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A LOGARITHMIC REGULARITY CRITERION FOR
3D NAVIER-STOKES SYSTEM IN A BOUNDED DOMAIN

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Abstract. This paper proves a logarithmic regularity criterion for 3D Navier-Stokes system in a bounded domain with the Navier-type boundary condition.

Keywords: regularity criterion; Navier-Stokes system; bounded domain

MSC 2010: 35Q30, 35Q35, 76D05

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with smooth boundary $\partial\Omega$, and n be the unit outward normal vector to $\partial\Omega$. We consider the regularity criterion of the Navier-Stokes system:

$$(1.1) \quad \operatorname{div} u = 0,$$

$$(1.2) \quad \partial_t u + u \cdot \nabla u + \nabla \pi = \Delta u \quad \text{in } \Omega \times (0, \infty),$$

$$(1.3) \quad u \cdot n = 0, \operatorname{rot} u \times n = 0 \quad \text{on } \partial\Omega \times (0, \infty),$$

$$(1.4) \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega \subset \mathbb{R}^3.$$

Here u and π denote the velocity field vector and pressure scalar of the fluid, respectively. We will denote the vorticity $\omega := \operatorname{rot} u$.

It is well-known that the problem has at least a global-in-time weak solution and a unique local-in-time strong solution [10], [13]. However, the regularity of weak solutions is still a very challenging open problem. On the other hand, the

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development of regularity criterion is of great importance for both theoretical and practical purpose. Giga [8], Berselli [6], Kim [12], and He, Ma and Wang [9] proved some regularity criteria. Very recently, Nakao and Taniuchi [16], [17] showed the regularity criterion

$$(1.5) \quad \int_0^T \frac{\|\omega\|_{L^\infty}}{\log(e + \|\omega\|_{L^\infty})} dt < \infty$$

with the no-slip boundary condition or (1.3). The aim of this note is to refine it as

$$(1.6) \quad \int_0^T \frac{\|\omega\|_{\text{BMO}}}{\log(e + \|\omega\|_{\text{BMO}})} dt < \infty.$$

Here BMO is the space of bounded mean oscillation whose norm is defined by

$$\|f\|_{\text{BMO}} := \|f\|_{L^2} + [f]_{\text{BMO}}$$

with

$$[f]_{\text{BMO}} := \sup_{\substack{x \in \Omega \\ r \in (0, d)}} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |f(y) - f_{\Omega_r(x)}| dy,$$

$$f_{\Omega_r(x)} := \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y) dy,$$

where $\Omega_r(x) := B_r(x) \cap \Omega$, where $B_r(x)$ is the ball with center x and radius r and d is the diameter of Ω . The symbol $|\Omega_r(x)|$ denotes the Lebesgue measure of $\Omega_r(x)$.

We will prove the following theorem.

Theorem 1.1. *Let $u_0 \in H_0^1 \cap H^3$ and $\operatorname{div} u_0 = 0$ in Ω . Let u be a local smooth solution to problem (1.1)–(1.4). If (1.6) holds with $0 < T < \infty$, then the solution u can be extended beyond $T > 0$.*

2. PRELIMINARIES

In this section we will collect some lemmas which will be used in the proof.

Lemma 2.1 (Poincaré inequality). *Let Ω be a bounded simple connected domain with smooth boundary and w be a smooth vector satisfying $w \cdot n = 0$ on the boundary $\partial\Omega$. Then*

$$(2.1) \quad \|w\|_{L^p} \leq C \|\nabla w\|_{L^p}$$

holds for $2 \leq p < \infty$.

Proof. If $p = 2$, then the proof was given in Lions [14], (6.47) on page 75. We assume $2 < p < \infty$. Using the Gagliardo-Nirenberg inequality and the case $p = 2$, we see that

$$\begin{aligned} \|w\|_{L^p} &\leq C\|w\|_{L^2}^{1-\theta}\|\nabla w\|_{L^p}^\theta + C\|w\|_{L^2} \\ &\leq C\|\nabla w\|_{L^2}^{1-\theta}\|\nabla w\|_{L^p}^\theta + C\|\nabla w\|_{L^2} \\ &\leq C\|\nabla w\|_{L^p}^{1-\theta}\|\nabla w\|_{L^p}^\theta + C\|\nabla w\|_{L^p} \leq C\|\nabla w\|_{L^p}. \end{aligned}$$

This completes the proof. \square

Lemma 2.2 ([19]). *For any smooth vector w satisfying $w \cdot n = 0$ or $w \times n = 0$ on $\partial\Omega$ and $1 < p < \infty$*

$$(2.2) \quad \|\nabla w\|_{L^p} \leq C(\|\operatorname{div} w\|_{L^p} + \|\operatorname{rot} w\|_{L^p}).$$

