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A NOTE ON q-PARTIAL DIFFERENCE EQUATIONS AND SOME APPLICATIONS TO GENERATING FUNCTIONS AND q-INTEGRALS

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Abstract. We study the condition on expanding an analytic several variables function in terms of products of the homogeneous generalized Al-Salam-Carlitz polynomials. As applications, we deduce bilinear generating functions for the homogeneous generalized Al-Salam-Carlitz polynomials. We also gain multilinear generating functions for the homogeneous generalized Al-Salam-Carlitz polynomials. Moreover, we obtain generalizations of Andrews-Askey integrals and Ramanujan q-beta integrals. At last, we derive $U(n+1)$ type generating functions for the homogeneous generalized Al-Salam-Carlitz polynomials.

Keywords: q-partial difference equation; homogeneous generalized Al-Salam-Carlitz polynomial; generating function; Andrews-Askey integral; Ramanujan q-beta integral

MSC 2010: 05A30, 11B65, 33D15, 33D45, 33D50, 35C11

1. Introduction

Polynomial solutions of q-partial difference equations often serve as a building block in algorithms for finding other types of closed-form solutions, computer algebra algorithms are usual tools for finding polynomials [1], [4]. In this paper, our attention is drawn to problems of products of the homogeneous generalized Al-Salam-Carlitz polynomials as the solutions of certain q-partial difference equations and to computing some related problems. For more information, please refer to [1], [4], [8], [9], [10], [20], [21], [22], [23].

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In this paper, we follow the notations and terminology of [15] and suppose that $0 < q < 1$. The basic hypergeometric series ${}_r\varphi_s$ (see [15], equation (1.2.22)),

$$(1.1)\qquad {}_r\varphi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} [(-1)^n q^{\binom{n}{2}}]^{1+s-r} z^n$$

converges absolutely for all z if $r \leqslant s$ and for $|z| < 1$ if $r = s + 1$ and for terminating. In the context, convergence of basic hypergeometric series is no issue at all because they are the terminating q-series. The compact factorials of ${}_r\varphi_s$ are defined respectively by

$$(1.2)\qquad (a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$$

and $(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \ldots (a_m;q)_n$, where $m \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The homogeneous Rogers-Szegő polynomials (see [23], page 3), are defined as

$$(1.3)\qquad h_n(b, c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} b^k c^{n-k} \quad \text{and} \quad g_n(b, c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} b^k c^{n-k}.$$

The Al-Salam-Carlitz polynomials were introduced by Al-Salam and Carlitz in 1965 as (see [2], equations (1.11) and (1.15))

$$(1.4)\qquad \varphi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a;q)_k x^k \quad \text{and}$$

$$\psi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} x^k (aq^{1-k};q)_k.$$

The homogeneous Hahn polynomials (also called Al-Salam-Carlitz polynomials) can be written as (see [23], Definition 1.5)

$$(1.5)\qquad \varphi_n^{(a)}(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a;q)_k x^k y^{n-k}.$$

The homogeneous generalized Al-Salam-Carlitz polynomials state (see [9], equation (4.7)):

$$(1.6)\qquad \varphi_n^{(a,b,c)}(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a, b;q)_k}{(c;q)_k} x^k y^{n-k} \quad \text{and}$$

$$\psi_n^{(a,b,c)}(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a, b;q)_k}{(c;q)_k} (-1)^k q^{\binom{k+1}{2} - nk} x^k y^{n-k}$$

with generating functions being (see [9], equations (4.10) and (4.11))

$$\sum_{n=0}^{\infty}\varphi_n^{(a,b,c)}(x,y|q)\frac{t^n}{(q;q)_n}=\frac{1}{(yt;q)_\infty}{}_2\varphi_1\left[\begin{matrix}a,b\\c\end{matrix};q,xt\right],\ \max\{|yt|,|xt|\}<1, \tag{1.7}$$

$$\sum_{n=0}^{\infty}\psi_n^{(a,b,c)}(x,y|q)\frac{(-1)^n q^{\binom{n}{2}}t^n}{(q;q)_n}=(yt;q)_\infty{}_2\varphi_1\left[\begin{matrix}a,b\\c\end{matrix};q,xt\right],\ |xt|<1. \tag{1.8}$$

These q-polynomials play important roles in the theory of q-orthogonal polynomials. In fact, there are two families of these polynomials: one with continuous orthogonality and one with discrete orthogonality. For more information, please refer to [12], [24], [28], [29]. These polynomials are also given explicitly in [17], [18], [19].

Liu in [22] deduced the following results using the method of q-partial difference equation (q-partial differential equation) for the homogeneous Rogers-Szegő polynomials and Al-Salam-Carlitz polynomials.

Proposition 1 ([22], Definition 1.4). *$h_n(x,y|q)$ and $g_n(x,y|q)$ satisfy the identites*

$$\partial_{q,x}\{h_n(x,y|q)\}=\partial_{q,y}\{h_n(x,y|q)\}=(1-q^n)h_{n-1}(x,y|q), \tag{1.9}$$

$$\partial_{q^{-1},x}\{g_n(x,y|q)\}=\partial_{q^{-1},y}\{g_n(x,y|q)\}=(1-q^{-n})g_{n-1}(x,y|q), \tag{1.10}$$

where

$$\partial_{q,x}\{f(x)\}=\frac{f(x)-f(xq)}{x},\quad \partial_{q^{-1},x}\{f(x)\}=\frac{f(q^{-1}x)-f(x)}{q^{-1}x}. \tag{1.11}$$

Proposition 2 ([22], Theorem 1.7). *If $f(x_1,y_1,\dots,x_k,y_k)$ is a $2k$-variable analytic function at $(0,0)\in\mathbb{C}^{2k}$, then we have*

(A) *f can be expanded in terms of $h_{n_1}(x_1,y_1|q)\dots h_{n_k}(x_k,y_k|q)$ if and only if f satisfies the q-partial difference equations*

$$\partial_{q,x_j}\{f\}=\partial_{q,y_j}\{f\}\quad \text{for } j=1,2,\dots,k; \tag{1.12}$$

(B) *f can be expanded in terms of $g_{n_1}(x_1,y_1|q)\dots g_{n_k}(x_k,y_k|q)$ if and only if f satisfies the q-partial difference equations*

$$\partial_{q^{-1},x_j}\{f\}=\partial_{q^{-1},y_j}\{f\}\quad \text{for } j=1,2,\dots,k. \tag{1.13}$$

Proposition 3 ([23], Theorem 1.8)**.** *If $f(x_1, y_1, \ldots, x_k, y_k)$ is a $2k$-variable analytic function at $(0,0) \in \mathbb{C}^{2k}$, then f can be expanded in terms of*

$$\varphi_{n_1}^{(\alpha_1)}(x_1, y_1|q) \ldots \varphi_{n_k}^{(\alpha_k)}(x_k, y_k|q) \tag{1.14}$$

if and only if f satisfies the q-partial difference equations

$$\partial_{q,x_j}\{f\} = \partial_{q,y_j}(1 - \alpha_j \eta_{x_j})\{f\} \quad \text{for } j = 1, 2, \ldots, k. \tag{1.15}$$

For more information about q-difference equation and q-partial difference equation, please refer to [9], [10], [11], [13], [14], [20], [21], [22], [23].

In this paper, motivated by the method and results of Liu, we make further study on the homogeneous generalized Al-Salam-Carlitz polynomials. We focus on the discrete q-polynomials in this paper, in fact, the results of continuous cases can be deduced using the same method, and so are omitted here.

Theorem 4. *If $f(x_1, y_1, \ldots, x_k, y_k)$ is a $2k$-variable analytic function at $(0, 0, \ldots, 0) \in \mathbb{C}^{2k}$, then we have*

(C) *f can be expanded in terms of*

$$\varphi_{n_1}^{(a_1,b_1,c_1)}(x_1, y_1|q) \ldots \varphi_{n_k}^{(a_k,b_k,c_k)}(x_k, y_k|q)$$

if and only if f satisfies the q-partial difference equations

$$\partial_{q,x_j}(1 - q^{-1}c_j\eta_{x_j})\{f\} = \partial_{q,y_j}(1 - (a_j + b_j)\eta_{x_j} + a_j b_j \eta_{x_j}^2)\{f\} \quad \text{for } j = 1, 2, \ldots, k; \tag{1.16}$$

(D) *f can be expanded in terms of*

$$\psi_{n_1}^{(a_1,b_1,c_1)}(x_1, y_1|q) \ldots \psi_{n_k}^{(a_k,b_k,c_k)}(x_k, y_k|q)$$

if and only if f satisfies the q-partial difference equations

$$\partial_{q^{-1},x_j}(1 - q^{-1}c_j\eta_{x_j})\{f\} = \partial_{q^{-1},y_j}(1 - (a_j + b_j)\eta_{x_j} + a_j b_j \eta_{x_j}^2)\{f\} \quad \text{for } j = 1, 2, \ldots, k. \tag{1.17}$$

Remark 5. For $b_1 = c_1 = \ldots = b_k = c_k = 0$ in Theorem 4, equation (1.16) reduces to (1.15). For $a_1 = b_1 = c_1 = \ldots = a_k = b_k = c_k = 0$ in Theorem 4, equations (1.16) and (1.17) reduce to (1.12) and (1.13), respectively.

