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# Artinianness of formal local cohomology modules

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*Abstract.* Let $\mathfrak{a}$ be an ideal of Noetherian local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module of dimension $d$. In this paper we investigate the Artinianness of formal local cohomology modules under certain conditions on the local cohomology modules with respect to $\mathfrak{m}$. Also we prove that for an arbitrary local ring $(R, \mathfrak{m})$ (not necessarily complete), we have $\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \mathrm{MinV}(\mathrm{Ann}_R\mathfrak{F}^d_{\mathfrak{a}}(M))$.

*Keywords:* formal local cohomology; local cohomology

*Classification:* 13D45, 13E99

## 1. Introduction

Throughout this paper, $R$ is a commutative Noetherian ring with identity, $\mathfrak{a}$ is an ideal of $R$ and $M$ is an $R$-module. Recall that the $i$th local cohomology module of $M$ with respect to $\mathfrak{a}$ is denoted by $\mathrm{H}^i_{\mathfrak{a}}(M)$. For basic facts about local cohomology refer to [3]. Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module.

$$\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_n \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \qquad \text{for each} \quad i \geq 0$$

is called the $i$th formal local cohomology of $M$ with respect to $\mathfrak{a}$.

It is known that if $(R, \mathfrak{m})$ is a regular local ring, then

$$\mathfrak{F}^i_{\mathfrak{a}}(R) \simeq \mathrm{Hom}_R(\mathrm{H}^{\dim R - i}_{\mathfrak{a}}(R), \mathrm{E}_R(R/\mathfrak{m}))$$

for all $i \geq 0$, see [8, III, Proposition 2.2], also when $(R, \mathfrak{m})$ is a quotient of a local Gorenstein ring formal local cohomology modules have been studied in [11]. The basic properties of formal local cohomology modules are found in [11], [1], [4], [2], [9] and [10].

A nonzero $R$-module $M$ is called secondary if its multiplication map by any element $a$ of $R$ is either surjective or nilpotent. A secondary representation for an $R$-module $M$ is an expression for $M$ as a finite sum of secondary modules. If such a representation exists, we will say that $M$ is representable. A prime ideal $\mathfrak{p}$ of $R$ is said to be an attached prime of $M$ if $\mathfrak{p} = (N :_R M)$ for some submodule $N$ of $M$. If $M$ admits a reduced secondary representation, $M = S_1 + S_2 + \cdots + S_n$, then the set of attached primes $\mathrm{Att}_R(M)$ of $M$ is equal to $\{\sqrt{(0 :_R S_i)} \colon i = 1, \ldots, n\}$, see [5].

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Recall that $\operatorname{Assh}(M)$ denotes the set $\{\mathfrak{p} \in \operatorname{Ass}(M)\colon \dim(R/\mathfrak{p}) = \dim(M)\}$. It is well known that Artinian modules are representable and the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(M)$ are Artinian for all $i \geq 0$ and $\operatorname{Att}_R(\mathrm{H}^{\dim M}_{\mathfrak{m}}(M)) = \operatorname{Assh}(M)$, see [6, Theorem 2.2].

In this paper we investigate some Artinianness properties of formal local cohomology modules under certain conditions on the local cohomology modules with respect to $\mathfrak{m}$. The following theorem is one of our main results:

**Theorem 1.1.** *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module. Let $i$ be an integer and $\mathrm{H}^{i+1}_{\mathfrak{m}}(M/\mathfrak{b}M)$ be finitely generated for any ideal $\mathfrak{b} \subseteq \mathfrak{a}$. Then there exists an integer $n_0$ such that $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathrm{H}^i_{\mathfrak{m}}(M)/(\mathfrak{a}^n \mathrm{H}^i_{\mathfrak{m}}(M))$ for all $n \geq n_0$. Therefore $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian and $\operatorname{Att}_R(\mathfrak{F}^i_{\mathfrak{a}}(M)) = \operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M)) \cap V(\mathfrak{a})$.*

Recall that $\operatorname{ara}(\mathfrak{a})$, the arithmetic rank of $\mathfrak{a}$, is the least number of elements of $R$ required to generate an ideal which has the same radical as $\mathfrak{a}$. Also the finiteness dimension of $M$ relative to $\mathfrak{a}$, denoted by $f_{\mathfrak{a}}(M)$, is the least integer $i$ such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is not finitely generated. Here we define $Lq_{\mathfrak{a}}(M) = \inf\{i\colon \mathfrak{F}^i_{\mathfrak{a}}(M) \text{ is not Artinian}\}$ and we show that $f_{\mathfrak{m}}(M) - \operatorname{ara}(\mathfrak{a}) \leq Lq_{\mathfrak{a}}(M)$.