Lemma 2.3 ([4]). *For any smooth vector f and $1 < p < \infty$*

$$(2.3) \quad \begin{aligned} -\int_{\Omega} \Delta f \cdot f |f|^{p-2} dx &= \int_{\Omega} |f|^{p-2} |\nabla f|^2 dx + 4 \frac{p-2}{p^2} \int_{\Omega} |\nabla |f|^{p/2}|^2 dx \\ &\quad - \int_{\partial\Omega} |f|^{p-2} (n \cdot \nabla) f \cdot f dS. \end{aligned}$$

Lemma 2.4 ([3], Lemma 2.2). *Assume that u is sufficiently smooth and satisfies boundary condition (1.3) on $\partial\Omega$. Then the following identity for $\omega := \operatorname{rot} u$ holds on $\partial\Omega$*

$$(2.4) \quad -\frac{\partial\omega}{\partial n} \cdot \omega = (\varepsilon_{1jk}\varepsilon_{1\beta\gamma} + \varepsilon_{2jk}\varepsilon_{2\beta\gamma} + \varepsilon_{3jk}\varepsilon_{3\beta\gamma})\omega_j\omega_\beta\partial_k n_\gamma,$$

where ε_{ijk} denotes the totally anti-symmetric tensor such that $(a \times b)_i = \varepsilon_{ijk}a_jb_k$.

Lemma 2.5 ([1], Lemma 7.44, and [15], Corollary 1.7). *For any smooth f and $1 < p < \infty$*

$$(2.5) \quad \|f\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\Omega)}^{1-1/p}\|f\|_{W^{1,p}(\Omega)}^{1/p}.$$

Proof. We have

$$\|f\|_{L^p(\partial\Omega)} \leq C\|f\|_{W^{1,p,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}^{1-1/p}\|f\|_{W^{1,p}(\Omega)}^{1/p}.$$

\square

Lemma 2.6 ([2]). For any smooth f and $1 \leq q < p < \infty$

$$(2.6) \quad \|f\|_{L^p(\Omega)} \leq C \|f\|_{L^q(\Omega)}^{q/p} \|f\|_{\text{BMO}(\Omega)}^{1-q/p}.$$

Lemma 2.7 ([18]). Let $1 \leq q, r \leq \infty$ and $0 \leq j < m$. Then the inequality

$$(2.7) \quad \|D^j u\|_{L^p} \leq C_1 \|D^m u\|_{L^r}^a \|u\|_{L^q}^{1-a} + C_2 \|u\|_{L^s}$$

holds for any function $u: \Omega \rightarrow \mathbb{R}$ defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, where

$$(2.8) \quad \frac{1}{p} = \frac{j}{N} + a \left(\frac{1}{r} - \frac{m}{N} \right) + (1-a) \frac{1}{q}, \quad \frac{i}{m} \leq a \leq 1,$$

$s > 0$ is arbitrary and C_1 and C_2 depend only on Ω , m and N .

Lemma 2.8 ([11]). Let b be a solution to the Poisson equation

$$-\Delta b = f \quad \text{in } \Omega$$

with boundary condition

$$b \cdot n = 0, \quad \text{rot } b \times n = 0 \quad \text{on } \partial\Omega.$$

Then

$$(2.9) \quad \|b\|_{H^2} \leq C \|f\|_{L^2} + C \|\nabla b\|_{L^2}.$$

Lemma 2.9. Let u be a solution to the Stokes system

$$(2.10) \quad -\Delta u + \nabla \pi = f \quad \text{and} \quad \text{div } u = 0 \quad \text{in } \Omega$$

with boundary condition

$$(2.11) \quad u \cdot n = 0, \quad \text{rot } u \times n = 0 \quad \text{on } \partial\Omega.$$

Then

$$(2.12) \quad \|u\|_{H^2} \leq C \|f\|_{L^2}.$$

Proof. First, using Lemma 2.8 we have

$$(2.13) \quad \|u\|_{H^2} \leq C\|f - \nabla\pi\|_{L^2} + C\|\nabla u\|_{L^2} \leq C\|f\|_{L^2} + C\|\nabla\pi\|_{L^2} + C\|\nabla u\|_{L^2}.$$