This paper is organized as follows. In Section 2, we give the proof of the main results. In section 3, we obtain bilinear generating functions for the homogeneous generalized Al-Salam-Carlitz polynomials. In section 4, we deduce multilinear generating functions for the homogeneous generalized Al-Salam-Carlitz polynomials. In section 5, we generalize Andrews-Askey integrals. In section 6, we generalize Ramanujan q-beta integrals. In section 7, we consider $U(n+1)$ type generating functions for the homogeneous generalized Al-Salam-Carlitz polynomials.

2. Proof of the main theorem

Before we prove the main results, the following lemmas are necessary.

Lemma 6 ([25], page 5, Proposition 1). *If $f(x_1, x_2, \ldots, x_k)$ is analytic at the origin $(0, 0, \ldots, 0) \in \mathbb{C}^k$, then f can be expanded in an absolutely convergent power series*

$$f(x_1, x_2, \ldots, x_k) = \sum_{n_1, n_2, \ldots, n_k = 0}^{\infty} \alpha_{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}.$$

Lemma 7. *The homogeneous polynomials $\varphi_n^{(a,b,c)}(x, y|q)$ and $\psi_n^{(a,b,c)}(x, y|q)$ satisfy the q-partial difference equations*

$$\begin{aligned}(2.1)\qquad &\partial_{q,x}\{\varphi_n^{(a,b,c)}(x, y|q) - q^{-1}c\varphi_n^{(a,b,c)}(qx, y|q)\}\\ &= \partial_{q,y}\{\varphi_n^{(a,b,c)}(x, y|q) - (a + b)\varphi_n^{(a,b,c)}(qx, y|q) + ab\varphi_n^{(a,b,c)}(q^2x, y|q)\}\end{aligned}$$

and

$$\begin{aligned}(2.2)\qquad &\partial_{q^{-1},x}\{\psi_n^{(a,b,c)}(x, y|q) - q^{-1}c\psi_n^{(a,b,c)}(qx, y|q)\}\\ &= \partial_{q^{-1},y}\{\psi_n^{(a,b,c)}(x, y|q) - (a + b)\psi_n^{(a,b,c)}(qx, y|q) + ab\psi_n^{(a,b,c)}(q^2x, y|q)\},\end{aligned}$$

respectively.

Proof of Lemma 7. Using the identity $\partial_{q,x}\{x^k\} = (1-q^k)x^{k-1}$, we immediately find that

(2.3)
$$\partial_{q,x}\{\varphi_n^{(a,b,c)}(x, y|q) - q^{-1}c\varphi_n^{(a,b,c)}(qx, y|q)\} = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (1 - q^k)\frac{(a, b; q)_k}{(c; q)_{k-1}} x^{k-1}y^{n-k}$$

and

$$\begin{aligned}(2.4)\qquad &\partial_{q,y}\{\varphi_n^{(a,b,c)}(x, y|q) - (a + b)\varphi_n^{(a,b,c)}(qx, y|q) + ab\varphi_n^{(a,b,c)}(q^2x, y|q)\}\\ &= \sum_{k=1}^{n} \begin{bmatrix} n \\ k-1 \end{bmatrix} \frac{(a, b; q)_k}{(c; q)_{k-1}} x^{k-1}y^{n-k}(1 - q^{n-k+1}).\end{aligned}$$

From the relations of the q-binomial coefficients we have

$$\begin{bmatrix} n \\ k \end{bmatrix} (1-q^k) = \begin{bmatrix} n \\ k-1 \end{bmatrix} (1-q^{n-k+1}). \tag{2.5}$$

Combing the above equations, we obtain equation (2.1). Similarly, we can deduce equation (2.2). The proof of Lemma 7 is complete. □

Lemma 8 (Hartogs' theorem [16], page 15). *If a complex-valued function is holomorphic (analytic) in each variable separately in an open domain $D \subseteq \mathbb{C}^n$, then it is holomorphic (analytic) in D.*

Now we begin to prove Theorem 4.

Proof of Theorem 4. The theorem can be proved by induction. We first prove the theorem for the case $k = 1$. Since f is analytic at $(0,0)$, by Lemma 6 we know that f can be expanded in an absolutely convergent power series in a neighbourhood of $(0,0)$. Thus, there exists a sequence $\lambda_{m,n}$ independent of x_1 and y_1 such that

$$f(x_1, y_1) = \sum_{m,n=0}^{\infty} \lambda_{m,n} x_1^m y_1^n = \sum_{m=0}^{\infty} x_1^m \sum_{n=0}^{\infty} \lambda_{m,m} y_1^n. \tag{2.6}$$

Substituting equation (2.6) into the q-partial difference equation, and using the relation

$$\begin{aligned} &\partial_{q,x_1}\{f(x_1,y_1) - q^{-1}c_1 f(qx_1,y_1)\} \\ &\quad = \partial_{q,y_1}\{f(x_1,y_1) - (a_1+b_1)f(qx_1,y_1) + abf(q^2x_1,y_1)\}, \end{aligned} \tag{2.7}$$

we get

$$\begin{aligned} &\sum_{m=1}^{\infty} (1-q^m)(1-c_1q^{m-1})x_1^{m-1} \sum_{n=0}^{\infty} \lambda_{m,n} y_1^n \\ &\quad = \sum_{m=0}^{\infty} (1-a_1q^{m-1})(1-b_1q^{m-1}) x_1^m \partial_{q,y_1}\Big\{\sum_{n=0}^{\infty} \lambda_{m,n} y_1^n\Big\}. \end{aligned} \tag{2.8}$$

Equating the coefficients of x_1^{m-1} on both sides of the above equation, we deduce

$$\sum_{n=0}^{\infty} \lambda_{m,n} y_1^n = \frac{(1-a_1q^{m-1})(1-b_1q^{m-1})}{(1-q^m)(1-c_1q^{m-1})} \partial_{q,y_1}\Big\{\sum_{n=0}^{\infty} \lambda_{m-1,n} y_1^n\Big\}. \tag{2.9}$$

Iterating the above recurrence relation we gain

$$\sum_{n=0}^{\infty} \lambda_{m,n} y_1^n = \frac{(a_1,b_1;q)_m}{(q,c_1;q)_m} \partial_{q,y_1}^m \Big\{\sum_{n=0}^{\infty} \lambda_{0,n} y_1^n\Big\}. \tag{2.10}$$

With the help of the identity $\partial_{q,y_1}^m\{y_1^n\} = y_1^{n-m}(q;q)_n/(q;q)_{n-m}$ we obtain

$$\sum_{n=0}^{\infty} \lambda_{m,n} y_1^n = \frac{(a_1,b_1;q)_m}{(c_1;q)_m} \sum_{n=m}^{\infty} \lambda_{0,n} \begin{bmatrix} n \\ m \end{bmatrix} y_1^{n-m}. \tag{2.11}$$

Noting that the series in equation (2.6) is absolutely convergent, substituting the above equation into (2.6) and interchanging the order of the summation, we deduce (2.12)

$$f(x_1,y_1) = \sum_{n=0}^{\infty} \lambda_{0,n} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(a_1,b_1;q)_m}{(c_1;q)_m} x_1^n y_1^{n-m} = \sum_{n=0}^{\infty} \lambda_{0,n} \cdot \varphi_n^{(a_1,b_1,c_1)}(x_1,y_1|q).$$

Conversely, if $f(a_1,b_1,c_1,x_1,y_1)$ can be expanded in terms of $\varphi_n^{(a_1,b_1,c_1)}(x_1,y_1|q)$, then using Lemma 7 and interchanging the order of the summation, we deduce

$$f(x_1,y_1,\ldots,x_k,y_k) = \sum_{n_1=0}^{\infty} d_{n_1}(x_2,y_2,\ldots,x_k,y_k)\varphi_{n_1}^{(a_1,b_1,c_1)}(x_1,y_1|q). \tag{2.13}$$

Setting $x_1 = 0$ in the above equation and using $\varphi_{n_1}^{(a_1,b_1,c_1)}(0,y_1|q) = y_1^{n_1}$, we obtain

$$f(0,y_1,x_2,y_2,\ldots,x_k,y_k) = \sum_{n_1=0}^{\infty} d_{n_1}(x_2,y_2,\ldots,x_k,y_k) y_1^{n_1}. \tag{2.14}$$

Using Maclaurin expansion theorem we immediately deduce

$$d_{n_1}(x_2,y_2,\ldots,x_k,y_k) = \frac{\partial^{n_1} f(0,y_1,x_2,y_2,\ldots,x_k,y_k)}{n!\partial y_1^{n_1}}\Big|_{y_1=0}. \tag{2.15}$$