In [10] we showed that, if $(R, \mathfrak{m})$ is a complete local ring, $\mathfrak{a}$ an ideal of $R$ and $M$ a finitely generated $R$-module of dimension $d$, then $\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Min} \mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M))$. In this paper, we eliminate the complete hypothesis entirely by proving the following:

**Theorem 1.2.** *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module of dimension $d$. Then $\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Min} \mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M))$.*

## 2. Main results

First we recall the following results which we will use in this paper.

**Lemma 2.1.** *Let $M$ be an Artinian $R$-module and $N$ a finitely generated $R$-module. Then $\operatorname{Att}_R(M \otimes_R N) = \operatorname{Att}_R M \cap \operatorname{Supp}_R N$.*

Proof: See [7, Proposition 5.2]. □

**Theorem 2.2.** *Let $M$ be an Artinian $R$-module and $S$ a multiplicative set of $R$. Then $\operatorname{Hom}_R(R_S, M)$ is a representable $R$-module and $\operatorname{Att}_R(\operatorname{Hom}_R(R_S, M)) = \{\mathfrak{p} \in \operatorname{Att}_R M \colon \mathfrak{p} \cap S = \phi\}$.*

Proof: By [7, Theorem 3.2], $\operatorname{Hom}_R(R_S, M)$ is a representable $R_S$-module and $\operatorname{Att}_{R_S}(\operatorname{Hom}_R(R_S, M)) = \{\mathfrak{p}R_S \colon \mathfrak{p} \in \operatorname{Att}_R M,\ \mathfrak{p} \cap S = \phi\}$. Now the result follows by [12, Lemma 4.6]. □

The following result shows that every representable formal local cohomology module with respect to $\mathfrak{a}$ is an $\mathfrak{a}$-torsion module.

**Theorem 2.3.** *Let $M$ be a finitely generated module and $(R,\mathfrak{m})$ a Noetherian local ring. If $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is nonzero and representable for some integer $i$ then* $\operatorname{Att}_R \mathfrak{F}^i_{\mathfrak{a}}(M) \subseteq V(\mathfrak{a})$ *and* $\mathfrak{a} \subseteq \sqrt{(0 :_R \mathfrak{F}^i_{\mathfrak{a}}(M))}$.

Proof: See [2, Theorem 2.3] and [2, Corollary 2.4]. □

We need the following lemma in the proof of the next theorem.

**Lemma 2.4.** *Let $(R,\mathfrak{m})$ be a local ring and $M$ an $R$-module. Then the module* $\operatorname{Hom}_R(R_x, M) = 0$ *for all* $x \in \sqrt{\operatorname{Ann}_R(M)}$.

Proof: Since $x \in \sqrt{\operatorname{Ann}_R(M)}$, there is an integer $t$ such that $x^t M = 0$. If $f \in \operatorname{Hom}_R(R_x, M)$ then $f(1/x^n) = x^t f(1/x^{t+n}) \in x^t M = 0$ for all $n \in \mathbb{N}$. Thus $f(1/x^n) = 0$ for all $n \in \mathbb{N}$. Therefore $f = 0$. □

**Theorem 2.5.** *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and $M$ a finitely generated $R$-module. Let $i$ be an integer and $x \in R$ be an element. If* $x \in \sqrt{(0 : \mathrm{H}^{i+1}_{\mathfrak{m}}(M))}$, *then there exists an integer $n_0$ such that* $\mathfrak{F}^i_{\langle x\rangle}(M) \simeq \mathrm{H}^i_{\mathfrak{m}}(M)/(\langle x\rangle^n \mathrm{H}^i_{\mathfrak{m}}(M))$ *for all $n \geq n_0$. Therefore $\mathfrak{F}^i_{\langle x\rangle}(M)$ is Artinian and* $\operatorname{Att}_R(\mathfrak{F}^i_{\langle x\rangle}(M)) = \operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M)) \cap V(\langle x\rangle)$.