Testing (2.10) by u and using $\operatorname{div} u = 0$, we get

$$\|\operatorname{rot} u\|_{L^2}^2 = \int_{\Omega} f u \, dx \leq \|f\|_{L^2} \|u\|_{L^2} \leq C\|f\|_{L^2} \|\operatorname{rot} u\|_{L^2},$$

which gives

$$(2.14) \quad \|\nabla u\|_{L^2} \leq C\|\operatorname{rot} u\|_{L^2} \leq C\|f\|_{L^2}.$$

Boundary condition

$$\operatorname{rot} u \times n = 0$$

leads to [5]

$$n \cdot \Delta u = n \cdot \operatorname{rot}^2 u = 0,$$

and thus

$$(2.15) \quad \frac{\partial \pi}{\partial n} = f \cdot n \quad \text{on } \partial\Omega.$$

Taking div in (2.10), we observe that

$$(2.16) \quad \Delta \pi = \operatorname{div} f \quad \text{in } \Omega.$$

Testing (2.16) by $(-\pi)$ and using (2.15), we have

$$\|\nabla \pi\|_{L^2}^2 = \int_{\Omega} f \cdot \nabla \pi \, dx \leq \|f\|_{L^2} \|\nabla \pi\|_{L^2}$$

which leads to

$$(2.17) \quad \|\nabla \pi\|_{L^2} \leq \|f\|_{L^2}.$$

Inequalities (2.13), (2.14), and (2.17) imply that (2.12) holds true.

This completes the proof. \square

Lemma 2.10 ([5], [7]). *Let s be a non-negative real number. If u belonging to $H^2(\Omega)$ is such that $\Delta u \in H^s(\Omega)$ and such that*

$$\operatorname{div} u = 0 \quad \text{in } \Omega \quad \text{and} \quad u \times n = 0 \quad \text{on } \partial\Omega,$$

$$\text{or such that } u \cdot n = 0, \operatorname{rot} u \times n = 0 \text{ on } \partial\Omega,$$

then

$$(2.18) \quad \|u\|_{H^{s+2}(\Omega)} \leq C(\|\Delta u\|_{H^s(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Using Lemma 2.10, we have

$$\begin{aligned} \|u\|_{H^3} &\leq C(\|\Delta u\|_{H^1} + \|u\|_{L^2}) = C(\|\operatorname{rot} \omega\|_{H^1} + \|u\|_{L^2}) \\ &\leq C(\|\omega\|_{H^2} + \|u\|_{L^2}) \leq C(\|\Delta \omega\|_{L^2} + \|\omega\|_{L^2} + \|u\|_{L^2}), \end{aligned}$$

which will be used in proving (3.10).

3. PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorem 1.1. Since it is easy to show the well-posedness of local strong solutions, we omit the details here and we only need to establish some a priori estimates.

First, testing (1.2) by u and using (1.1) and (1.3), we find that

$$\frac{1}{2} \frac{d}{dt} \int |u|^2 dx + \int |\operatorname{rot} u|^2 dx = 0,$$

which gives

$$(3.1) \quad \|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \leq C.$$

Taking rot to (1.2) and using (1.1), we get the well-known equation

$$(3.2) \quad \partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u - \Delta \omega = 0.$$

Testing the above equation by $|\omega|^{p-2} \omega$ ($2 \leq p < \infty$) and using (1.1), (2.3), (2.4), (2.5), and (2.6), we obtain

$$\begin{aligned} &\frac{1}{p} \frac{d}{dt} \int_{\Omega} |\omega|^p dx + \int_{\Omega} |\omega|^{p-2} |\nabla \omega|^2 dx + 4 \frac{p-2}{p^2} \int_{\Omega} |\nabla |\omega|^{p/2}|^2 dx \\ &= \int_{\partial \Omega} |\omega|^{p-2} (n \cdot \nabla) \omega \cdot \omega dS + \int_{\Omega} (\omega \cdot \nabla) u \cdot |\omega|^{p-2} \omega dx \\ &= - \int_{\partial \Omega} |\omega|^{p-2} \sum_i \varepsilon_{ijk} \varepsilon_{i\beta\gamma} \omega_j \omega_\beta \partial_k n_\gamma dS + \int_{\Omega} (\omega \cdot \nabla) u \cdot |\omega|^{p-2} \omega dx \\ &\leq C \int_{\partial \Omega} |\omega|^p dS + \|\omega\|_{L^{p+1}}^p \|\nabla u\|_{L^{p+1}} \\ &\leq C \int_{\partial \Omega} f^2 dS + C \|\omega\|_{L^{p+1}}^{p+1} \quad (f := |\omega|^{p/2}) \\ &\leq C \|f\|_{L^2(\Omega)} \|f\|_{H^1(\Omega)} + C \|\omega\|_{L^{p+1}}^{p+1} \\ &\leq 2 \frac{p-2}{p^2} \int_{\Omega} |\nabla f|^2 dx + C \|\omega\|_{L^p}^p + C \|\omega\|_{\text{BMO}} \|\omega\|_{L^p}^p, \end{aligned}$$