Since $f(x_1,y_1,\ldots,x_k,y_k)$ is analytic near $(x_1,y_1,\ldots,x_k,y_k) = (0,0,\ldots,0) \in \mathbb{C}^{2k}$, from the above equation we know that $d_{n_1}(x_2,y_2,\ldots,x_k,y_k)$ is analytic near $(x_2,y_2,\ldots,x_k,y_k) = (0,0,\ldots,0) \in \mathbb{C}^{2k-2}$. Substituting equation (2.13) into the q-partial difference equation in Theorem 4, we have

$$\begin{aligned}
&\sum_{n_1=0}^{\infty} \partial_{q,x_j}(1-q^{-1}c_j\eta_{x_j})\{d_{n_1}(x_2,y_2,\ldots,x_j,y_j,\ldots,x_k,y_k)\}\varphi_{n_1}^{(a_1,b_1,c_1)}(x_1,y_1|q) \\
&\quad = \sum_{n_1=0}^{\infty} \partial_{q,y_j}(1-(a_j+b_j)\eta_{x_j} + a_jb_j\eta_{x_j}^2) \\
&\qquad \times \{d_{n_1}(x_2,y_2,\ldots,x_j,y_j,\ldots,x_k,y_k)\}\varphi_{n_1}^{(a_1,b_1,c_1)}(x_1,y_1|q).
\end{aligned} \tag{2.16}$$

By equating the coefficients of $\varphi_{n_1}^{(a_1,b_1,c_1)}(x_1,y_1|q)$ on both sides of equation (2.16), we have

$$\begin{aligned}
&\partial_{q,x_j}(1-q^{-1}c_j\eta_{x_j})\{d_{n_1}(x_2,y_2,\ldots,x_j,y_j,\ldots,x_k,y_k)\} \\
&\quad = \partial_{q,y_j}(1-(a_j+b_j)\eta_{x_j} + a_jb_j\eta_{x_j}^2)\{d_{n_1}(x_2,y_2,\ldots,x_j,y_j,\ldots,x_k,y_k)\}.
\end{aligned} \tag{2.17}$$

Thus, by the inductive hypothesis there exists a sequence $\lambda_{n_1,n_2,\ldots,n_k}$ independent of $x_2, y_2, \ldots, x_k, y_k$ such that

$$\begin{aligned} (2.18) \qquad & d_{n_1}(x_2, y_2, \ldots, x_j, y_j, \ldots, x_k, y_k) \\ & \quad = \sum_{n_2,\ldots,n_k=0}^{\infty} \alpha_{n_1,n_2,\ldots,n_k} \cdot \varphi_{n_2}^{(a_2,b_2,c_2)}(x_2, y_2|q) \ldots \varphi_{n_k}^{(a_k,b_k,c_k)}(x_k, y_k|q). \end{aligned}$$

Substituting this equation into (2.13), we find that f can be expanded in terms of

$$\varphi_{n_1}^{(a_1,b_1,c_1)}(x_1, y_1|q) \ldots \varphi_{n_k}^{(a_k,b_k,c_k)}(x_k, y_k|q).$$

Conversely, if f can be expanded in terms of

$$\varphi_{n_1}^{(a_1,b_1,c_1)}(x_1, y_1|q) \ldots \varphi_{n_k}^{(a_k,b_k,c_k)}(x_k, y_k|q),$$

using Lemma 7 we find that $\partial_{q,x_j}(1 - q^{-1}c_j\eta_{x_j})\{f\} = \partial_{q,y_j}(1 - (a_j + b_j)\eta_{x_j} + a_jb_j\eta_{x_j}^2)\{f\}$ for $j = 1, 2, \ldots, k$. Similarly, we can gain (1.17). The proof of Theorem 4 is complete. $\square$

3. Bilinear generating functions for the homogeneous generalized Al-Salam-Carlitz polynomials

In this section, we deduce bilinear generating functions for the homogeneous generalized Al-Salam-Carlitz polynomials using the method of q-partial difference equation.

Theorem 9. *For $M, N \in \mathbb{N}$, $b = q^{-M}$ and $s = q^{-N}$ we have*

$$\begin{aligned} (3.1) \qquad & \sum_{n=0}^{\infty} \varphi_n^{(a,b,c)}(x, y|q)\varphi_n^{(r,s,t)}(u, v|q)\frac{w^n}{(q;q)_n} \\ & \quad = \frac{1}{(ywv;q)_\infty} \sum_{n=0}^{\infty} \frac{(a,b;q)_n(xwv)^n}{(q,c;q)_n} \\ & \qquad \times \sum_{m=0}^{\infty} \frac{(r,s;q)_m(ywu)^m}{(q,t;q)_m} {}_3\varphi_0\left[\begin{matrix} q^{-m}, q^{-n}, ywv \\ - \end{matrix}; q, \frac{q^{m+n}}{ywv}\right], \end{aligned}$$

where $|ywv| < 1$. *For* $M, N \in \mathbb{N}$, $b = q^{-M}$ *and* $s = q^{-N}$ *we have*

$$
\begin{aligned}
(3.2)\qquad & \sum_{n=0}^{\infty} \psi_n^{(a,b,c)}(x,y|q)\psi_n^{(r,s,t)}(u,v|q)\frac{(-1)^n q^{\binom{n}{2}} w^n}{(q;q)_n} \\
&= (ywv;q)_\infty \sum_{n=0}^{\infty} \frac{(a,b;q)_n (xwv)^n}{(q,c;q)_n} \\
&\quad \times \sum_{m=0}^{\infty} \frac{(r,s;q)_m (ywu)^m}{(q,t;q)_m} {}_3\varphi_2\left[\begin{matrix} q^{-m}, q^{-n}, q/(ywv) \\ 0, 0 \end{matrix}; q, q\right].
\end{aligned}
$$

Corollary 10 ([23], equation (3.2))**.** *We have*

$$
\begin{aligned}
(3.3)\qquad & \sum_{n=0}^{\infty} \varphi_n^{(a)}(x,y|q)\varphi_n^{(r)}(u,v|q)\frac{w^n}{(q;q)_n} \\
&= \frac{(axwv, rywu;q)_\infty}{(xwv, ywv, ywu;q)_\infty} {}_3\varphi_2\left[\begin{matrix} a, r, ywv \\ axwv, rywu \end{matrix}; q, xwu\right],
\end{aligned}
$$

where $\max\{|xwv|, |ywv|, |ywu|, |xwu|\} < 1$. *For* $\max\{|xwv|, |ywu|\} < 1$ *we have*

$$
\begin{aligned}
(3.4)\qquad & \sum_{n=0}^{\infty} \psi_n^{(a)}(x,y|q)\psi_n^{(r)}(u,v|q)\frac{(-1)^n q^{\binom{n}{2}} w^n}{(q;q)_n} \\
&= \frac{(axwv, rywu, ywv;q)_\infty}{(xwv, ywu;q)_\infty} {}_3\varphi_2\left[\begin{matrix} a, r, q/(ywv) \\ q/(xwv), q/(ywu) \end{matrix}; q, q\right].
\end{aligned}
$$

Corollary 11 (q-Mehler formula)**.** *We have*

$$
(3.5)\qquad \sum_{n=0}^{\infty} h_n(x,y|q)h_n(u,v|q)\frac{w^n}{(q;q)_n} = \frac{(xyuvw^2;q)_\infty}{(xwv, ywv, xwu, ywu;q)_\infty}, \quad \max\{|xwv|, |ywv|, |xwu|, |ywu|\} < 1,
$$

$$
(3.6)\qquad \sum_{n=0}^{\infty} g_n(x,y|q)g_n(u,v|q)\frac{(-1)^n q^{\binom{n}{2}} w^n}{(q;q)_n} = \frac{(xwv, ywv, xwu, ywu;q)_\infty}{(xyuvw^2/q;q)_\infty}, \quad |xyuvw^2/q| < 1.
$$

Remark 12. For $c = t = 0$, $b \to 0$, $s \to 0$ in Theorem 9, equations (3.1) and (3.2) reduce to (3.3) and (3.4), respectively. For $(a, x, r, u) = (1/a, ax, 1/r, ru)$ in (3.4) letting $a \to 0$ and $r \to 0$ in Corollary 10, equations (3.3) and (3.4) reduce to (3.5) and (3.6), respectively.

Before the proof of Theorem 9, the following two lemmas are necessary.

