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# A BINARY OPERATION-BASED REPRESENTATION OF A LATTICE

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In this paper, we study and characterize some properties of a given binary operation on a lattice. More specifically, we show necessary and sufficient conditions under which a binary operation on a lattice coincides with its meet (resp. its join) operation. Importantly, we construct two new posets based on a given binary operation on a lattice and investigate some cases that these two posets have a lattice structure. Moreover, we provide some representations of a given lattice based on these new constructed lattices.

*Keywords:* lattice, binary operation, neutral element, lattice representation

*Classification:* 06B05, 06B15

## 1. INTRODUCTION

Binary operations are among the oldest fundamental concepts in algebraic structures. A binary operation on a non-empty set $X$ is any function from the Cartesian product $X \times X$ into $X$. Since their introduction, they have become the key notion in the definitions of groups, monoids, semigroups, rings, and in more algebraic structures studied in abstract algebra [5, 16, 18, 26]. Binary operations have become essential tools in lattice theory and its applications, several notions and properties, and the notion of the lattice itself can be interpreted in terms of binary operations on it [7, 10, 11, 20, 24, 25]. Furthermore, it is not surprising that binary operations with specific properties appear in various theoretical and application fields. For instance, for aggregation functions (as specific binary operations) on bounded lattices and their wide use in various fields of applied sciences, including, computer and information sciences, economics, and social sciences (see, e. g. [12, 13, 14, 15, 17, 21, 22]). Also, they play an important role (as generalization of the basic connectives between fuzzy sets) in theories of fuzzy sets and fuzzy logic [2].

The aim of the present paper is to study and characterize some important properties of a binary operation on an arbitrary lattice (not necessarily bounded). More specifically, we show necessary and sufficient conditions under which a given binary operation on a lattice coincides with its meet (resp. its join) operation. These results generalize remarkable results of Cooman and Kerre (Proposition 3.7, [6]), and of Mesiar and Komorníková

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(Proposition 2, [22]) to any lattice. Furthermore, we construct two new posets based on a binary operation on a lattice, and investigate some conditions under which these two posets have a lattice structure. In some cases, we provide representations of a given lattice in terms of these new constructed lattices.

In addition to the theoretical importance of the obtained results in lattice theory, there exist many more reasons for addressing this study. For instance, several extensions of the notion of a derivation on a lattice have been introduced based on the lattice meet and join operations. This notion of derivation was first appeared on the ring structures, and it has many applications (see, e. g. [1]). Later, Szász [28, 29] has extended the notion of derivation to a lattice structure, and Ferrari [9] has investigated some properties of this notion and he has provided some interesting examples in particular classes of lattices. After that, Xin et al. [30] have ameliorated the notion of derivation on a lattice by reducing two required conditions to one condition based on the properties of the lattice meet and join operations. We anticipate that these properties and characterizations of binary operations on an arbitrary lattice are prerequisite to introduce the different notions of $(F, G)$-derivation on a lattice as generalization of the notion of derivation introduced by Xin et al. [30], where $F$ and $G$ are binary operations. Also, to avoid making the precedent studies used the lattice meet and join operations.

The rest of the paper is organized as follows. In Section 2, we recall the necessary basic concepts of binary operations and lattices. In Section 3, after recalling some basic properties of binary operations on a lattice, we show necessary and sufficient conditions under which a given binary operation coincides with the lattice meet (resp. join) operation. In Section 4, we construct two new lattices based on a given binary operation on a lattice. In Section 5, we provide some representations of a given lattice in terms of the lattices constructed in Section 4. In order to provide deeper insights, we also study the links between the different conditions of these representation theorems. Finally, we present some conclusions and discuss future research in Section 6.

## 2. BASIC CONCEPTS

In this section, we recall some definitions and properties of binary operations and lattices that will be needed throughout this paper.

### 2.1. Binary operations

A *binary operation* $F$ on a nonempty set $X$ is a function $F : X \times X \to X$. The following properties of binary operations are of interest in this paper:

(i) commutativity: $F(x, y) = F(y, x)$, for any $x, y \in X$;

(ii) associativity: $F(x, F(y, z)) = F(F(x, y), z)$, for any $x, y, z \in X$;

An element $e \in X$ is called

(iii) a left-neutral (resp. right-neutral) element of $F$, if $F(e, x) = x$ (resp. $F(x, e) = x$), for any $x \in X$;

(iv) a neutral element of $F$, if it is left- and right-neutral element, i. e., $F(e, x) = F(x, e) = x$, for any $x \in X$.

For further details, we refer to [8, 16, 18, 19].

### 2.2. Lattices

In this subsection, we recall the notions of ordered sets and lattices. Further information can be found in [4, 7, 25, 27].

An *order relation* $\leqslant$ on a set $X$ is a binary relation on $X$ that is *reflexive* (i. e., $x \leqslant x$, for any $x \in X$), *antisymmetric* (i. e., $x \leqslant y$ and $y \leqslant x$ implies $x = y$, for any $x, y \in X$) and *transitive* (i. e., $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$, for any $x, y, z \in X$). A set $X$ equipped with an order relation $\leqslant$ is called a *partially ordered set* (a *poset*, for short) and denoted by $(X, \leqslant)$.

Let $(X, \leqslant)$ be a poset and $A$ be a subset of $X$. An element $x_0 \in X$ is called a *lower bound* of $A$ if $x_0 \leqslant x$, for any $x \in A$. $x_0$ is called the *greatest lower bound* (or the *infimum*) of $A$ if $x_0$ is a lower bound and $m \leqslant x_0$, for any lower bound $m$ of $A$. *Upper bound* and *least upper bound* (or *supermum*) are defined dually.

Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be two posets. A mapping $\varphi$ from $X$ into $Y$ is called an *order isomorphism* if it is surjuctive and satisfies

$$x \leqslant_X y \text{ if and only if } \varphi(x) \leqslant_Y \varphi(y), \text{ for any } x, y \in X\,.$$

A poset $(L, \leqslant)$ is called a *lattice* if any two elements $x$ and $y$ have a *greatest lower bound*, denoted $x \wedge y$ and called the *meet* (*infimum*) of $x$ and $y$, as well as a *least upper bound*, denoted $x \vee y$ and called the *join* (*supremum*) of $x$ and $y$. A lattice can also be defined as an algebraic structure: a set $L$ equipped with two binary operations $\wedge$ and $\vee$ that are idempotent, commutative, associative and satisfy the absorption laws ($x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$, for any $x, y \in L$). The order relation and the binary operations $\wedge$ and $\vee$ are then related as: $x \leqslant y$ if and only if $x \wedge y = x$; $x \leqslant y$ if and only if $x \vee y = y$. Usually, the notation $(L, \leqslant, \wedge, \vee)$ is used.