Proof: By [11, Corollary 3.16], there exists a long exact sequence

$$\cdots \to \operatorname{Hom}_R(R_x, \mathrm{H}^i_{\mathfrak{m}}(M)) \xrightarrow{\varphi} \mathrm{H}^i_{\mathfrak{m}}(M) \to \mathfrak{F}^i_{\langle x\rangle}(M) \to \operatorname{Hom}_R(R_x, \mathrm{H}^{i+1}_{\mathfrak{m}}(M)) \to \cdots.$$

If $\mathrm{H}^{i+1}_{\mathfrak{m}}(M) = 0$, then $\operatorname{Hom}_R(R_x, \mathrm{H}^{i+1}_{\mathfrak{m}}(M)) = 0$. Thus from the above long exact sequence we see that $\mathfrak{F}^i_{\langle x\rangle}(M)$ is a homomorphic image of $\mathrm{H}^i_{\mathfrak{m}}(M)$, and so $\mathfrak{F}^i_{\langle x\rangle}(M)$ is Artinian. Now assume that $\mathrm{H}^{i+1}_{\mathfrak{m}}(M) \neq 0$. By assumption $x \in \sqrt{(0 : \mathrm{H}^{i+1}_{\mathfrak{m}}(M))}$ and so $\operatorname{Hom}_R(R_x, \mathrm{H}^{i+1}_{\mathfrak{m}}(M)) = 0$ by Lemma 2.4. Thus from the above long exact sequence, we conclude that there exists an exact sequence

$$0 \to \operatorname{Im}(\varphi) \to \mathrm{H}^i_{\mathfrak{m}}(M) \to \mathfrak{F}^i_{\langle x\rangle}(M) \to 0.$$

Hence $\mathfrak{F}^i_{\langle x\rangle}(M)$ is homomorphic image of an Artinian module and so is Artinian. Thus there is an integer $n_0$ such that $\langle x\rangle^n \mathfrak{F}^i_{\langle x\rangle}(M) = 0$ for all $n \geq n_0$ by Theorem 2.3. Let $n \geq n_0$. Then from the above exact sequence we have the following exact sequence:

$$\to \frac{\operatorname{Im}\varphi}{\langle x\rangle^n \operatorname{Im}\varphi} \to \frac{\mathrm{H}^i_{\mathfrak{m}}(M)}{\langle x\rangle^n \mathrm{H}^i_{\mathfrak{m}}(M)} \to \frac{\mathfrak{F}^i_{\langle x\rangle}(M)}{\langle x\rangle^n \mathfrak{F}^i_{\langle x\rangle}(M)} \to 0,$$

and so we have:

$$\to \frac{\operatorname{Im}\varphi}{\langle x\rangle^n \operatorname{Im}\varphi} \to \frac{\mathrm{H}^i_{\mathfrak{m}}(M)}{\langle x\rangle^n \mathrm{H}^i_{\mathfrak{m}}(M)} \to \mathfrak{F}^i_{\langle x\rangle}(M) \to 0.$$

Since

$$\operatorname{Att}_R\Big(\frac{\operatorname{Im}\varphi}{\langle x\rangle^k \operatorname{Im}\varphi}\Big) = V(\langle x\rangle)\cap \operatorname{Att}_R(\operatorname{Im}\varphi) \subseteq V(\langle x\rangle)\cap \operatorname{Att}_R(\operatorname{Hom}_R(R_x, \mathrm{H}^i_{\mathfrak{m}}(M))$$

and by Theorem 2.2, $V(\langle x\rangle)\cap \operatorname{Att}_R(\operatorname{Hom}_R(R_x, \mathrm{H}^i_{\mathfrak{m}}(M)) = \phi$ we have $\operatorname{Att}_R\big(\operatorname{Im}\varphi/ (\langle x\rangle^n \operatorname{Im}\varphi)\big) = \phi$ and so $\operatorname{Im}\varphi/(\langle x\rangle^n \operatorname{Im}\varphi) = 0$. Now from the above exact sequence we conclude that $\mathfrak{F}^i_{\langle x\rangle}(M) \simeq \mathrm{H}^i_{\mathfrak{m}}(M)/(\langle x\rangle^n \mathrm{H}^i_{\mathfrak{m}}(M))$ for all $n \geq n_0$. But $\mathrm{H}^i_{\mathfrak{m}}(M)$ is Artinian and so $\mathfrak{F}^i_{\langle x\rangle}(M)$ is Artinian. Now Lemma 2.1 completes the proof. □