Lemma 13 (q-Leibniz formula)**.** *For $n \in \mathbb{N}_0$ we have*

$$D_a^n\{f(a)g(a)\} = \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_a^k\{f(a)\} D_a^{n-k}\{g(aq^k)\}, \tag{3.7}$$

$$\theta_a^n\{f(a)g(a)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \theta_a^k\{f(a)\} \theta_a^{n-k}\{g(aq^{-k})\}. \tag{3.8}$$

Lemma 14. *For $m, n \in \mathbb{N}_0$ we have*

$$D_y^n\Big\{\frac{y^m}{(yvw;q)_\infty}\Big\} = \frac{y^m(vw)^n}{(yvw;q)_\infty} {}_3\varphi_0\left[\begin{matrix} q^{-m}, q^{-n}, ywv \\ - \end{matrix}; q, \frac{q^{m+n}}{ywv}\right], \quad |yvw| < 1. \tag{3.9}$$

P r o o f. Using formula (3.7), the left-hand side (LHS) of (3.9) equals

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} \frac{(q;q)_m}{(q;q)_{m-k}} y^{m-k} \frac{(vwq^k)^{n-k}}{(yvwq^k;q)_\infty}$$
$$= \frac{y^m(vw)^n}{(yvw;q)_\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q;q)_m}{(q;q)_{m-k}} \frac{(yvw;q)_k}{(yvw)^k},$$

which is the right-hand side (RHS) of (3.9) after simplification. The proof of Lemma 14 is complete. □

Now we begin to prove Theorem 9.

P r o o f of Theorem 9. We will use $f(x, y)$ to denote the RHS of equation (3.1). The RHS of equation (3.1) is terminational and convergent, $f(x, y)$ is analytic at the origin $(0, 0) \in \mathbb{C}^2$. Noting that the RHS of equation (3.1) is absolutely convergent, interchanging the order of the summation and using Lemma 14, we have

$$\text{RHS of (3.1)} = \sum_{m=0}^{\infty} \frac{(r,s;q)_m u^m}{(q,t;q)_m} D_v^m\Big\{\frac{1}{(yvw;q)_\infty} {}_2\varphi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, xvw\right]\Big\},$$

then we check that

$$\partial_{q,x}(1 - q^{-1}c\eta_x)\{f(x,y)\}$$
$$= \sum_{m=0}^{\infty} \frac{(r,s;q)_m u^m}{(q,t;q)_m} D_v^m\Big\{\frac{1}{(yvw;q)_\infty} \sum_{k=0}^{\infty} \frac{(a,b;q)_k (vw)^k}{(q,c;q)_k}(1-cq^{k-1})(1-q^k)x^{k-1}\Big\}$$
$$= \sum_{m=0}^{\infty} \frac{(r,s;q)_m u^m}{(q,t;q)_m} D_v^m\Big\{\frac{1}{(yvw;q)_\infty} \sum_{k=0}^{\infty} \frac{(a,b;q)_k (vw)^k x^{k-1}}{(q,c;q)_{k-1}}\Big\}$$
$$= \sum_{m=0}^{\infty} \frac{(r,s;q)_m u^m}{(q,t;q)_m} D_v^m\Big\{\frac{(1-a)(1-b)vw}{(yvw;q)_\infty} {}_2\varphi_1\left[\begin{matrix} aq, bq \\ c \end{matrix}; q, xvw\right]\Big\}$$

and

$$\begin{aligned}
&\partial_{q,y}(1-(a+b)\eta_x+ab\eta_x^2)\{f(x,y)\}\\
&\quad=\sum_{m=0}^{\infty}\frac{(r,s;q)_m u^m}{(q,t;q)_m}D_v^m\left\{\frac{vw}{(yvw;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b;q)_k(vw)^k}{(q,c;q)_k}[1-(a+b)q^k+abq^{2k}]\right\}\\
&\quad=\sum_{m=0}^{\infty}\frac{(r,s;q)_m u^m}{(q,t;q)_m}D_v^m\left\{\frac{vw}{(yvw;q)_\infty}\sum_{k=0}^{\infty}\frac{(a,b;q)_{k+1}(xvw)^k}{(q,c;q)_k}\right\}\\
&\quad=\sum_{m=0}^{\infty}\frac{(r,s;q)_m u^m}{(q,t;q)_m}D_v^m\left\{\frac{(1-a)(1-b)vw}{(yvw;q)_\infty}{}_2\varphi_1\left[\begin{matrix}aq,bq\\c\end{matrix};q,xvw\right]\right\},
\end{aligned}$$

that is, function $f(x,y)$ satisfies

$$\partial_{q,x}(1-q^{-1}c\eta_x)\{f(x,y)\}=\partial_{q,y}(1-(a+b)\eta_x+ab\eta_x^2)\{f(x,y)\}. \tag{3.10}$$

Thus, by Theorem 4 there exists a sequence $\{\gamma_n\}$ independent of (x,y) such that

$$\text{RHS of (3.1)}=\sum_{n=0}^{\infty}\gamma_n\varphi_n^{(a,b,c)}(x,y|q). \tag{3.11}$$

Setting $x=0$ in equation (3.11) and using equation (1.6), we obtain

$$\sum_{n=0}^{\infty}\gamma_n y^n=\frac{1}{(ywv;q)_\infty}\sum_{k=0}^{\infty}\frac{(r,s;q)_k(ywu)^k}{(q,t;q)_k}.$$

From equation (1.7) we see that

$$\sum_{n=0}^{\infty}\varphi_n^{(r,s,t)}(u,v|q)\frac{(yw)^n}{(q;q)_n}=\frac{1}{(ywv;q)_\infty}\sum_{k=0}^{\infty}\frac{(r,s;q)_k(ywu)^k}{(q,t;q)_k}.$$

Thus we have

$$\sum_{n=0}^{\infty}\gamma_n y^n=\sum_{n=0}^{\infty}\varphi_n^{(r,s,t)}(u,v|q)\frac{w^n y^n}{(q;q)_n},$$

which gives

$$\gamma_n=\varphi_n^{(r,s,t)}(u,v|q)\frac{w^n}{(q;q)_n}. \tag{3.12}$$

Substituting γ_n into equation (3.11), we get equation (3.1). Equation (3.2) can be proved similarly, here we omit the proof for simplicity. The proof of Theorem 9 is complete. □

P r o o f of Corollary 10. Let $c = t = 0$, $b \to 0$, $s \to 0$ in Theorem 9, equation (3.1) becomes the LHS of (3.3) and equals

$$\frac{1}{(ywv;q)_\infty}\sum_{n=0}^{\infty}\frac{(a;q)_n(xwv)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(r;q)_m(ywu)^m}{(q;q)_m}{}_3\varphi_0\left[\begin{matrix}q^{-m},q^{-n},ywv\\ -\end{matrix};q,\frac{q^{m+n}}{ywv}\right]$$
$$=\frac{1}{(ywv;q)_\infty}\sum_{m=0}^{\infty}\frac{(r;q)_m(ywu)^m}{(q;q)_m}\sum_{j=0}^{m}\begin{bmatrix}m\\ j\end{bmatrix}\frac{(ywv;q)_j}{(ywv)^j}\sum_{n=0}^{\infty}\frac{(a;q)_{n+j}(xwv)^{n+j}}{(q;q)_n}$$
$$=\frac{(axwv;q)_\infty}{(ywv,xwv;q)_\infty}\sum_{m=0}^{\infty}\frac{(r;q)_m(ywu)^m}{(q;q)_m}\sum_{j=0}^{m}\begin{bmatrix}m\\ j\end{bmatrix}\frac{(a,ywv;q)_j}{(axwv;q)_j}\Big(\frac{x}{y}\Big)^j$$
$$=\frac{(axwv;q)_\infty}{(ywv,xwv;q)_\infty}\sum_{j=0}^{\infty}\frac{(a,r,ywv;q)_j(xwu)^j}{(q,axwv;q)_j}\sum_{m=0}^{\infty}\frac{(rq^j;q)_m}{(q;q)_m}(ywu)^m,$$

which is the RHS of equation (3.3). Similarly, we can deduce equation (3.4). The proof of Corollary 10 is complete. □

P r o o f of Corollary 11. Noting the following relations

$$\lim_{a\to 0}\varphi_n^{(a)}(x,y|q)=h_n(x,y|q),\quad \lim_{a\to 0}\psi_n^{(1/a)}(ax,y|q)=g_n(x,y|q), \tag{3.13}$$

we can deduce equations (3.5) and (3.6), respectively. The proof of Corollary 11 is complete. □

4. Multilinear generating functions for the homogeneous generalized Al-Salam-Carlitz polynomials

Andrews in [3] proved the following formula for the q-Lauricella function.

Proposition 15 ([3], equation (4.1)). *If* $\max\{|\alpha|,|\gamma|,|y_1|,\dots,|y_k|\}<1$, *then we have*

$$\sum_{n_1,n_2,\dots,n_k=0}^{\infty}\frac{(\alpha;q)_{n_1+n_2+\dots+n_k}(\beta_1;q)_{n_1}(\beta_2;q)_{n_2}\dots(\beta_k;q)_{n_k}y_1^{n_1}y_2^{n_2}\dots y_k^{n_k}}{(\gamma;q)_{n_1+n_2+\dots+n_k}(q;q)_{n_1}(q;q)_{n_2}\dots(q;q)_{n_k}}$$
$$=\frac{(\alpha,\beta_1y_1,\beta_2y_2,\dots,\beta_ky_k;q)_\infty}{(\gamma,y_1,y_2,\dots,y_k;q)_\infty}{}_{k+1}\varphi_k\left[\begin{matrix}\gamma/\alpha,y_1,y_2,\dots,y_k\\ \beta_1y_1,\beta_2y_2,\dots,\beta_ky_k\end{matrix};q,\alpha\right]. \tag{4.1}$$

Liu [23] generalized the above Andrews' results (4.1) for $\beta_i = 0$ ($1 \leqslant i \leqslant k$) as follows.