A bounded lattice is a lattice $(L, \leqslant, \wedge, \vee)$ that additionally has a *least element* denoted by 0 and a *greatest element* denoted by 1 satisfying $0 \leqslant x \leqslant 1$ for any $x \in L$. For a bounded lattice the notation $(L, \leqslant, \wedge, \vee, 0, 1)$ is used.

Let $(L, \leqslant, \wedge, \vee)$ and $(M, \preceq, \frown, \smile)$ be two lattices. A mapping $\varphi$ from $L$ into $M$ is called a $\wedge$-*homomorphism* (resp. a $\vee$-*homomorphism*), if it satisfies $\varphi(x \wedge y) = \varphi(x) \frown \varphi(y)$ (resp. $\varphi(x \vee y) = \varphi(x) \smile \varphi(y)$), for any $x, y \in L$. A *lattice homomorphism* is both $\wedge$- and $\vee$-homomorphism. Also, a *lattice isomorphism* is a bijective lattice homomorphism.

The following proposition shows that an order isomorphism between to lattices is equivalent to a lattice isomorphism.

**Proposition 2.1.** [7] Let $L$, $M$ be two lattices and $\varphi$ be a mapping from $L$ into $M$. The following statements are equivalent:

(i) $\varphi$ is an order isomorphism;

(ii) $\varphi$ is a lattice isomorphism.

## 3. BINARY OPERATIONS ON A LATTICE

In this section, we study and characterize some properties of binary operations on a lattice.

### 3.1. Properties of binary operations on a lattice

In this subsection, we present the properties of conjunctive, disjunctive, idempotent and increasing binary operations on a lattice. Some of these properties are adopted from those of aggregation functions on bounded lattices (see, e. g. [12, 15, 17, 22]).

**Definition 3.1.** A binary operation $F$ on a lattice $(L, \leqslant, \wedge, \vee)$ is called:

(i) *idempotent*, if $F(x, x) = x$, for any $x \in L$;

(ii) *conjunctive* (resp. *disjunctive*), if $F(x, y) \leqslant x \wedge y$ (resp. $x \vee y \leqslant F(x, y)$), for any $x, y \in L$;

(iii) *left-increasing* (resp. *right-increasing*), if $x \leqslant y$, implies $F(x, z) \leqslant F(y, z)$ (resp. $F(z, x) \leqslant F(z, y)$), for any $x, y, z \in L$;

(iv) *increasing*, if $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$ imply $F(x_1, y_1) \leqslant F(x_2, y_2)$, for any $x_1, y_1, x_2, y_2 \in L$.

We note that a binary operation $F$ on a lattice is increasing if and only if it is left- and right-increasing. Also, if $F$ is commutative, then it holds that $F$ is left-increasing if and only if $F$ is right-increasing.

**Example 3.1.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice. The binary operations $\wedge$ and $\vee$ are idempotent and increasing. Moreover, $\wedge$ is conjunctive and $\vee$ is disjunctive.

Inspired by the notion of transpose of a binary aggregation function on the real interval $[0, 1]$ (see, e. g. [3]), the following definition extends this notion to any binary operation on a lattice.

**Definition 3.2.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. The binary operation $F^t$ defined as $F^t(x, y) = F(y, x)$, for any $x, y \in L$, is called the *transpose* of $F$.

The following proposition expresses the relationships between the properties of a binary operation and those of its transpose. The proof is straightforward.

**Proposition 3.1.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. The following equivalences hold:

(i) $F$ is commutative if and only if $F^t = F$;

(ii) $F$ is conjunctive if and only if $F^t$ is conjunctive;

(iii) $F$ is disjunctive if and only if $F^t$ is disjunctive;

(iv) $F$ is left-increasing if and only if $F^t$ is right-increasing;

(v) $F$ is increasing if and only if $F^t$ is increasing;

(vi) An element $e \in L$ is a left-neutral element of $F$ if and only if $e$ is a right-neutral element of $F^t$;

(vii) An element $e \in L$ is a neutral element of $F$ if and only if $e$ is a neutral element of $F^t$.

### 3.2. Characterizations of some properties of binary operations on a lattice

In this subsection, we study and characterize some properties of a binary operation on a lattice. More precisely, we give necessary and sufficient conditions that a binary operation on a lattice coincides with its meet (resp. its join) operation. These characterizations give us alternative conditions to those known for aggregation functions on bounded lattices (see, e. g. [3, 6, 22]).

**Proposition 3.2.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. Then it holds that

(i) If $F$ is increasing, idempotent and conjunctive, then $F(x,y)\wedge\alpha = F(x\wedge\alpha, y\wedge\alpha) = F(x\wedge\alpha, y) = F(x, y\wedge\alpha)$ for any $x, y, \alpha \in L$;

(ii) If $F$ is increasing, idempotent and disjunctive, then $F(x,y)\vee\alpha = F(x\vee\alpha, y\vee\alpha) = F(x\vee\alpha, y) = F(x, y\vee\alpha)$ for any $x, y, \alpha \in L$.

Proof. Let $x, y, \alpha \in L$.