**Corollary 2.6.** *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and $M$ a finitely generated $R$-module. Let $i$ be an integer. If $\mathrm{H}^{i+1}_{\mathfrak{m}}(M)$ is finitely generated, then $\mathfrak{F}^i_{\langle x\rangle}(M)$ is Artinian for all $x \in R$ and $\operatorname{Att}_R(\mathfrak{F}^i_{\langle x\rangle}(M)) = \operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M)) \cap V(\langle x\rangle)$.*

Proof: If $x \in R\setminus \mathfrak{m}$, then $\mathfrak{F}^i_{\langle x\rangle}(M) = 0$. Thus we can assume that $x \in \mathfrak{m}$. By assumption $\mathrm{H}^{i+1}_{\mathfrak{m}}(M)$ is finitely generated and so there exists $k \in \mathbb{N}$, such that $\mathfrak{m}^k \mathrm{H}^{i+1}_{\mathfrak{m}}(M) = 0$. This implies that $x \in \sqrt{(0 : \mathrm{H}^{i+1}_{\mathfrak{m}}(M))}$, the claim follows by Theorem 2.5. □

**Lemma 2.7.** *Let $(R,\mathfrak{m})$ be a local ring and $M$ a finitely generated $R$-module. Then $\operatorname{Hom}_R(R_x, M) = 0$ for all $x \in \mathfrak{m}$.*

Proof: If $f \in \operatorname{Hom}_R(R_x, M)$ then $f(1/x^n) = x^k f(1/x^{k+n}) \in x^k M$ for all $k, n \in \mathbb{N}$. Thus $f(1/x^n) \in \bigcap_k x^k M = 0$ for all $n \in \mathbb{N}$ by Krull's theorem. Therefore $f = 0$. □

**Theorem 2.8.** *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and $M$ a finitely generated $R$-module. Let $i$ be an integer and $\mathrm{H}^{i+1}_{\mathfrak{m}}(M/\mathfrak{b}M)$ be finitely generated for any ideal $\mathfrak{b}$ of $R$ with $\mathfrak{b} \subseteq \mathfrak{a}$. Then there exists an integer $n_0$ such that $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathrm{H}^i_{\mathfrak{m}}(M)/(\mathfrak{a}^n \mathrm{H}^i_{\mathfrak{m}}(M))$ for all $n \geq n_0$. Therefore $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian and $\operatorname{Att}_R(\mathfrak{F}^i_{\mathfrak{a}}(M)) = \operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M)) \cap V(\mathfrak{a})$.*

Proof: Assume that $\mathfrak{a} = (a_1, \ldots, a_t)$. We use induction on $t$. If $t = 1$ then the result follows by Corollary 2.6. Now suppose that $t > 1$ and that the result has been proved for $t-1$. Set $\mathfrak{b} := (a_1, \ldots, a_{t-1})$. By [11, Theorem 3.15], there exists an exact sequence

$$\cdots \to \operatorname{Hom}_R(R_{a_t}, \mathfrak{F}^i_{\mathfrak{b}}(M)) \xrightarrow{\varphi} \mathfrak{F}^i_{\mathfrak{b}}(M) \to \mathfrak{F}^i_{\mathfrak{a}}(M) \to \operatorname{Hom}_R(R_{a_t}, \mathfrak{F}^{i+1}_{\mathfrak{b}}(M)) \to \cdots.$$

By assumption $\mathrm{H}^{i+1}_{\mathfrak{m}}(M/\mathfrak{b}^n M)$ is finitely generated for all $n \in \mathbb{N}$ and so by Lemma 2.7 we have $\operatorname{Hom}_R(R_{a_t}, \mathrm{H}^{i+1}_{\mathfrak{m}}(M/\mathfrak{b}^n M)) = 0$ for all $n \in \mathbb{N}$. On the other hand

$$\begin{aligned}\operatorname{Hom}_R(R_{a_t},\mathfrak{F}_{\mathfrak{b}}^{i+1}(M)) &\simeq \operatorname{Hom}_R\Big(R_{a_t},\varprojlim_n \operatorname{H}_m^{i+1}\Big(\frac{M}{\mathfrak{b}^n M}\Big)\Big)\\ &\simeq \varprojlim_n \operatorname{Hom}_R\Big(R_{a_t},\operatorname{H}_m^{i+1}\Big(\frac{M}{\mathfrak{b}^n M}\Big)\Big).\end{aligned}$$