Proposition 16 ([23], Theorem 12.1). *If* $\max\{|a|, |c|, |x_1|, |y_1|, \ldots, |x_k|, |y_k|\} < 1$, *then we have the following multilinear generating function for the Rogers-Szegő polynomials*

$$\sum_{n_1,n_2,\ldots,n_k=0}^{\infty} \frac{(a;q)_{n_1+n_2+\ldots+n_k} h_{n_1}(x_1,y_1|q) h_{n_2}(x_2,y_2|q) \ldots h_{n_k}(x_k,y_k|q)}{(c;q)_{n_1+n_2+\ldots+n_k}(q;q)_{n_1}(q;q)_{n_2}\ldots(q;q)_{n_k}} \tag{4.2}$$
$$= \frac{(a;q)_\infty}{(c,x_1,y_1,x_2,y_2,\ldots,x_k,y_k;q)_\infty} {}_{2k+1}\varphi_{2k}\left[\begin{matrix} c/a, x_1, y_1, x_2, y_2, \ldots, x_k, y_k \\ 0, 0, \ldots, 0 \end{matrix}; q, a\right].$$

In this section, we gain multilinear generating functions for the homogeneous generalized Al-Salam-Carlitz polynomials.

Theorem 17. *For* $M_1, \ldots, M_k \in \mathbb{N}$ *and* $a_i = q^{-M_i}$, *if* $\max\{|\alpha|, |\gamma|, |x_1 t_1|, |y_1 t_1|, \ldots, |x_k t_k|, |y_k t_k|\} < 1$, *we have*

$$\sum_{n_1,n_2,\ldots,n_k=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2+\ldots+n_k} \varphi_{n_1}^{(a_1,b_1,c_1)}(x_1,y_1|q) \varphi_{n_2}^{(a_2,b_2,c_2)}(x_2,y_2|q)\ldots}{(\gamma;q)_{n_1+n_2+\ldots+n_k}(q;q)_{n_1}(q;q)_{n_2}\ldots(q;q)_{n_k}} \tag{4.3}$$
$$\times \varphi_{n_k}^{(a_k,b_k,c_k)}(x_k,y_k|q) t_1^{n_1} t_2^{n_2}\ldots t_k^{n_k}$$
$$= \frac{(\alpha;q)_\infty}{(\gamma, y_1t_1, y_2t_2, \ldots, y_kt_k;q)_\infty} \sum_{i=1}^{k} \frac{(\gamma/\alpha, y_1t_1, y_2t_2, \ldots, y_kt_k;q)_i \alpha^i}{(q;q)_i}$$
$$\times \prod_{j=1}^{k} {}_2\varphi_1\left[\begin{matrix} a_j, b_j \\ c_j \end{matrix}; q, x_j t_j q^i\right]$$

and

$$\sum_{n_1,n_2,\ldots,n_k=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2+\ldots+n_k} \psi_{n_1}^{(a_1,b_1,c_1)}(x_1,y_1|q) \psi_{n_2}^{(a_2,b_2,c_2)}(x_2,y_2|q)\ldots}{(\gamma;q)_{n_1+n_2+\ldots+n_k}(q;q)_{n_1}(q;q)_{n_2}\ldots(q;q)_{n_k}} \tag{4.4}$$
$$\times \psi_{n_k}^{(a_k,b_k,c_k)}(x_k,y_k|q) t_1^{n_1} t_2^{n_2}\ldots t_k^{n_k}$$
$$= \frac{(\alpha;q)_\infty}{(\gamma, y_1t_1, y_2t_2, \ldots, y_kt_k;q)_\infty} \sum_{i=1}^{k} \frac{(\gamma/\alpha, y_1t_1, y_2t_2, \ldots, y_kt_k;q)_i \alpha^i}{(q;q)_i}$$
$$\times \prod_{j=1}^{k} {}_3\varphi_2\left[\begin{matrix} a_j, b_j, 0 \\ c_j, q^{1-i}/(y_j t_j) \end{matrix}; q, \frac{q x_j}{y_j}\right].$$

Remark 18. For $a_i \to 0$, $b_i = c_i = 0$ $(1 \leqslant i \leqslant k)$ in Theorem 17, equation (4.3) reduces to (4.2).

P r o o f of Theorem 17. Letting $\beta_1 = \ldots = \beta_k = 0$ in Proposition 15, we have

$$\sum_{n_1,n_2,\ldots,n_k=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2+\ldots+n_k} y_1^{n_1} y_2^{n_2} \ldots y_k^{n_k}}{(\gamma;q)_{n_1+n_2+\ldots+n_k}(q;q)_{n_1}(q;q)_{n_2}\ldots(q;q)_{n_k}} = \frac{(\alpha;q)_\infty}{(\gamma,y_1,y_2,\ldots,y_k;q)_\infty} {}_{k+1}\varphi_k\left[\begin{matrix}\gamma/\alpha, y_1, y_2, \ldots, y_k \\ 0,\ldots,0\end{matrix};q,\alpha\right]. \tag{4.5}$$

If we use $f(a_1,b_1,c_1,x_1,y_1,\ldots,a_k,b_k,c_k,x_k,y_k)$ to denote the RHS of equation (4.3), using the ratio test, we find that f is an analytic function of $a_1,b_1,c_1,x_1,y_1,\ldots,a_k$, b_k,c_k,x_k,y_k for

$$\max\{|\alpha|,|\gamma|,|x_1t_1|,|y_1t_1|,\ldots,|x_kt_k|,|y_kt_k|\} < 1.$$

By direct computation, we deduce that

$$\partial_{q,x_j}(1-q^{-1}c_j\eta_{x_j})\{f\} = \partial_{q,y_j}(1-(a+b)\eta_{x_j} + a_jb_j\eta_{x_j}^2)\{f\}. \tag{4.6}$$

By Theorem 4 there exists a sequence $\{\lambda_{n_1,n_2,\ldots,n_k}\}$ independent of a_1,b_1,c_1,x_1, $y_1,\ldots,a_k,b_k,c_k,x_k,y_k$ such that

$$\begin{aligned} f(a_1,b_1,c_1,x_1,y_1,\ldots,a_k,b_k,c_k,x_k,y_k) &= \sum_{n_1,n_2,\ldots,n_k=0}^{\infty} \lambda_{n_1,n_2,\ldots,n_k}\cdot\varphi_{n_1}^{(a_1,b_1,c_1)}(x_1,y_1|q) \\ &\quad\times\varphi_{n_2}^{(a_2,b_2,c_2)}(x_2,y_2|q)\ldots\varphi_{n_k}^{(a_k,b_k,c_k)}(x_k,y_k|q). \end{aligned} \tag{4.7}$$

Setting $x_1 = x_2 = \ldots = x_k = 0$ in equation (4.7) we immediately gain

$$\sum_{n_1,n_2,\ldots,n_k=0}^{\infty} \lambda_{n_1,n_2,\ldots,n_k}\cdot y_1^{n_1}y_2^{n_2}\ldots y_k^{n_k} = \frac{(\alpha;q)_\infty}{(\gamma,y_1t_1,y_2t_2,\ldots,y_kt_k;q)_\infty} {}_{k+1}\varphi_k\left[\begin{matrix}\gamma/\alpha, y_1t_1, y_2t_2, \ldots, y_kt_k \\ 0,\ldots,0\end{matrix};q,\alpha\right]. \tag{4.8}$$

Equating the coefficients of the above equation we have

$$\lambda_{n_1,n_2,\ldots,n_k} = \frac{(\alpha;q)_{n_1+n_2+\ldots+n_k} t_1^{n_1}t_2^{n_2}\ldots t_k^{n_k}}{(\gamma;q)_{n_1+n_2+\ldots+n_k}(q;q)_{n_1}(q;q)_{n_2}\ldots(q;q)_{n_k}}. \tag{4.9}$$

Substituting equation (4.9) into (4.7) we deduce equation (4.3). Similarly, we obtain (4.4). The proof of Theorem 17 is complete. □

5. Generalizations of Andrews-Askey integrals

The Jackson q-integral defined as [15], equations (1.11.1)–(1.11.3),

$$\int_a^b f(x)\,\mathrm{d}_q x = (1-q)\sum_{n=0}^{\infty}[bf(bq^n) - af(aq^n)]q^n. \tag{5.1}$$

Andrews and Askey [5] deduced the following Andrews-Askey integral.