(i) We only prove that $F(x,y)\wedge\alpha = F(x\wedge\alpha, y\wedge\alpha)$, as the other equalities can be proved similarly. Since $F$ is increasing, it follows that $F(x\wedge\alpha, y\wedge\alpha) \leqslant F(x,y)$. The fact that $F$ is conjunctive implies that $F(x\wedge\alpha, y\wedge\alpha) \leqslant \alpha$. Hence, $F(x\wedge\alpha, y\wedge\alpha)$ is a lower bound of $\{F(x,y), \alpha\}$. On the other hand, let $m \in L$ another lower bound of $\{F(x,y), \alpha\}$, i. e., $m \leqslant F(x,y)$ and $m \leqslant \alpha$. Since $F$ is conjunctive, it follows that $m \leqslant x$ and $m \leqslant y$. Hence, $m \leqslant x\wedge\alpha$ and $m \leqslant y\wedge\alpha$. Since $F$ is idempotent and increasing, it holds that $m = F(m,m) \leqslant F(x\wedge\alpha, y\wedge\alpha)$. Thus, $F(x\wedge\alpha, y\wedge\alpha)$ is the greatest lower bound of $\{F(x,y), \alpha\}$, i. e., $F(x,y)\wedge\alpha = F(x\wedge\alpha, y\wedge\alpha)$.

(ii) The proof is analogous to that of (i).

$\square$

The following lemma generalizes a remarkable result of Cooman and Kerre (Proposition 3.7 in [6]) to an arbitrary lattice.

**Lemma 3.1.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice. If an idempotent binary operation $F$ on $L$ is increasing and conjunctive (resp. disjunctive), then $F$ can not be other than the meet operation $\wedge$ (resp. the join operation $\vee$) of $L$.

P r o o f. Suppose that $F$ is idempotent, increasing and conjunctive binary operation on $L$. From Proposition 3.2 it follows that $F(x,y) \wedge (x \wedge y) = F(x \wedge (x \wedge y), y \wedge (x \wedge y)) = F(x \wedge y, x \wedge y) = x \wedge y$, for any $x, y \in L$. Hence, $(x \wedge y) \leqslant F(x,y)$, for any $x, y \in L$. Now, the fact that $F$ is conjunctive guarantees that $F(x,y) = x \wedge y$, for any $x, y \in L$. Thus, $F = \wedge$.

In similar way, we prove that $F$ is the join operation $\vee$ of $L$. □

The following proposition shows an equivalent condition to both properties of increasing and conjunctive binary operation on a lattice.

**Proposition 3.3.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. If $F$ is idempotent, then the following conditions are equivalent:

(i) $F$ is increasing and conjunctive;

(ii) $F(x,y) \wedge \alpha = F(x \wedge \alpha, y \wedge \alpha) = F(x \wedge \alpha, y) = F(x, y \wedge \alpha)$, for any $x, y, \alpha \in L$.

P r o o f. $(i) \Rightarrow (ii)$: The proof is directly from Proposition 3.2.
$(ii) \Rightarrow (i)$: First, we prove that $F$ is conjunctive. Let $x, y \in L$, from $(ii)$ it follows that $F(x,y) \wedge x = F(x \wedge x, y) = F(x,y)$ and $F(x,y) \wedge y = F(x, y \wedge y) = F(x,y)$. Hence, $F(x,y) \leqslant x$ and $F(x,y) \leqslant y$. Thus, $F(x,y) \leqslant x \wedge y$. Consequently, $F$ is conjunctive. Second, we prove that $F$ is increasing. Let $x, y, t, z \in L$ such that $x \leqslant y$. Since $F$ is idempotent, it follows that $F(x,z) \wedge F(y,z) = F(F(x,z) \wedge y, F(x,z) \wedge z) = F(F(x \wedge y, z), F(x, z \wedge z)) = F(F(x,z), F(x,z)) = F(x,z)$. Hence, $F(x,z) \leqslant F(y,z)$. Thus, $F$ is left-increasing. Moreover, $F(t,x) \wedge F(t,y) = F(F(t,x) \wedge t, F(t,x) \wedge y) = F(F(t \wedge t, x), F(t, x \wedge y)) = F(F(t,x), F(t,x)) = F(t,x)$. This implies that $F(t,x) \leqslant F(t,y)$. Hence, $F$ is right-increasing. Now, since $F$ is left- and right-increasing, it holds that $F$ is increasing. □

In the same line, we obtain the following proposition.

**Proposition 3.4.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. If $F$ is idempotent, then the following conditions are equivalent:

(i) $F$ is increasing and disjunctive;

(ii) $F(x,y) \vee \alpha = F(x \vee \alpha, y \vee \alpha) = F(x \vee \alpha, y) = F(x, y \vee \alpha)$, for any $x, y, \alpha \in L$.

The following theorem shows a necessary and sufficient condition that a binary operation on a lattice coincides with its meet (resp. its join) operation. We note that this result generalizes the well known result that a binary operation $F$ on a bounded lattice is an idempotent semi-norm (resp. semi-conorm) if and only if $F$ is the lattice meet (resp. join) operation (see, Propositions 3.7 in [6] and 2 in [22]). Furthermore, it establishes a simple criterion to check; if a binary operation on a given lattice coincides with its meet (resp. its join) operation.

**Theorem 3.1.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be an idempotent binary operation on $L$. Then it holds that

(i) $F$ is the meet operation $\wedge$ if and only if $F(x,y)\wedge\alpha = F(x\wedge\alpha, y\wedge\alpha) = F(x\wedge\alpha, y) = F(x, y\wedge\alpha)$, for any $x, y, \alpha \in L$;

(ii) $F$ is the join operation $\vee$ if and only if $F(x,y)\vee\alpha = F(x\vee\alpha, y\vee\alpha) = F(x\vee\alpha, y) = F(x, y\vee\alpha))$, for any $x, y, \alpha \in L$.

Proof. The proof can be easily obtained from Lemma 3.1, Propositions 3.3 and 3.4. □

Next, we need the following remark.

**Remark 3.1.** In view of Lemma 3.1, if a binary operation $F$ on a lattice $(L, \leqslant, \wedge, \vee)$ is idempotent, conjunctive (resp. disjunctive) and increasing, then $F = \wedge$ (resp. $F = \vee$). But, if $F$ is only left- or right-increasing, then $F$ is not necessarily the meet (resp. the join) operation of $L$. Indeed, let $(L = \{0,1,2\}, \leqslant)$ be the lattice given by the Hasse diagram in Figure 1, and $F, G$ be two binary operations on $L$ defined by the following tables:

| $F(x,y)$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 2 | 0 | 1 | 2 |

| $G(x,y)$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 2 | 1 | 2 |
| 2 | 2 | 2 | 2 |

One can easily verify that the binary operation $F$ is left-increasing, conjunctive and idempotent, but it is not the lattice meet operation $\wedge$. Analogously, $F^t$ is right-increasing, conjunctive and idempotent, but also it is not the meet operation $\wedge$. Moreover, the binary operation $G$ (resp. $G^t$) is left-increasing (resp. right-increasing), disjunctive and idempotent, but it is not the join operation $\vee$.