Therefore $\operatorname{Hom}_R(R_{a_t},\mathfrak{F}_{\mathfrak{b}}^{i+1}(M))=0$ and we get the following exact sequence:

$$0\to \operatorname{Im}\varphi\to \mathfrak{F}_{\mathfrak{b}}^{i}(M)\to \mathfrak{F}_{\mathfrak{a}}^{i}(M)\to 0.$$

But by the inductive hypothesis $\mathfrak{F}_{\mathfrak{b}}^{i}(M)$ is Artinian and from the above exact sequence we conclude that $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is Artinian. Thus by Theorem 2.3, there exists an integer $k_0$ such that $\mathfrak{a}^k\mathfrak{F}_{\mathfrak{a}}^{i}(M)=0$ for all $k\geq k_0$. Let $k\geq k_0$. Then from the above exact sequence we have the following exact sequence:

$$\to \frac{\operatorname{Im}\varphi}{\mathfrak{a}^k\operatorname{Im}\varphi}\to \frac{\mathfrak{F}_{\mathfrak{b}}^{i}(M)}{\mathfrak{a}^k\mathfrak{F}_{\mathfrak{b}}^{i}(M)}\to \frac{\mathfrak{F}_{\mathfrak{a}}^{i}(M)}{\mathfrak{a}^k\mathfrak{F}_{\mathfrak{a}}^{i}(M)}\to 0,$$

and so we have:

$$\to \frac{\operatorname{Im}\varphi}{\mathfrak{a}^k\operatorname{Im}\varphi}\to \frac{\mathfrak{F}_{\mathfrak{b}}^{i}(M)}{\mathfrak{a}^k\mathfrak{F}_{\mathfrak{b}}^{i}(M)}\to \mathfrak{F}_{\mathfrak{a}}^{i}(M)\to 0.$$

On the other hand,

$$\operatorname{Att}_R\Big(\frac{\operatorname{Im}\varphi}{\mathfrak{a}^k\operatorname{Im}\varphi}\Big)=V(\mathfrak{a})\cap\operatorname{Att}_R(\operatorname{Im}\varphi)\subseteq V(\mathfrak{a})\cap\operatorname{Att}_R(\operatorname{Hom}_R(R_{a_t},\mathfrak{F}_{\mathfrak{b}}^{i}(M)).$$

Since $a_t\in\mathfrak{a}$ by Theorem 2.2, we have $V(\mathfrak{a})\cap\operatorname{Att}_R(\operatorname{Hom}_R(R_{a_t},\mathfrak{F}_{\mathfrak{b}}^{i}(M))=\phi$. Hence $\operatorname{Att}_R(\operatorname{Im}\varphi/(\mathfrak{a}^k\operatorname{Im}\varphi))=\phi$ and so $\operatorname{Im}\varphi/(\mathfrak{a}^k\operatorname{Im}\varphi)=0$. Now from the above exact sequence we have $\mathfrak{F}_{\mathfrak{a}}^{i}(M)\simeq\mathfrak{F}_{\mathfrak{b}}^{i}(M)/(\mathfrak{a}^k\mathfrak{F}_{\mathfrak{b}}^{i}(M))$ for all $k\geq k_0$. But by the inductive hypothesis there exists an integer $u_0$ such that $\mathfrak{F}_{\mathfrak{b}}^{i}(M)\simeq \operatorname{H}_{\mathfrak{m}}^{i}(M)/(\mathfrak{b}^u\operatorname{H}_{\mathfrak{m}}^{i}(M))$ for all $u\geq u_0$. Assume that $n_0=\max\{k_0,u_0\}$. Thus we have $\mathfrak{F}_{\mathfrak{a}}^{i}(M)\simeq\operatorname{H}_{\mathfrak{m}}^{i}(M)/(\mathfrak{b}^n\operatorname{H}_{\mathfrak{m}}^{i}(M)+\mathfrak{a}^n\operatorname{H}_{\mathfrak{m}}^{i}(M))$ for all $n\geq n_0$ and since $\mathfrak{b}\subseteq\mathfrak{a}$ we get $\mathfrak{F}_{\mathfrak{a}}^{i}(M)\simeq\operatorname{H}_{\mathfrak{m}}^{i}(M)/(\mathfrak{a}^n\operatorname{H}_{\mathfrak{m}}^{i}(M))$ for all $n\geq n_0$. Thus $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is an Artinian module and Lemma 2.1 completes the proof. □