Proposition 19 ([5], Theorem 1)**.** *If there are no zero factors in the denominator of the integral, then we have*

$$\int_u^v \frac{(qt/u, qt/v; q)_\infty}{(st, yt; q)_\infty}\,\mathrm{d}_q t = \frac{(1-q)v(q, u/v, qv/u, ysuv; q)_\infty}{(su, yu, sv, yv; q)_\infty}. \tag{5.2}$$

For more information about Andrews-Askey integral, please refer to [5], [22], [27].

In this section, we generalize Andrews-Askey integral using the method of q-partial difference equation.

Theorem 20. *We have*

$$\begin{aligned}\int_u^v \frac{(qt/u, qt/v; q)_\infty}{(st, yt; q)_\infty}\sum_{n=0}^{\infty}\frac{(a,b;q)_n (xsuv)^n}{(q,c;q)_n}{}_2\varphi_0\left[\begin{matrix} q^{-n}, ysuv \\ - \end{matrix}; q, \frac{tq^n}{suv}\right]\mathrm{d}_q t \\ = \frac{(1-q)v(q, u/v, qv/u, ysuv; q)_\infty}{(su, sv, yu, yv; q)_\infty}\sum_{n=0}^{\infty}\frac{(a,b;q)_n (xv)^n}{(q,c;q)_n}{}_2\varphi_0\left[\begin{matrix} q^{-n}, yv \\ - \end{matrix}; q, \frac{uq^n}{v}\right],\end{aligned} \tag{5.3}$$

where $\max\{|su|, |sv|, |yu|, |yv|, |xv|\} < 1$.

Remark 21. For $x = 0$ in Theorem 20, equation (5.3) reduces to equation (5.2).

Before the proof of Theorem 20, the following lemma is necessary.

Lemma 22. *We have*

$$\begin{aligned}\sum_{n=0}^{\infty}\varphi_n^{(a,b;c)}(x,y|q)h_n(r,s|q)\frac{t^n}{(q;q)_n} \\ = \frac{1}{(yst, yrt; q)_\infty}\sum_{n=0}^{\infty}\frac{(a,b;q)_n (xrt)^n}{(q,c;q)_n}{}_2\varphi_0\left[\begin{matrix} q^{-n}, yrt \\ - \end{matrix}; q, \frac{sq^n}{r}\right],\end{aligned} \tag{5.4}$$

where $\max\{|yst|, |yrt|, |xrt|\} < 1$. *For* $|xrt| < 1$ *we have*

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_n^{(a,b;c)}(x,y|q)g_n(r,s|q)\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} \\ = (yst, yrt; q)_\infty\sum_{n=0}^{\infty}\frac{(a,b;q)_n (xrt)^n}{(q,c;q)_n}{}_2\varphi_1\left[\begin{matrix} q^{-n}, q/(yrt) \\ 0 \end{matrix}; q, yst\right].\end{aligned} \tag{5.5}$$

Proof of Lemma 22. Let $f(x,y)$ denote the RHS of equation (5.4); it is obvious that $f(x,y)$ is analytic in x, y separately, so by Hartogs' theorem, function $f(x,y)$ is analytic at $(0,0)$. Direct computation yields

$$\partial_{q,x}(1-q^{-1}c\eta_x)\{f(x,y)\} = \partial_{q,y}(1-(a+b)\eta_x + ab\eta_x^2)\{f(x,y)\}. \tag{5.6}$$

By Theorem 4 there exists a sequence $\{\gamma_n\}$ independent of x and y such that

$$\frac{1}{(yst, yrt; q)_\infty}\sum_{n=0}^{\infty}\frac{(a,b;q)_n (xrt)^n}{(q,c;q)_n}{}_2\varphi_0\left[\begin{matrix} q^{-n}, yrt \\ - \end{matrix}; q, \frac{sq^n}{r}\right] = \sum_{n=0}^{\infty}\gamma_n \cdot \varphi_n^{(a,b;c)}(x,y|q). \tag{5.7}$$

Putting $x=0$ in the above equation, and using the fact $\varphi_n^{(a,b;c)}(0,y|q) = y^n$, we find that

$$\sum_{n=0}^{\infty}\gamma_n \cdot y^n = \frac{1}{(yst, yrt; q)_\infty}. \tag{5.8}$$

Equating the coefficients of y^n on both sides of the above equation, we deduce that $\gamma_n = h_n(r,s|q)t^n/(q;q)_n$. Substituting this into (5.7) we complete the proof of equation (5.4). Similarly, we can gain equation (5.5). The proof of Lemma 22 is complete. □

Proof of Theorem 20. We can rewrite equation (5.3) as

$$\begin{aligned}\int_u^v \frac{(qt/u, qt/v; q)_\infty}{(st;q)_\infty}\cdot\frac{1}{(yt, ysuv;q)_\infty} \\ \times\sum_{n=0}^{\infty}\frac{(a,b;q)_n (xsuv)^n}{(q,c;q)_n}{}_2\varphi_0\left[\begin{matrix} q^{-n}, ysuv \\ - \end{matrix}; q, \frac{tq^n}{suv}\right] \mathrm{d}_q t \\ = \frac{(1-q)v(q, u/v, qv/u; q)_\infty}{(su, sv; q)_\infty}\cdot\frac{1}{(yu, yv; q)_\infty} \\ \times\sum_{n=0}^{\infty}\frac{(a,b;q)_n (xv)^n}{(q,c;q)_n}{}_2\varphi_0\left[\begin{matrix} q^{-n}, yv \\ - \end{matrix}; q, \frac{uq^n}{v}\right].\end{aligned} \tag{5.9}$$

If we use $f(x,y)$ to denote the RHS of equation (5.9), then we find that $f(x,y)$ satisfies equation (1.16), so we have

$$f(x,y) = \sum_{n=0}^{\infty}\gamma_n \cdot \varphi_n^{(a,b;c)}(x,y|q). \tag{5.10}$$

Setting $x = 0$ in the above equation we have

$$\sum_{n=0}^{\infty} \gamma_n \cdot y^n = \frac{(1-q)v(q, u/v, qv/u; q)_\infty}{(su, sv, yu, yv; q)_\infty} = \frac{1}{(ysuv; q)_\infty} \int_u^v \frac{(qt/u, qt/v; q)_\infty}{(st, yt; q)_\infty} \mathrm{d}_q t$$
$$= \sum_{n=0}^{\infty} \frac{y^n}{(q; q)_n} \int_u^v \frac{(qt/u, qt/v; q)_\infty h_n(suv/t, 1|q) t^n}{(st; q)_\infty} \mathrm{d}_q t,$$

which implies

$$\gamma_n = \frac{1}{(q; q)_n} \int_u^v \frac{(qt/u, qt/v; q)_\infty h_n(suv/t, 1|q) t^n}{(st; q)_\infty} \mathrm{d}_q t. \tag{5.11}$$

Substituting (5.11) into (5.10) gives

$$f(x, y) = \sum_{n=0}^{\infty} \frac{\varphi_n^{(a,b;c)}(x, y|q)}{(q; q)_n} \int_u^v \frac{(qt/u, qt/v; q)_\infty h_n(suv/t, 1|q) t^n}{(st; q)_\infty} \mathrm{d}_q t$$
$$= \int_u^v \frac{(qt/u, qt/v; q)_\infty}{(st; q)_\infty} \sum_{n=0}^{\infty} \varphi_n^{(a,b;c)}(x, y|q) h_n(suv/t, 1|q) \frac{(yt)^n}{(q; q)_n} \mathrm{d}_q t,$$

which equals the LHS of equation (5.9) after using equation (5.4). The proof is complete. □

6. Generalizations of Ramanujan q-beta integrals

The following two integrals of Ramanujan [6] are quite famous.

Proposition 23 ([6], equations (2) and (3)). *Let* $0 < q = \exp(-2k^2) < 1$ *and* $m \in \mathbb{R}$ *and suppose that* $|yzq| < 1$. *We have*

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\theta^2 + 2m\theta}}{(yq^{1/2}\mathrm{e}^{2\mathrm{i}k\theta}, zq^{1/2}\mathrm{e}^{-2\mathrm{i}k\theta}; q)_\infty} \mathrm{d}\theta = \sqrt{\pi}\mathrm{e}^{m^2} \frac{(-yq\mathrm{e}^{2mk\mathrm{i}}, -zq\mathrm{e}^{-2mk\mathrm{i}}; q)_\infty}{(yzq; q)_\infty}. \tag{6.1}$$

Suppose that $\max\{|yq^{1/2}\mathrm{e}^{2mk}|, |zq^{1/2}\mathrm{e}^{-2mk}|\} < 1$. *We have*

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\theta^2 + 2m\theta}(-yq\mathrm{e}^{2k\theta}, -zq\mathrm{e}^{-2k\theta}; q)_\infty \mathrm{d}\theta = \frac{\sqrt{\pi}\mathrm{e}^{m^2}(yzq; q)_\infty}{(yq^{1/2}\mathrm{e}^{2mk}, zq^{1/2}\mathrm{e}^{-2mk}; q)_\infty}. \tag{6.2}$$

Derivations of (6.1) and (6.2) for real values of parameter m have been deduced by Askey in [6]. Later on, it has become clear that these integrals are in fact valid for arbitrary complex values of parameter m and they are thus instances of the standard Fourier transform with the exponential kernel by Atakishiyev and Feinsilver in [7].