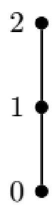


**Fig. 1.** The Hasse diagram of the lattice $(L = \{0,1,2\}, \leqslant)$.

## 4. LATTICE STRUCTURES CONSTRUCTED BY A BINARY OPERATION

In this section, we construct two posets based on a binary operation on a lattice. Furthermore, we study some cases that these posets have lattice structures. These lattices will turn out to be useful technical tools in the following section.

### 4.1. Posets constructed by a binary operation on a lattice

Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. For any $\alpha \in L$, we define two specific $F$-based functions $f_\alpha$ and $f^\alpha$ from $L$ into $L$ as:

$$f_\alpha(x) = F(\alpha, x) \text{ and } f^\alpha(x) = F(x, \alpha), \text{ for any } x \in L\,.$$

We denote by $\mathcal{A}^\ell_F(L)$ to the set of $f_\alpha$ functions and by $\mathcal{A}^\ell_F(L)$ to the set of $f^\alpha$ functions, i. e.,

$$\mathcal{A}^\ell_F(L) = \{f_\alpha \mid \alpha \in L\} \text{ and } \mathcal{A}^r_F(L) = \{f^\alpha \mid \alpha \in L\}.$$

The following proposition is immediate.

**Proposition 4.1.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. If $F$ is commutative, then $\mathcal{A}^\ell_F(L) = \mathcal{A}^r_F(L)$.

**Remark 4.1.** The converse of the above proposition does not hold in general. Indeed, consider the lattice $(L = \{0, 1, 2\}, \leqslant)$ given by the Hasse diagram in Figure 1. Let $F$ be the binary operation on $L$ defined by the following table:

| $F(x, y)$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | 1 | 2 |
| 1 | 2 | 1 | 1 |
| 2 | 1 | 2 | 1 |

From the above table, one can see that $f_0 = f^1$, $f_1 = f^2$ and $f_2 = f^0$. Indeed,

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $f_0(x)$ | $F(0, 0) = 1$ | $F(0, 1) = 1$ | $F(0, 2) = 2$ |
| $f^1(x)$ | $F(0, 1) = 1$ | $F(1, 1) = 1$ | $F(2, 1) = 2$ |
| $f_1(x)$ | $F(1, 0) = 2$ | $F(1, 1) = 1$ | $F(1, 2) = 1$ |
| $f^2(x)$ | $F(0, 2) = 2$ | $F(1, 2) = 1$ | $F(2, 2) = 1$ |
| $f_2(x)$ | $F(2, 0) = 1$ | $F(2, 1) = 2$ | $F(2, 2) = 1$ |
| $f^0(x)$ | $F(0, 0) = 1$ | $F(1, 0) = 2$ | $F(2, 0) = 1$ |

Then $\mathcal{A}^\ell_F(L) = \mathcal{A}^r_F(L)$. But, since $F(0, 1) = 1 \neq F(1, 0) = 2$, it holds that $F$ is not commutative.

**Proposition 4.2.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice, $F$ be a binary operation on $L$ and $\alpha, \beta \in L$. If $F$ has a neutral element $e \in L$ and $f_\alpha = f^\beta$, then $\alpha = \beta$.

P r o o f. Suppose that $F$ has a neutral element $e \in L$. Let $\alpha, \beta \in L$ such that $f_\alpha = f^\beta$. Then $F(\alpha, x) = F(x, \beta)$, for any $x \in L$. Setting $x = e$, we obtain that $\alpha = \beta$. □

In view of Remark 4.1, the following proposition states a sufficient condition that the converse implication in Proposition 4.1 holds.

**Proposition 4.3.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. If $F$ has a neutral element $e \in L$ and $\mathcal{A}^\ell_F(L) = \mathcal{A}^r_F(L)$, then $F$ is commutative.

P r o o f. Suppose that $F$ has a neutral element $e \in L$ and $\mathcal{A}_F^\ell(L) = \mathcal{A}_F^r(L)$. Let $\alpha, \beta \in L$, we will show that $F$ is commutative, i. e., $F(\alpha, \beta) = F(\beta, \alpha)$. Since $f_\alpha \in \mathcal{A}_F^\ell(L)$ and $\mathcal{A}_F^\ell(L) = \mathcal{A}_F^r(L)$, it follows that exists $\gamma \in L$ such that $f_\alpha = f^\gamma$. From Proposition 4.2, it follows that $\alpha = \gamma$. This implies that $f_\alpha = f^\alpha$. Hence, $F(\alpha, \beta) = f_\alpha(\beta) = f^\alpha(\beta) = F(\beta, \alpha)$. Thus, $F$ is commutative. □

The sets $\mathcal{A}_F^\ell(L)$ and $\mathcal{A}_F^r(L)$ equipped with the usual order of functions have poset structures. We denote by $\leqslant^\ell$ the usual order of functions defined on $\mathcal{A}_F^\ell(L)$ as:

$$f_\alpha \leqslant^\ell f_\beta \text{ if and only if } f_\alpha(x) \leqslant f_\beta(x), \text{ for any } x \in L\,.$$

In the same way, we denote by $\leqslant^r$ the usual order of functions defined on $\mathcal{A}_F^r(L)$ as:

$$f^\alpha \leqslant^r f^\beta \text{ if and only if } f^\alpha(x) \leqslant f^\beta(x), \text{ for any } x \in L\,.$$

**Remark 4.2.** In general, the posets $(\mathcal{A}_F^\ell(L), \leqslant^\ell)$ and $(\mathcal{A}_F^r(L), \leqslant^r)$ are not necessarily lattices. Indeed, consider the lattice $(D(6), |, gcd, lcm)$ given by the Hasse diagram in Figure 2, and $F$ the binary operation on $D(6)$ defined by the following table:

| $F(x,y)$ | 1 | 2 | 3 | 6 |
|---|---|---|---|---|
| 1 | 3 | 2 | 3 | 2 |
| 2 | 2 | 1 | 6 | 3 |
| 3 | 6 | 2 | 6 | 6 |
| 6 | 2 | 3 | 6 | 6 |

The Hasse diagrams given in Figure 2 show that $(\mathcal{A}_F^\ell(D(6)), \leqslant^\ell)$ and $(\mathcal{A}_F^r(D(6)), \leqslant^r)$ are not lattices.