which gives

$$\begin{aligned} \frac{d}{dt} \|\omega\|_{L^p} &\leq C \|\omega\|_{L^p} (1 + \|\omega\|_{\text{BMO}}) \\ &\leq C \|\omega\|_{L^p} \log(e + \|\omega\|_{\text{BMO}}) \frac{1 + \|\omega\|_{\text{BMO}}}{\log(e + \|\omega\|_{\text{BMO}})} \\ &\leq C \|\omega\|_{L^p} \log(e + \|u\|_{H^3}) \frac{\|\omega\|_{\text{BMO}}}{\log(e + \|\omega\|_{\text{BMO}})} \end{aligned}$$

and therefore,

$$(3.3) \quad \int_{\Omega} |\omega|^p dx \leq C(e + y)^{C_0 \varepsilon}$$

provided that

$$(3.4) \quad \int_{t_0}^t \frac{\|\omega\|_{\text{BMO}}}{\log(e + \|\omega\|_{\text{BMO}})} ds \leq \varepsilon \ll 1$$

and $y(t) := \sup_{[t_0, t]} \|u\|_{H^3}$ for any $0 < t_0 \leq t \leq T$ and C_0 is an absolute constant.

Here we have used the estimate

$$\|\omega(\cdot, t_0)\|_{L^p} \leq C$$

by the standard energy estimate and we omit the details.

Testing (1.2) by $\partial_t u$ and using (1.1) and (3.3), we derive

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\text{rot } u|^2 dx + \int_{\Omega} |\partial_t u|^2 dx \\ &= - \int (u \cdot \nabla) u \cdot \partial_t u dx \leq \|u\|_{L^6} \|\nabla u\|_{L^3} \|\partial_t u\|_{L^2} \\ &\leq C \|\nabla u\|_{L^2} \|\nabla u\|_{L^3} \|\partial_t u\|_{L^2} \leq \frac{1}{2} \|\partial_t u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^3}^2 \\ &\leq \frac{1}{2} \|\partial_t u\|_{L^2}^2 + C \|\omega\|_{L^3}^4, \end{aligned}$$

which implies

$$(3.5) \quad \int_{t_0}^t \|\partial_t u\|_{L^2}^2 ds \leq C(e + y)^{C_0 \varepsilon}.$$

Here we have used the facts

$$\int_{\Omega} \nabla \pi \cdot \partial_t u dx = 0$$

and

$$-\int_{\Omega} \Delta u \cdot \partial_t u \, dx = \int_{\Omega} \operatorname{rot}^2 u \cdot \partial_t u \, dx = \int_{\Omega} \operatorname{rot} u \cdot \operatorname{rot} \partial_t u \, dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\operatorname{rot} u|^2 \, dx$$

and

$$-\Delta u = \operatorname{rot}^2 u,$$

since

$$\operatorname{div} u = 0.$$

Applying ∂_t to (1.2), we have

$$(3.6) \quad \partial_t^2 u + u \cdot \nabla \partial_t u + \nabla \partial_t \pi - \Delta \partial_t u = -\partial_t u \cdot \nabla u.$$

Testing (3.6) by $\partial_t u$ and using (1.1), (3.3), and (3.5), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\partial_t u|^2 \, dx + \int_{\Omega} |\operatorname{rot} \partial_t u|^2 \, dx &= - \int_{\Omega} \partial_t u \cdot \nabla u \cdot \partial_t u \, dx \\ &\leq \|\nabla u\|_{L^6} \|\partial_t u\|_{L^2} \|\partial_t u\|_{L^3} \leq C \|\omega\|_{L^6} \|\partial_t u\|_{L^2}^{3/2} \|\operatorname{rot} \partial_t u\|_{L^2}^{1/2} \\ &\leq \frac{1}{2} \|\operatorname{rot} \partial_t u\|_{L^2}^2 + C \|\omega\|_{L^6}^{4/3} \|\partial_t u\|_{L^2}^2, \end{aligned}$$

which gives

$$(3.7) \quad \int_{\Omega} |\partial_t u|^2 \, dx + \int_{t_0}^t \int_{\Omega} |\operatorname{rot} \partial_t u|^2 \, dx \, ds \leq C(e+y)^{C_0 \varepsilon}.$$