In this section, we generalize Ramanujan's q-beta integrals by q-partial difference equation as follows.

Theorem 24. *We have*

$$
\begin{aligned}
(6.3)\quad &\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\theta^2+2m\theta}}{(yq^{1/2}\mathrm{e}^{2\mathrm{i}k\theta}, zq^{1/2}\mathrm{e}^{-2\mathrm{i}k\theta};q)_\infty} \\
&\quad\times \sum_{n=0}^{\infty} \frac{(a,b;q)_n(-xq\mathrm{e}^{2mk\mathrm{i}})^n}{(q,c;q)_n} {}_2\varphi_0\left[\begin{matrix} q^{-n}, -yq\mathrm{e}^{2mk\mathrm{i}} \\ - \end{matrix};q,-q^{n+1/2}\mathrm{e}^{2k\mathrm{i}(\theta-m)}\right] \mathrm{d}\theta \\
&= \sqrt{\pi}\mathrm{e}^{m^2} \frac{(-yq\mathrm{e}^{2mki}, -zq\mathrm{e}^{-2mk\mathrm{i}};q)_\infty}{(yzq;q)_\infty} {}_2\varphi_1\left[\begin{matrix} a,b \\ c \end{matrix};q,xzq\right],
\end{aligned}
$$

where $\max\{|yzq|,|xzq|\}<1$*. For* $\max\{|yq^{1/2}\mathrm{e}^{2mk}|, |zq^{1/2}\mathrm{e}^{-2mk}|, |xzq|\}<1$*, we have*

$$
\begin{aligned}
(6.4)\quad &\int_{-\infty}^{\infty} \mathrm{e}^{-\theta^2+2m\theta}(-yq\mathrm{e}^{2k\theta}, -zq\mathrm{e}^{-2k\theta};q)_\infty \\
&\quad\times \sum_{n=0}^{\infty} \frac{(a,b;q)_n(xq^{1/2}\mathrm{e}^{2mk})^n}{(q,c;q)_n} {}_2\varphi_1\left[\begin{matrix} q^{-n}, q^{1/2}\mathrm{e}^{-2mk}/y \\ 0 \end{matrix};q,-yq\mathrm{e}^{2k\theta}\right] \mathrm{d}\theta \\
&= \sqrt{\pi}\mathrm{e}^{m^2} \frac{(yzq;q)_\infty}{(yq^{1/2}\mathrm{e}^{2mk}, zq^{1/2}\mathrm{e}^{-2mk};q)_\infty} {}_2\varphi_1\left[\begin{matrix} a,b \\ c \end{matrix};q,xzq\right].
\end{aligned}
$$

Remark 25. For $x=0$ in Theorem 24, equations (6.3) and (6.4) reduce to (6.1) and (6.2), respectively.

P r o o f of Theorem 24. We rewrite equation (6.3) equivalently as

$$
\begin{aligned}
(6.5)\quad &\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\theta^2+2m\theta}}{(zq^{1/2}\mathrm{e}^{-2\mathrm{i}k\theta};q)_\infty} \frac{1}{(yq^{1/2}\mathrm{e}^{2\mathrm{i}k\theta}, -yq\mathrm{e}^{2mk\mathrm{i}};q)_\infty} \\
&\quad\times \sum_{n=0}^{\infty} \frac{(a,b;q)_n(-xq\mathrm{e}^{2mk\mathrm{i}})^n}{(q,c;q)_n} {}_2\varphi_0\left[\begin{matrix} q^{-n}, -yq\mathrm{e}^{2mk\mathrm{i}} \\ - \end{matrix};q,-q^{n+1/2}\mathrm{e}^{2k\mathrm{i}(\theta-m)}\right] \mathrm{d}\theta \\
&= \sqrt{\pi}\mathrm{e}^{m^2} \frac{(-zq\mathrm{e}^{-2mk\mathrm{i}};q)_\infty}{(yzq;q)_\infty} {}_2\varphi_1\left[\begin{matrix} a,b \\ c \end{matrix};q,xzq\right].
\end{aligned}
$$

Denoting the RHS of equation (6.5) by $f(x,y)$, we check that $f(x,y)$ satisfies equation

$$
(6.6)\qquad \partial_{q,x}(1-q^{-1}c\eta_x)\{f\} = \partial_{q,y}(1-(a+b)\eta_x + ab\eta_x^2)\{f\}.
$$

Using Theorem 4, we have

$$f(x,y)=\sum_{n=0}^{\infty}\gamma_n\cdot\varphi_n^{(a,b;c)}(x,y|q). \tag{6.7}$$

Setting $x=0$ in the above equation gives

$$\begin{aligned}\sum_{n=0}^{\infty}\gamma_n y^n &= \sqrt{\pi}\mathrm{e}^{m^2}\frac{(-zq\mathrm{e}^{-2mk\mathrm{i}};q)_\infty}{(yzq;q)_\infty}\\ &=\frac{1}{(-yq\mathrm{e}^{2mk\mathrm{i}};q)_\infty}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\theta^2+2m\theta}}{(yq^{1/2}\mathrm{e}^{2\mathrm{i}k\theta},zq^{1/2}\mathrm{e}^{-2\mathrm{i}k\theta};q)_\infty}\,\mathrm{d}\theta\\ &=\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\theta^2+2m\theta}}{(zq^{1/2}\mathrm{e}^{-2\mathrm{i}k\theta};q)_\infty}\frac{h_n(q\mathrm{e}^{2mk\mathrm{i}},q^{1/2}\mathrm{e}^{2k\mathrm{i}\theta}|q)y^n}{(q;q)_n}\,\mathrm{d}\theta.\end{aligned}$$

So we have

$$\gamma_n=\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\theta^2+2m\theta}}{(zq^{1/2}\mathrm{e}^{-2\mathrm{i}k\theta};q)_\infty}\frac{h_n(q\mathrm{e}^{2mk\mathrm{i}},q^{1/2}\mathrm{e}^{2k\mathrm{i}\theta}|q)}{(q;q)_n}\,\mathrm{d}\theta. \tag{6.8}$$

The the RHS of equation (6.5) is equal to

$$\begin{aligned}&\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\theta^2+2m\theta}}{(zq^{1/2}\mathrm{e}^{-2\mathrm{i}k\theta};q)_\infty}\frac{h_n(q\mathrm{e}^{2mk\mathrm{i}},q^{1/2}\mathrm{e}^{2k\mathrm{i}\theta}|q)}{(q;q)_n}\varphi_n^{(a,b;c)}(x,y|q)\,\mathrm{d}\theta\\ &=\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\theta^2+2m\theta}}{(zq^{1/2}\mathrm{e}^{-2\mathrm{i}k\theta};q)_\infty}\sum_{n=0}^{\infty}h_n(q\mathrm{e}^{2mk\mathrm{i}},q^{1/2}\mathrm{e}^{2k\mathrm{i}\theta}|q)\varphi_n^{(a,b;c)}(x,y|q)\frac{\mathrm{d}\theta}{(q;q)_n},\end{aligned}$$

which is the LHS of equation (6.5) after using equation (5.4). Similarly, we gain equation (6.4) by using equation (5.5). The proof of Theorem 24 is complete. □

7. $U(n+1)$ type generating functions for generalized Al-Salam-Carlitz polynomials

Multiple basic hypergeometric series associated to the unitary $U(n+1)$ group have been studied by various authors, see [26], [30]. In [26], Milne initiated the theory and application of the $U(n+1)$ generalization of the classical Bailey Transform and Bailey Lemma, which involves the following nonterminating $U(n+1)$ generalizations of the q-binomial theorem.

Proposition 26 ([26], Theorem 5.42). *Let b, z and $x_1, \ldots, x_n$ be indeterminate, and let $n \geqslant 1$. Suppose that none of the denominators in the following identity vanishes, and that $0 < |q| < 1$ and $|z| < |x_1 \ldots x_n||x_m|^{-n}|q|^{(n-1)/2}$ for $m = 1, 2 \ldots, n$. Then*

(7.1)

$$\sum_{\substack{y_k \geqslant 0\\ k=1,2\ldots,n}} \prod_{1 \leqslant r < s \leqslant n} \frac{1 - q^{y_r - y_s} x_r/x_s}{1 - x_r/x_s} \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s}; q\right)_{y_r}^{-1}$$
$$\times \prod_{i=1}^{n} (x_i)^{ny_i - (y_1+\ldots+y_n)} (-1)^{(n-1)(y_1+\ldots+y_n)}$$
$$\times q^{y_2+2y_3+\ldots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\ldots+\binom{y_n}{2}\right]-e_2(y_1,\ldots,y_n)} (b;q)_{y_1+\ldots+y_n} z^{y_1+\ldots+y_n}$$
$$= \frac{(bz;q)_\infty}{(z;q)_\infty},$$

where $e_2(y_1, \ldots, y_n)$ is the second elementary symmetric function of $\{y_1, \ldots, y_n\}$.

