, xy_n, y_iy_j - xz_{i,j}) \quad \text{for } 1 \leqslant i < j \leqslant n.$$

In [5] it is proven that I_3^2 has a nonlinear resolution and it is conjectured that this holds for $n > 3$. Note that the I_n define Cohen-Macaulay rings with minimal multiplicity. We prove:

Theorem 3.3. *For I_n $(n > 3)$ as above, $\operatorname{reg}(I_n) = 2$ and $\operatorname{reg}(I_n^2) > 4$. Therefore I_n^2 has a nonlinear resolution and we get an infinite family of ideals with regularity jumps at $k = 2$.*

P r o o f. In order to avoid complicated indices, we prefer to work with the first description of I_n given above. Then one can easily check that I_n is weakly stable (Definition 2.3) and hence $\operatorname{reg}(I_n) = \mu(I_n) = 2$. Note however that I_n^2 is *not* weakly stable. In order to show that $\operatorname{reg}(I_n^2) > 4$, setting $J_n = I_n^2$, we show that there exists an integer l such that $H_m^l(S/J_n)_{4-l} \neq 0$. By Remark 3.2, it follows that $\operatorname{reg}(J_n) > 4$. To this end, we use the fact that local cohomology can be computed via Čech complex. Note that since $x_1^4 = \ldots = x_{n+1}^4 = 0$ in S/J_n, the cohomology can in fact be computed by the complex

$$\begin{aligned} 0 \to S/J_n \xrightarrow{d^0} \bigoplus_{n+2 \leqslant i \leqslant s} (S/J_n)_{x_i} \xrightarrow{d^1} \bigoplus_{n+2 \leqslant i < j \leqslant s} (S/J_n)_{x_ix_j} \xrightarrow{d^2} \ldots\\ \xrightarrow{d^{s-n-1}} (S/J_n)_{x_{n+2} \ldots x_s} \to 0. \end{aligned}$$

For $n \geqslant 3$, one can check the following equalities in S/J_n (note that we abuse the notation and denote the image of an element $\beta \in S$ in S/J_n again by β):

$$x_1x_2x_3x_4 = x_1^2x_2x_{2n+1} = x_1^2x_3x_{n+3} = x_1^2x_4x_{n+2}$$

If $x_ix_j - x_1x_r$ is a generator of I_n such that $\{i,j\} \cap \{2,3,4\} \neq \emptyset$, then one sees that

$$x_1^2x_tx_r = 0 \quad \forall t \in \{i,j\} \cap \{2,3,4\}.$$

Now let $x_{i_1}x_{j_1} - x_1x_{r_1}, \ldots, x_{i_l}x_{j_l} - x_1x_{r_l}$ be the set of all generators of I_n of the form $x_ix_j - x_1x_r$ such that $\{i,j\} \cap \{2,3,4\} = \emptyset$. Note that $l = \frac{1}{2}(n-3)(n-4)$. Set $\alpha := x_1x_2x_3x_4$. Then combining the above two series of equalities shows that

$$\text{for } r \in \{n+2, \ldots, s\} \setminus \{r_1, \ldots, r_l\},\ \alpha x_r = 0.$$

This implies that the element

$$\kappa := \Big(0, \ldots, \frac{\alpha}{x_{r_1} \ldots x_{r_l}}, \ldots, 0\Big) \in C^l(S/J_n)$$

lies in $\ker(d^l)_{4-l}$ (note that for $n = 4$, $l = 0$ and one defines $\kappa := (0, \ldots, \alpha, \ldots, 0)$). It follows that $\kappa \in H_m^l(S/J_n)_{4-l}$ which, as one sees, is nonzero in this cohomology module (for example by induction on n and using the structure of the maps d^i) and hence $H_m^l(S/J_n)_{4-l} \neq 0$. □

Example. Let $n = 4$. Note that for $n = 3$ we get the example of Conca. One can compute (for example using CoCoA) that $\operatorname{reg}(I_4) = 2$ and $\operatorname{reg}(I_4^2) = 5$. In this case $H_m^0(S/J_4)_4 \neq 0$ as one gets from the above proof.

4. Future works

It would be interesting to find results analogous to Theorems 2.12, 2.13 and 2.14 for nonmonomial ideals. One could also try to extend Theorem 2.10 and get some information about the regularity of powers of more general monomial ideals using their primary decomposition.

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Authors' addresses: Keivan Borna, Faculty of Mathematics and Computer Science, Kharazmi University, No. 43. South Mofatteh Ave., Tehran, 15719-14911, Iran, e-mail: `borna@khu.ac.ir`; Abolfazl Mohajer, Johannes Gutenberg-Universität Mainz, Staudingerweg 9, D 55128 Mainz, Germany, e-mail: `mojaher@uni-mainz.de`.