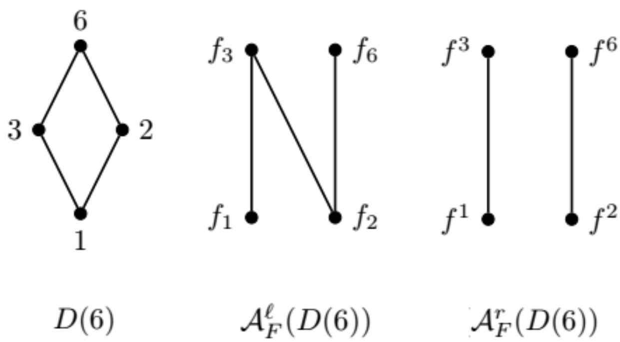


**Fig. 2.** The Hasse diagrams of the lattice $(D(6), |, gcd, lcm)$, and the posets $(\mathcal{A}_F^\ell(D(6)), \leqslant^\ell)$ and $(\mathcal{A}_F^r(D(6)), \leqslant^r)$

### 4.2. First lattice structure of the posets $\mathcal{A}_F^\ell(L)$ and $\mathcal{A}_F^r(L)$

In this subsection, we show some cases that the posets $(\mathcal{A}_F^\ell(L), \leqslant^\ell)$ and $(\mathcal{A}_F^r(L), \leqslant^r)$ have a lattice structure. First, we prove the following key results.

**Proposition 4.4.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. Then it holds that

(i) If $F$ is left-increasing and having a right-neutral element $e \in L$, then for any $\alpha, \beta \in L$, we obtain that

$$\alpha \leqslant \beta \text{ if and only if } f_\alpha \leqslant^\ell f_\beta\,;$$

(ii) If $F$ is right-increasing and having a left-neutral element $e \in L$, then for any $\alpha, \beta \in L$, we obtain that

$$\alpha \leqslant \beta \text{ if and only if } f^\alpha \leqslant^r f^\beta\,.$$

Proof.

(i) The fact that $F$ is left-increasing guarantees the first implication. Conversely, let $f_\alpha, f_\beta \in \mathcal{A}^\ell_F(L)$ such that $f_\alpha \leqslant^\ell f_\beta$. This implies that $F(\alpha, x) \leqslant F(\beta, x)$, for any $x \in L$. Using the fact that $e$ is a right-neutral element of $F$ and setting $x = e$, it follows that $\alpha = F(\alpha, e) \leqslant F(\beta, e) = \beta$. Thus, $\alpha \leqslant \beta$.

(ii) Follows from Proposition 3.1 and (i).

□

Proposition 4.4 leads to the following corollary.

**Corollary 4.1.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. If $F$ is increasing and having a neutral element $e \in L$, then for any $\alpha, \beta \in L$, the following statements are equivalent:

(i) $\alpha \leqslant \beta$;

(ii) $f_\alpha \leqslant^\ell f_\beta$;

(iii) $f^\alpha \leqslant^r f^\beta$.

In the following result, we provide sufficient conditions that the posets $(\mathcal{A}^\ell_F(L), \leqslant^\ell)$ and $(\mathcal{A}^r_F(L), \leqslant^r)$ have a lattice structure.

**Theorem 4.1.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. Then it holds that

(i) If $F$ is left-increasing and having a right-neutral element $e \in L$, then $(\mathcal{A}^\ell_F(L), \leqslant^\ell)$ is a lattice;

(ii) If $F$ is right-increasing and having a left-neutral element $e \in L$, then $(\mathcal{A}^r_F(L), \leqslant^r)$ is a lattice.

Proof.

(i) Let $f_\alpha, f_\beta \in \mathcal{A}^\ell_F(L)$. Using Proposition 4.4, we obtain that $f_{\alpha\wedge\beta}$ (resp. $f_{\alpha\vee\beta}$) is the greatest lower bound (resp. the least upper bound) of $f_\alpha$ and $f_\beta$. Thus, $(\mathcal{A}^\ell_F(L), \leqslant^\ell)$ is a lattice.

(ii) Let $f^\alpha, f^\beta \in \mathcal{A}^r_F(L)$. From Proposition 3.1 and (i), we obtain that $f^{\alpha\wedge\beta}$ and $f^{\alpha\vee\beta}$ are respectively the greatest lower bound and the least upper bound of $f^\alpha$ and $f^\beta$. Thus, $(\mathcal{A}^r_F(L), \leqslant^r)$ is a lattice.

□

Next, we denote by $(\mathcal{A}^\ell_F(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ the lattice structure associated to the poset $(\mathcal{A}^\ell_F(L), \leqslant^\ell)$, where $f_\alpha \wedge^\ell f_\beta = f_{\alpha\wedge\beta}$ and $f_\alpha \vee^\ell f_\beta = f_{\alpha\vee\beta}$, for any $f_\alpha, f_\beta \in \mathcal{A}^\ell_F(L)$. Also, $(\mathcal{A}^r_F(L), \leqslant^r, \wedge^r, \vee^r)$ denote the lattice structure associated to the poset $(\mathcal{A}^r_F(L), \leqslant^r)$, where $f^\alpha \wedge^r f^\beta = f^{\alpha\wedge\beta}$ and $f^\alpha \vee^r f^\beta = f^{\alpha\vee\beta}$, for any $f^\alpha, f^\beta \in \mathcal{A}^r_F(L)$.

In the following we give a counter example to show that the converse implications of Theorem 4.1 are not necessarily hold.

**Example 4.1.** Let $(L = \{0, 1, 2\}, \leqslant, \min, \max)$ be the lattice given by the Hasse diagram in Figure 3 and $F$ be a binary operation on $L$ defined by the following table:

| $F(x, y)$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | 0 | 2 |
| 1 | 0 | 0 | 0 |
| 2 | 2 | 0 | 2 |

From Figure 3, it follows that:

(i) $(\mathcal{A}^\ell_F(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ is a lattice, but $F$ is not left-increasing and has not a right-neutral element $e \in L$;

(ii) $(\mathcal{A}^r_F(L), \leqslant^r, \wedge^r, \vee^r)$ is a lattice, but $F$ is not right-increasing and has not a left-neutral element $e \in L$.