**Corollary 2.9.** *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and $M$ a finitely generated $R$-module of dimension $d$. Let $i\geq\dim\mathfrak{a}M$ be an integer and $\operatorname{H}_{\mathfrak{m}}^{i}(M)$ be finitely generated. Then $\mathfrak{F}_{\mathfrak{a}}^{i-1}(M)$ is Artinian and $\operatorname{Att}_R(\mathfrak{F}_{\mathfrak{a}}^{i-1}(M))=\operatorname{Att}_R(\operatorname{H}_{\mathfrak{m}}^{i-1}(M))\cap V(\mathfrak{a})$.*

Proof: Let $\mathfrak{b}\subseteq\mathfrak{a}$ be an ideal of $R$. The exact sequence

$$0\to\mathfrak{b}M\to M\to M/\mathfrak{b}M\to 0$$

induces the exact sequence

$$\cdots\to\operatorname{H}_{\mathfrak{m}}^{i}(M)\to\operatorname{H}_{\mathfrak{m}}^{i}(M/\mathfrak{b}M)\to\operatorname{H}_{\mathfrak{m}}^{i+1}(\mathfrak{b}M)\to\cdots.$$

Since $\dim \mathfrak{b}M \leq \dim \mathfrak{a}M < i+1$, by the Grothendieck's vanishing theorem [3, Theorem 6.1.2], $\mathrm{H}^{i+1}_{\mathfrak{m}}(\mathfrak{b}M) = 0$. From the above exact sequence we conclude that $\mathrm{H}^{i}_{\mathfrak{m}}(M/\mathfrak{b}M)$ is finitely generated. Now the result follows by Theorem 2.8. □

Now we can obtain the following corollary which is an improvement of [2, Theorem 3.1].

**Corollary 2.10.** *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module of dimension $d$. Then there exists an integer $n_0$ such that $\mathfrak{F}^d_{\mathfrak{a}}(M) \simeq \mathrm{H}^d_{\mathfrak{m}}(M)/(\mathfrak{a}^n \mathrm{H}^d_{\mathfrak{m}}(M))$ for all $n \geq n_0$. Thus $\mathfrak{F}^d_{\mathfrak{a}}(M)$ is Artinian and*

$$\mathrm{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M) = \mathrm{Assh}(M) \cap \mathrm{V}(\mathfrak{a}) = \Big\{\mathfrak{p} \in \mathrm{Ass}\, M \colon \dim \frac{\mathrm{R}}{\mathfrak{p}} = \dim \mathrm{M},\ \mathfrak{p} \supseteq \mathfrak{a}\Big\}.$$

Proof: By the Grothendieck's vanishing theorem [3, Theorem 6.1.2] we have $\mathrm{H}^{d+1}_{\mathfrak{m}}(M/N) = 0$ for any submodule $N$ of $M$. Thus by Theorem 2.8 there exists an integer $n_0$ such that $\mathfrak{F}^d_{\mathfrak{a}}(M) \simeq \mathrm{H}^d_{\mathfrak{m}}(M)/(\mathfrak{a}^n \mathrm{H}^d_{\mathfrak{m}}(M))$ for all $n \geq n_0$. Since $\mathrm{H}^d_{\mathfrak{m}}(M)$ is Artinian, $\mathfrak{F}^d_{\mathfrak{a}}(M)$ is Artinian and $\mathrm{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M) = \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{m}}(M)/(\mathfrak{a}^n \mathrm{H}^d_{\mathfrak{m}}(M))) = \mathrm{Att}_R \mathrm{H}^d_{\mathfrak{m}}(M) \cap V(\mathfrak{a})$. But $\mathrm{Att}_R \mathrm{H}^d_{\mathfrak{m}}(M) = \{\mathfrak{p} \in \mathrm{Ass}\, M \colon \dim \mathrm{R}/\mathfrak{p} = \dim \mathrm{M}\}$ and so we have $\mathrm{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M) = \{\mathfrak{p} \in \mathrm{Ass}\, M \colon \dim \mathrm{R}/\mathfrak{p} = \dim \mathrm{M}\} \cap \mathrm{V}(\mathfrak{a})$. Therefore the proof is complete. □

**Theorem 2.11.** *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module. Suppose that $\mathfrak{a}$ can be generated by $t$ elements. For every $i \geq 0$, if $\mathrm{H}^{i+1}_{\mathfrak{m}}(M), \mathrm{H}^{i+2}_{\mathfrak{m}}(M), \dots, \mathrm{H}^{i+t}_{\mathfrak{m}}(M)$ are finitely generated, then $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian.*