In this section, we give the $U(n+1)$ generalizations of generating functions for generalized Al-Salam-Carlitz polynomials.

Theorem 27. *Let z and $x_1, \ldots, x_n$ be indeterminate, and let $n \geqslant 1$. Suppose that none of the denominators in the following identity vanishes, and that $0 < |q| < 1$ and $|z| < |x_1 \ldots x_n||x_m|^{-n}|q|^{(n-1)/2}$ for $m = 1, 2 \ldots, n$. Then*

(E) *we have*

(7.2)

$$\sum_{\substack{y_k \geqslant 0\\ k=1,2\ldots,n}} \prod_{1 \leqslant r < s \leqslant n} \left[\frac{1 - q^{y_r - y_s} x_r/x_s}{1 - x_r/x_s}\right] \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s}; q\right)_{y_r}^{-1}$$
$$\times \prod_{i=1}^{n} (x_i)^{ny_i - (y_1+\ldots+y_n)} (-1)^{(n-1)(y_1+\ldots+y_n)}$$
$$\times q^{y_2+2y_3+\ldots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\ldots+\binom{y_n}{2}\right]-e_2(y_1,\ldots,y_n)} \varphi^{(a,b,c)}_{y_1+\ldots+y_n}(x, y|q)$$
$$\times \varphi^{(r,s,t)}_{y_1+\ldots+y_n}(u, v|q) z^{y_1+\ldots+y_n}$$
$$= \frac{1}{(vyz;q)_\infty} \sum_{n=0}^{\infty} \frac{(a,b;q)_n (vxz)^n}{(q,c;q)_n} \sum_{m=0}^{\infty} \frac{(r,s;q)_m (uyz)^m}{(q,t;q)_m} {}_3\varphi_0$$
$$\times \left[\begin{matrix} q^{-m}, q^{-n}, vyz \\ - \end{matrix}; q, \frac{q^{m+n}}{vyz}\right],$$

where $\max\{|vyz|, |vxz|, |uyz|\} < 1$;

(F) *we have*

$$\sum_{\substack{y_k\geqslant 0\\ k=1,2\dots,n}}\prod_{1\leqslant r<s\leqslant n}\Big[\frac{1-q^{y_r-y_s}x_r/x_s}{1-x_r/x_s}\Big]\prod_{r,s=1}^{n}\Big(q\frac{x_r}{x_s};q\Big)_{y_r}^{-1} \tag{7.3}$$
$$\times\prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\dots+y_n)}(-1)^{(n-1)(y_1+\dots+y_n)}$$
$$\times q^{y_2+2y_3+\dots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\dots+\binom{y_n}{2}\right]-e_2(y_1,\dots,y_n)}(-1)^{y_1+\dots+y_n}$$
$$\times q^{\binom{y_1+\dots+y_n}{2}}\psi_n^{(a,b,c)}(x,y|q)\psi_n^{(r,s,t)}(u,v|q)z^{y_1+\dots+y_n}$$
$$=(vyz;q)_\infty\sum_{n=0}^{\infty}\frac{(a,b;q)_n(vxz)^n}{(q,c;q)_n}\sum_{m=0}^{\infty}\frac{(r,s;q)_m(uyz)^m}{(q,t;q)_m}{}_3\varphi_2$$
$$\times\left[\begin{matrix}q^{-m},q^{-n},q/(vyz)\\ 0,0\end{matrix};q,q\right],$$

where $\max\{|vxz|,|uyz|\}<1$.

Remark 28. For $n=1$ in Theorem 27, equations (7.2) and (7.3) reduce to (3.1) and (3.2), respectively.

P r o o f of Theorem 27. Let $(b,z)=(0,yzv)$ in Proposition 26, equation (7.1) becomes

$$\sum_{\substack{y_k\geqslant 0\\ k=1,2\dots,n}}\prod_{1\leqslant r<s\leqslant n}\Big[\frac{1-q^{y_r-y_s}x_r/x_s}{1-x_r/x_s}\Big]\prod_{r,s=1}^{n}\Big(q\frac{x_r}{x_s};q\Big)_{y_r}^{-1} \tag{7.4}$$
$$\times\prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\dots+y_n)}(-1)^{(n-1)(y_1+\dots+y_n)}$$
$$\times q^{y_2+2y_3+\dots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\dots+\binom{y_n}{2}\right]-e_2(y_1,\dots,y_n)}(zyv)^{y_1+\dots+y_n}$$
$$=\frac{1}{(vyz;q)_\infty}.$$

We first prove the following equation (7.5)

$$\sum_{\substack{y_k\geqslant 0\\ k=1,2\dots,n}}\prod_{1\leqslant r<s\leqslant n}\Big[\frac{1-q^{y_r-y_s}x_r/x_s}{1-x_r/x_s}\Big]\prod_{r,s=1}^{n}\Big(q\frac{x_r}{x_s};q\Big)_{y_r}^{-1} \tag{7.5}$$
$$\times\prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\dots+y_n)}(-1)^{(n-1)(y_1+\dots+y_n)}$$
$$\times q^{y_2+2y_3+\dots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\dots+\binom{y_n}{2}\right]-e_2(y_1,\dots,y_n)}$$
$$\times\varphi_{y_1+\dots+y_n}^{(r,s,t)}(u,v|q)(zy)^{y_1+\dots+y_n}=\frac{1}{(vyz;q)_\infty}\sum_{n=0}^{\infty}\frac{(r,s;q)_n(uyz)^n}{(q,t;q)_n}.$$

If we use $f(u,v)$ to denote the RHS of equation (7.5), it is obvious that function $f(u,v)$ is analytic at origin, at the same time $f(u,v)$ satisfies

$$\partial_{q,u}(1-q^{-1}t\eta_u)\{f\} = \partial_{q,v}(1-(s+t)\eta_u + rs\eta_u^2)\{f\}. \tag{7.6}$$

Thus, by Theorem 4 there exists a sequence $\{\alpha_n\}$ independent of (u,v) such that

$$\text{RHS of (7.2)} = \sum_{n=0}^{\infty} \alpha_n \cdot \varphi_n^{(r,s,t)}(u,v|q). \tag{7.7}$$

Taking $u=0$ in equation (7.7) and making use of (7.4), we have

$$\begin{aligned} \text{RHS of (7.5)}\Big|_{u=0} &= \sum_{n=0}^{\infty} \alpha_n v^n = \frac{1}{(vyz;q)_\infty} \\ &= \sum_{\substack{y_k \geqslant 0 \\ k=1,2\ldots,n}} \prod_{1\leqslant r<s\leqslant n} \left[\frac{1-q^{y_r-y_s}x_r/x_s}{1-x_r/x_s}\right] \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s};q\right)_{y_r}^{-1} \\ &\quad \times \prod_{i=1}^{n} (x_i)^{ny_i-(y_1+\ldots+y_n)} (-1)^{(n-1)(y_1+\ldots+y_n)} \\ &\quad \times q^{y_2+2y_3+\ldots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\ldots+\binom{y_n}{2}\right]-e_2(y_1,\ldots,y_n)} (vyz)^{y_1+\ldots+y_n}, \end{aligned} \tag{7.8}$$

so we gain equation (7.5) after equating the coefficients of (7.8). Next we will prove equation (7.2). Let $g(x,y)$ denote the RHS of equation (7.2), it is easy to verify that $g(x,y)$ satisfies

$$\partial_{q,x}(1-q^{-1}c\eta_x)\{g\} = \partial_{q,y}(1-(a+b)\eta_x + ab\eta_x^2)\{g\}. \tag{7.9}$$

Thus, there exists a sequence $\{\gamma_n\}$ independent of (x,y) such that

$$\text{RHS of (7.2)} = \sum_{n=0}^{\infty} \gamma_n \cdot \varphi_n^{(a,b,c)}(x,y|q). \tag{7.10}$$

Taking $x=0$ in equation (7.11) we have

$$\text{RHS of (7.2)}\Big|_{x=0} = \sum_{n=0}^{\infty} \gamma_n y^n = \frac{1}{(vyz;q)_\infty} \sum_{n=0}^{\infty} \frac{(r,s;q)_n (uyz)^n}{(q,t;q)_n}. \tag{7.11}$$

Combining equation (7.12) with equation (7.5) and then equating the coefficients of equation (7.12) on both sides, we get equation (7.2). Similarly, we can deduce equation (7.3). The proof of Theorem 27 is complete. □

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