**Remark 4.3.** If $(L, \leqslant, \wedge, \vee, 0, 1)$ is a bounded lattice, then the lattices $(\mathcal{A}^\ell_F(L), \leqslant^\ell, \wedge^\ell, \vee^\ell, f_0, f_1)$ and $(\mathcal{A}^r_F(L), \leqslant^r, \wedge^r, \vee^r, f^0, f^1)$ are also bounded.

### 4.3. Second lattice structure of the posets $\mathcal{A}^\ell_F(L)$ and $\mathcal{A}^r_F(L)$

In this subsection, we show other conditions on the binary operation $F$ that the posets $(\mathcal{A}^\ell_F(L), \leqslant^\ell)$ and $(\mathcal{A}^r_F(L), \leqslant^r)$ have a lattice structure. We start by proving the following key results.

**Proposition 4.5.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be an idempotent and conjunctive or disjunctive binary operation on $L$. Then it holds that:

(i) If $F$ is left-increasing, then for any $\alpha, \beta \in L$, we obtain that

$$\alpha \leqslant \beta \text{ if and only if } f_\alpha \leqslant^\ell f_\beta\,;$$

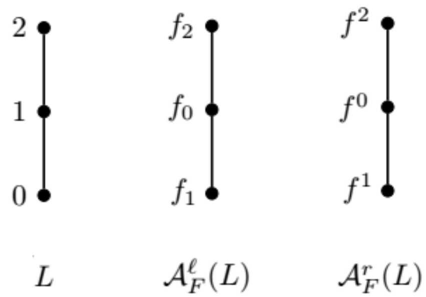


**Fig. 3.** The Hasse diagrams of the lattices $(L = \{0, 1, 2\}, \leqslant, \min, \max)$, $(\mathcal{A}^\ell_F(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ and $(\mathcal{A}^r_F(L), \leqslant^r, \wedge^r, \vee^r)$.

(ii) If $F$ is right-increasing, then for any $\alpha, \beta \in L$, we obtain that

$$\alpha \leqslant \beta \text{ if and only if } f^\alpha \leqslant^r f^\beta .$$

Proof.

(i) The proof of the direct implication is straightforward. For the converse implication, let $f_\alpha, f_\beta \in \mathcal{A}^\ell_F(L)$ such that $f_\alpha \leqslant^\ell f_\beta$. Then $F(\alpha, x) \leqslant F(\beta, x)$, for any $x \in L$. Setting $x = \alpha$ (resp. $x = \beta$). Since $F$ is idempotent and conjunctive (resp. idempotent and disjunctive), it follows that $\alpha = F(\alpha, \alpha) \leqslant F(\beta, \alpha) \leqslant \beta$ (resp. $\alpha \leqslant F(\alpha, \beta) \leqslant F(\beta, \beta) = \beta$). Thus, $\alpha \leqslant \beta$.

(ii) Follows from Proposition 3.1 and (i).

□

Similarly to Theorem 4.1, the following result shows other sufficient conditions under which the posets $(\mathcal{A}^\ell_F(L), \leqslant^\ell)$ and $(\mathcal{A}^r_F(L), \leqslant^r)$ have lattice structures.

**Theorem 4.2.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be an idempotent and conjunctive or disjunctive binary operation on $L$. Then it holds that

(i) If $F$ is left-increasing, then $(\mathcal{A}^\ell_F(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ is a lattice;

(ii) If $F$ is right-increasing, then $(\mathcal{A}^r_F(L), \leqslant^r, \wedge^r, \vee^r)$ is a lattice.

Proof.

(i) Let $f_\alpha, f_\beta \in \mathcal{A}^\ell_F(L)$. From Proposition 4.5, one can easily verify that $f_{\alpha\wedge\beta}$ and $f_{\alpha\vee\beta}$ are respectively the greatest lower bound and the least upper bound of $f_\alpha$ and $f_\beta$. Thus, $(\mathcal{A}^\ell_F(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ is a lattice.

(ii) Let $f^{\alpha}, f^{\beta} \in \mathcal{A}^{r}_{F}(L)$. From Proposition 3.1 and (i), it holds that $f^{\alpha\wedge\beta}$ and $f^{\alpha\vee\beta}$ are respectively the greatest lower bound and the least upper bound of $f^{\alpha}$ and $f^{\beta}$. Thus, $(\mathcal{A}^{r}_{F}(L), \leqslant^{r}, \wedge^{r}, \vee^{r})$ is a lattice.

□

Note that the converse implications of Theorem 4.2 do not necessarily hold, as can be see the following example.

**Example 4.2.** Let $(L = \{0,1,2,3\}, \leqslant, \min, \max)$ be the lattice given by the Hasse diagram in Figure 4. Let $F$ and $G$ be two idempotent binary operations on $L$ such that $F$ is conjunctive and $G$ is disjunctive, which are given by the following tables:

| $F(x,y)$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 |
| 2 | 0 | 1 | 2 | 0 |
| 3 | 0 | 1 | 2 | 3 |

| $G(x,y)$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 2 | 1 | 3 | 3 |
| 2 | 2 | 3 | 2 | 3 |
| 3 | 3 | 3 | 3 | 3 |

Figure 4 guarantees that $(\mathcal{A}^{\ell}_{F}(L), \leqslant^{\ell}, \wedge^{\ell}, \vee^{\ell})$ and $(\mathcal{A}^{\ell}_{G}(L), \leqslant^{\ell}, \wedge^{\ell}, \vee^{\ell})$ are lattices, but $F$ and $G$ are not left- or right-increasing. Indeed, $1 \leqslant 2$, but

(i) $F(1,3) = 1 \nleqslant F(2,3) = 0$ and $G(1,2) = 3 \nleqslant G(2,2) = 2$;

(ii) $F(1,1) = 1 \nleqslant F(1,2) = 0$ and $G(2,1) = 3 \nleqslant G(2,2) = 2$.