Proof: We use induction on $t$. When $t = 1$, the claim follows by Corollary 2.6. Now suppose, inductively, that $t > 1$ and the result has been proved for ideals that can be generated by fewer than $t$ elements. Suppose that $\mathfrak{a} = \langle a_1, \dots, a_t \rangle$. Set $\mathfrak{b} := Ra_1 + \cdots + Ra_{t-1}$ and $\mathfrak{c} := Ra_t$. By [11, Theorem 5.1], there exists a long exact sequence

$$\cdots \to \mathfrak{F}^i_{\mathfrak{b}\cap\mathfrak{c}}(M) \to \mathfrak{F}^i_{\mathfrak{b}}(M) \oplus \mathfrak{F}^i_{\mathfrak{c}}(M) \to \mathfrak{F}^i_{\mathfrak{b}+\mathfrak{c}}(M) \to \mathfrak{F}^{i+1}_{\mathfrak{b}\cap\mathfrak{c}}(M) \to \cdots.$$

By the inductive hypothesis, $\mathfrak{F}^i_{\mathfrak{b}}(M)$ and $\mathfrak{F}^i_{\mathfrak{c}}(M)$ are Artinian. Since $\mathfrak{bc}$ can be generated by $t-1$ elements, it follows from the inductive hypothesis that $\mathfrak{F}^{i+1}_{\mathfrak{bc}}(M)$ is Artinian. Since the $\mathfrak{bc}$-adic and the $\mathfrak{b} \cap \mathfrak{c}$-adic topology on $M$ are equivalent, by [11, Lemma 3.8] it follows that $\mathfrak{F}^{i+1}_{\mathfrak{b}\cap\mathfrak{c}}(M) \cong \mathfrak{F}^{i+1}_{\mathfrak{bc}}(M)$, also we have $\mathfrak{a} = \mathfrak{b} + \mathfrak{c}$. Now the above long exact sequence completes the inductive step. □

Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module. Recall that the finiteness dimension $f_{\mathfrak{a}}(M)$ of $M$ relative to $\mathfrak{a}$ is defined by

$$f_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 \colon\ \mathrm{H}^i_{\mathfrak{a}}(M) \text{ is not finitely generated}\}.$$

We define $Lq_{\mathfrak{a}}(M)$, the lower Artinianness dimension of $M$ with respect to $\mathfrak{a}$, as the least integer $i$ such that $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is not Artinian. In the next result we obtain a lower bound for $Lq_{\mathfrak{a}}(M)$.

**Theorem 2.12.** *Let $\mathfrak{a}$ be an ideal of local ring $(R,\mathfrak{m})$ and $M$ a finitely generated $R$-module. Then $f_{\mathfrak{m}}(M) - \operatorname{ara}(\mathfrak{a}) \leq Lq_{\mathfrak{a}}(M)$.*

Proof: Set $t := \operatorname{ara}(\mathfrak{a})$. Suppose that $\mathfrak{b}$ is an ideal of $R$ such that $\mathfrak{b}$ can be generated by $t$ elements and $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$. By [11, Lemma 3.8], $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathfrak{F}^i_{\mathfrak{b}}(M)$ for all $i \geq 0$. Hence $Lq_{\mathfrak{a}}(M) = Lq_{\mathfrak{b}}(M)$. Since $\mathrm{H}^0_{\mathfrak{m}}(M), \mathrm{H}^1_{\mathfrak{m}}(M), \cdots, \mathrm{H}^{f_{\mathfrak{m}}(M)-1}_{\mathfrak{m}}(M)$ are finitely generated, by Theorem 2.11, $\mathfrak{F}^i_{\mathfrak{b}}(M)$ is Artinian for all $i < f_{\mathfrak{m}}(M) - t$. Therefore $f_{\mathfrak{m}}(M) - t \leq Lq_{\mathfrak{b}}(M) = Lq_{\mathfrak{a}}(M)$, as required. □

**Theorem 2.13.** *Let $\mathfrak{a}$ be an ideal of local ring $(R,\mathfrak{m})$ and $M$ a finitely generated $R$-module and $i \geq 0$ an integer. Then $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathfrak{F}^i_{\mathfrak{a}+\bigcap_{j\geq i}\sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}}(M)$.*