Here we have used the fact that

$$-\int_{\Omega} \Delta \partial_t u \cdot \partial_t u \, dx = \int_{\Omega} \operatorname{rot}^2 \partial_t u \cdot \partial_t u \, dx = \int_{\Omega} |\operatorname{rot} \partial_t u|^2 \, dx.$$

On the other hand, thanks to the H^2 -theory of the Stokes system (see Lemma 2.7), it follows from (1.2), (3.3), and (3.7) that

$$(3.8) \quad \begin{aligned} \|u\|_{H^2} &\leq C \|-\Delta u + \nabla \pi\|_{L^2} \leq C \|\partial_t u + u \cdot \nabla u\|_{L^2} \\ &\leq C \|\partial_t u\|_{L^2} + C \|u\|_{L^6} \|\nabla u\|_{L^3} \leq C \|\partial_t u\|_{L^2} + C \|\nabla u\|_{L^2} \|\nabla u\|_{L^3} \\ &\leq C \|\partial_t u\|_{L^2} + C \|\omega\|_{L^2} \|\omega\|_{L^3} \leq C(e+y)^{C_0 \varepsilon}. \end{aligned}$$

Testing (3.6) by $-\Delta \partial_t u + \nabla \partial_t \pi$ and using (1.1), (3.7), and (3.8), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\operatorname{rot} \partial_t u|^2 \, dx + \int_{\Omega} |-\Delta \partial_t u + \nabla \partial_t \pi|^2 \, dx \\ &= \int_{\Omega} (-\partial_t u \cdot \nabla u - u \cdot \nabla \partial_t u) (-\Delta \partial_t u + \nabla \partial_t \pi) \, dx \\ &\leq (\|\nabla u\|_{L^3} \|\partial_t u\|_{L^6} + \|u\|_{L^\infty} \|\nabla \partial_t u\|_{L^2}) \|-\Delta \partial_t u + \nabla \partial_t \pi\|_{L^2} \\ &\leq C \|u\|_{H^2} \|\operatorname{rot} \partial_t u\|_{L^2} \|-\Delta \partial_t u + \nabla \partial_t \pi\|_{L^2} \\ &\leq \frac{1}{2} \|-\Delta \partial_t u + \nabla \partial_t \pi\|_{L^2}^2 + C \|u\|_{H^2}^2 \|\operatorname{rot} \partial_t u\|_{L^2}^2, \end{aligned}$$

which leads to

$$(3.9) \quad \int_{\Omega} |\operatorname{rot} \partial_t u|^2 dx + \int_{t_0}^t \|\partial_t u\|_{H^2}^2 ds \leq C(e + y)^{C_0 \varepsilon}.$$

Here we have used the fact that

$$\begin{aligned} \int_{\Omega} \partial_t^2 u (-\Delta \partial_t u + \nabla \partial_t \pi) dx &= \int_{\Omega} \partial_t^2 u (\operatorname{rot}^2 \partial_t u + \nabla \partial_t \pi) dx \\ &= \int_{\Omega} \partial_t^2 u \cdot \operatorname{rot}^2 \partial_t u dx = \int_{\Omega} \operatorname{rot} \partial_t^2 u \cdot \operatorname{rot} \partial_t u dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\operatorname{rot} \partial_t u|^2 dx \end{aligned}$$

due to

$$\int_{\Omega} \partial_t^2 u \cdot \nabla \partial_t \pi dx = 0.$$

Here we also have used the fact that

$$\|\operatorname{rot} \partial_t u(\cdot, t_0)\|_{L^2} \leq C$$

by the standard energy method and we omit the details.

On the other hand, it follows from (2.18), (3.2), (3.3), (3.8), and (3.9) that

$$\begin{aligned} \|u\|_{H^3} &\leq C(\|u\|_{L^2} + \|\omega\|_{L^2} + \|\Delta \omega\|_{L^2}) \\ &\leq C(1 + \|\partial_t \omega + u \cdot \nabla \omega - \omega \cdot \nabla u\|_{L^2}) \\ &\leq C + C\|\partial_t \omega\|_{L^2} + C\|u\|_{L^\infty} \|\nabla \omega\|_{L^2} + C\|\omega\|_{L^4} \|\nabla u\|_{L^4} \\ &\leq C + C\|\partial_t \omega\|_{L^2} + C\|u\|_{H^2}^2 \\ &\leq C(e + y)^{C_0 \varepsilon}, \end{aligned}$$

which gives

$$(3.10) \quad \|u\|_{L^\infty(0, T; H^3)} \leq C.$$

This completes the proof. □

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