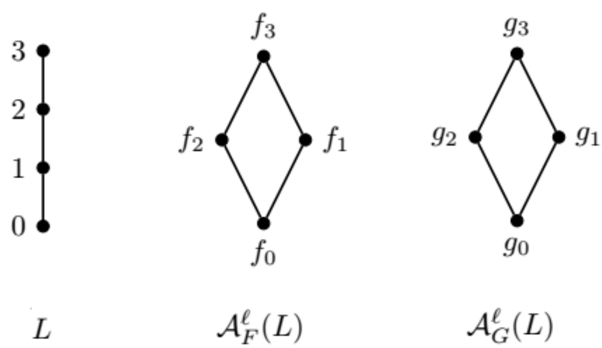


**Fig. 4.** The Hasse diagrams of the lattices $(L = \{0,1,2,3\}, \leqslant, \min, \max)$, $(\mathcal{A}^{\ell}_{F}(L), \leqslant^{\ell}, \wedge^{\ell}, \vee^{\ell})$ and $(\mathcal{A}^{\ell}_{G}(L), \leqslant^{\ell}, \wedge^{\ell}, \vee^{\ell})$.

## 5. REPRESENTATIONS OF A LATTICE BASED ON A BINARY OPERATION

In this section, we provide some representations of a given lattice in terms of a binary operation defined on it.

### 5.1. First representation of a lattice based on a binary operation

In this subsection, we provide a representation of a given lattice in terms of the lattice structures constructed based on a binary operation on it. More precisely, based on Theorem 4.1, we show that any lattice $(L, \leqslant, \wedge, \vee)$ is isomorphic with its associated lattices $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ and $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$.

**Theorem 5.1.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. Then it holds that

(i) If $F$ is left-increasing and having a right-neutral element $e \in L$, then the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ are isomorphic;

(ii) If $F$ is right-increasing and having a left-neutral element $e \in L$, then the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$ are isomorphic.

Proof.

(i) Theorem 4.1 guarantees that $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ is a lattice. Next, let $\varphi$ be a mapping from $L$ into $\mathcal{A}_F^\ell(L)$ such that $\varphi(\alpha) = f_\alpha$, for any $\alpha \in L$. Obvious that $\varphi$ is surjective. Moreover, since $F$ is left-increasing and having a right-neutral element $e \in L$, it follows from Proposition 4.4 that

$$\alpha \leqslant \beta \text{ if and only if } \varphi(\alpha) \leqslant^\ell \varphi(\beta), \text{ for any } \alpha, \beta \in L\,.$$

Hence, $\varphi$ is an order isomorphism between these two lattices $L$ and $\mathcal{A}_F^\ell(L)$. Then Proposition 2.1 guarantees that $\varphi$ is a lattice isomorphism. Thus, the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ are isomorphic.

(ii) Follows from Proposition 3.1 and (i).

$\square$

**Remark 5.1.** Note that Example 4.1 shows that the converse implications of Theorem 5.1 do not necessarily hold.

In view of Theorem 5.1, we obtain the following result.

**Theorem 5.2.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. If $F$ is increasing having a neutral element $e \in L$, then the lattices $(L, \leqslant, \wedge, \vee)$, $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ and $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$ are isomorphic.

**Example 5.1.** Let $(L = \{0, 1, 2\}, \leqslant, \min, \max)$ be the lattice given by the Hasse diagram in Figure 5 and $F$ be a binary operation on $L$ defined by the following table:

| $F(x,y)$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 2 |

The above table guarantees that $F$ is increasing and has 1 as a neutral element, then from Theorem 5.2, it holds that the lattices $(L = \{0, 1, 2\}, \leqslant, \min, \max)$, $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ and $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$ are isomorphic. Figure 5 shows this isomorphism.

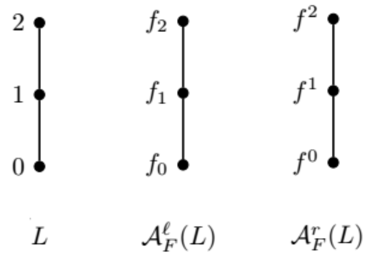


**Fig. 5.** The Hasse diagrams of the lattices $(L = \{0,1,2\}, \leqslant, \min, \max)$, $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ and $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$.

### 5.2. Second representation of a lattice based on a binary operation

In this subsection, we provide a second representation theorem of a given lattice based on the second lattice structures constructed based on a binary operation on it.

**Theorem 5.3.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be an idempotent and conjunctive or disjunctive binary operation on $L$. Then it holds that

(i) If $F$ is left-increasing, then the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ are isomorphic.

(ii) If $F$ is right-increasing, then the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$ are isomorphic.

Proof.

(i) Theorem 4.2 guarantees that $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ is a lattice. Let $\varphi$ be a mapping from $L$ into $\mathcal{A}_F^\ell(L)$ such that $\varphi(\alpha) = f_\alpha$, for any $\alpha \in L$. Then the surjectivity of $\varphi$ is immediate. Moreover, from Proposition 4.5, we obtain that

$$\alpha \leqslant \beta \text{ if and only if } \varphi(\alpha) \leqslant^\ell \varphi(\beta), \text{ for any } \alpha, \beta \in L\,.$$

Hence, $\varphi$ is an order isomorphism between these two lattices. Then, Proposition 2.1 guarantees that $\varphi$ is a lattice isomorphism. Therefore, the lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$ are isomorphic.

(ii) Follows from Proposition 3.1 and (i).

□

Theorem 5.3 leads to the following result.

**Theorem 5.4.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. If $F$ is the meet operation $\wedge$ or the join operation $\vee$ of $L$, then the lattices $(\mathcal{A}_F^\ell(L), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ and $(\mathcal{A}_F^r(L), \leqslant^r, \wedge^r, \vee^r)$ are identical and isomorphic with $(L, \leqslant, \wedge, \vee)$.

In the following, we illustrate two examples of Theorem 5.3 justifying the condition of left-increasing (resp. right-increasing).