Proof: Let $x \in \bigcap_{j\geq i}\sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}$ . Thus $\operatorname{Hom}_R(R_x, \mathfrak{F}^j_{\mathfrak{a}}(M)) = 0$ for all $j \geq i$ by Lemma 2.4. Hence the exact sequence

$$\cdots \to \operatorname{Hom}_R(R_x, \mathfrak{F}^i_{\mathfrak{a}}(M)) \to \mathfrak{F}^i_{\mathfrak{a}}(M) \to \mathfrak{F}^i_{\langle\mathfrak{a},x\rangle}(M) \to \operatorname{Hom}_R(R_x, \mathfrak{F}^{i+1}_{\mathfrak{a}}(M)) \to \cdots$$

implies that $\mathfrak{F}^j_{\mathfrak{a}}(M) \simeq \mathfrak{F}^j_{\langle\mathfrak{a},x\rangle}(M)$ for all $j \geq i$. For any $y \in \bigcap_{j\geq i}\sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}$, by replacing $\mathfrak{F}^j_{\mathfrak{a}}(M)$ with $\mathfrak{F}^j_{\langle\mathfrak{a},x\rangle}(M)$ for all $j \geq i$ in the above long exact sequence, we get $\mathfrak{F}^j_{\langle\mathfrak{a},x\rangle}(M) \simeq \mathfrak{F}^j_{\langle\mathfrak{a},x,y\rangle}(M)$. Continuing in this way completes the proof. □

**Corollary 2.14.** *Let $\mathfrak{a}$ be an ideal of local ring $(R,\mathfrak{m})$ and $M$ a finitely generated $R$-module and $i \geq 0$ an integer. If $\mathfrak{F}^j_{\mathfrak{a}}(M)$ is Artinian for all $j \geq i$ then $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathfrak{F}^i_{\bigcap_{j\geq i}\sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}}(M)$.*

Proof: By [2, Theorem 2.9], $\mathfrak{a} \subseteq \sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}$ for all $j \geq i$. Thus $\mathfrak{a} \subseteq \bigcap_{j\geq i}\sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}$ for all $j \geq i$. Therefore Theorem 2.13 implies that

$$\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathfrak{F}^i_{\mathfrak{a}+\bigcap_{j\geq i}\sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}}(M) \simeq \mathfrak{F}^i_{\bigcap_{j\geq i}\sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}}(M),$$

as required. □

The following result is an extension of [10, Corollary 2.7] for an arbitrary Noetherian local ring $(R,\mathfrak{m})$. Here $R$ is not necessarily complete.

**Corollary 2.15.** *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and $M$ a finitely generated $R$-module of dimension $d$. Then $\mathfrak{F}^d_{\mathfrak{a}}(M) \simeq \mathfrak{F}^d_{\operatorname{Ann}_R(\mathfrak{F}^d_{\mathfrak{a}}(M))}(M)$.*

Proof: By [1, Lemma 2.2] the module $\mathfrak{F}^d_{\mathfrak{a}}(M)$ is Artinian and by [11, Theorem 4.5] the module $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $i > d$. Thus by Corollary 2.14 we have

$$\mathfrak{F}^d_{\mathfrak{a}}(M) \simeq \mathfrak{F}^d_{\sqrt{\operatorname{Ann}_R(\mathfrak{F}^d_{\mathfrak{a}}(M))}}(M) \simeq \mathfrak{F}^d_{\operatorname{Ann}_R(\mathfrak{F}^d_{\mathfrak{a}}(M))}(M),$$

as required. □

In the next result we provide a generalization of [10, Theorem 2.11 (ii)] by eliminating the complete hypothesis.

**Theorem 2.16.** *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and $M$ a finitely generated $R$-module of dimension $d$. Then* $\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Min} \operatorname{V}(\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M))$.

Proof: Since $\mathfrak{F}^d_{\mathfrak{a}}(M)$ is Artinian we have

$$\operatorname{Min} \operatorname{V}(\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Min} \operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) \subseteq \operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)).$$

On the other hand, by Corollary 2.15 and Corollary 2.10

$$\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Att}_R(\mathfrak{F}^d_{\operatorname{Ann}_R(\mathfrak{F}^d_{\mathfrak{a}}(M))}(M)) = \operatorname{V}(\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M)) \cap \operatorname{Assh}(M).$$

It is easy to see that, by using definition of $\operatorname{Assh}(M)$, $\operatorname{V}(\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M)) \cap \operatorname{Assh}(M) \subseteq \operatorname{Min} \operatorname{V}(\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M))$ and so the proof is complete. □

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