**Example 5.2.** Let $(D(6), |, gcd, lcm)$ be the lattice given by the Hasse diagram in Figure 6 and $F$ be a binary operation on $D(6)$ defined by the following table:

| $F(x,y)$ | 1 | 2 | 3 | 6 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 1 | 1 |
| 3 | 1 | 1 | 3 | 1 |
| 6 | 1 | 2 | 3 | 6 |

One easily verifies that $F$ is idempotent, conjunctive and left-increasing. Then from Theorem 5.3, we obtain the lattice $(\mathcal{A}_F^\ell(D(6)), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ which is isomorphic with $(D(6), |, gcd, lcm)$. But, the poset $(\mathcal{A}_F^r(D(6)), \leqslant^r)$ has not the structure of a lattice, and then is not isomorphic with $(D(6), |, gcd, lcm)$. The Hasse diagrams of the lattices $(D(6), |, gcd, lcm)$, $(\mathcal{A}_F^\ell(D(6)), \leqslant^\ell, \wedge^\ell, \vee^\ell)$ and the poset $(\mathcal{A}_F^r(D(6)), \leqslant^r)$ are shown in Figure 6.

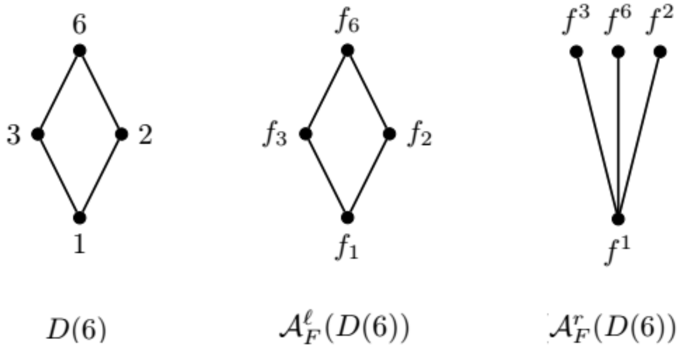


**Fig. 6.** The Hasse diagrams of the lattices $(D(6), |, gcd, lcm)$, $(\mathcal{A}_F^\ell(D(6)), \leqslant^\ell, \wedge^\ell, \vee^\ell)$, and the poset $(\mathcal{A}_F^r(D(6)), \leqslant^r)$.

**Example 5.3.** Let $(D(12), |, gcd, lcm)$ be the lattice given by the Hasse diagram in Figure 7 and $F$ be a binary operation on $D(12)$ defined by the following table:

| $F(x,y)$ | 1 | 2 | 3 | 4 | 6 | 12 |
|---|---|---|---|---|---|---|
| 1 | 1 | 4 | 6 | 4 | 12 | 12 |
| 2 | 2 | 2 | 12 | 4 | 12 | 12 |
| 3 | 3 | 6 | 3 | 12 | 6 | 12 |
| 4 | 4 | 4 | 12 | 4 | 12 | 12 |
| 6 | 6 | 6 | 6 | 12 | 6 | 12 |
| 12 | 12 | 12 | 12 | 12 | 12 | 12 |

One easily verifies that $F$ is idempotent, disjunctive and right-increasing. Then Theorem 5.3 guarantees that the lattice $(\mathcal{A}^r_F(D(12)), \leqslant^r, \wedge^r, \vee^r)$ is isomorphic with $(D(12), |, gcd, lcm)$. But, the poset $(\mathcal{A}^\ell_F(D(12)), \leqslant^\ell)$ has not the structure of a lattice, and then is not isomorphic with $(D(12), |, gcd, lcm)$. The Hasse diagrams of the lattices $(D(12), |, gcd, lcm)$, $(\mathcal{A}^r_F(D(12)), \leqslant^r, \wedge^r, \vee^r)$ and the poset $(\mathcal{A}^\ell_F(D(12)), \leqslant^\ell)$ are shown in Figure 7.

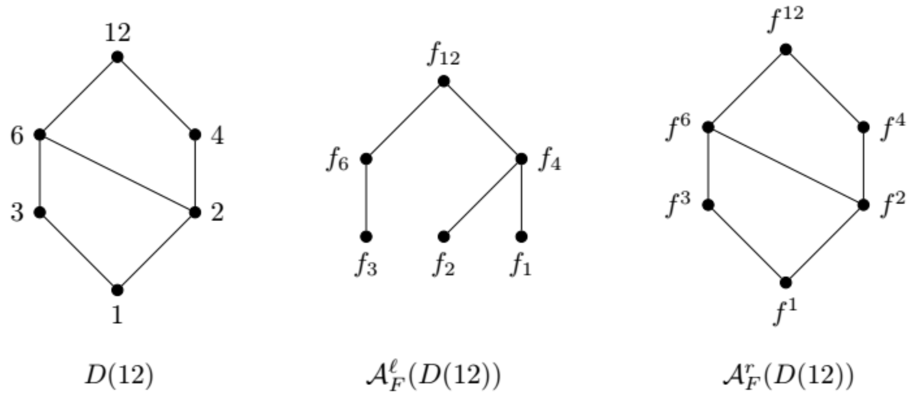


**Fig. 7.** The Hasse diagrams of the lattices $(D(12), |, gcd, lcm)$, $(\mathcal{A}^r_F(D(12)), \leqslant^r, \wedge^r, \vee^r)$, and the poset $(\mathcal{A}^\ell_F(D(12)), \leqslant^\ell)$.

### 5.3. Links between the different conditions of the representation theorems of a lattice based on a binary operation

In this subsection, we show the links between the different conditions of the above representation theorems of a given lattice based on a binary operation on it. First, we need to show the following sufficient condition for the uniqueness of a left-neutral (resp. a right-neutral) element of a binary operation on a lattice.

**Proposition 5.1. (Uniqueness)** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a conjunctive or a disjunctive binary operation on $L$. If $F$ has a left-neutral (resp. a right-neutral) element $e \in L$, then $e$ is unique.

Proof. Let $e_1, e_2 \in L$ be two left-neutral (resp. two right-neutral) elements of $F$. Two cases to consider.

(i) If $F$ is conjunctive, then it holds that $e_2 = F(e_1, e_2) \leqslant e_1$ and $e_1 = F(e_2, e_1) \leqslant e_2$ (resp. $e_2 = F(e_2, e_1) \leqslant e_1$ and $e_1 = F(e_1, e_2) \leqslant e_2$). Hence, $e_1 = e_2$.

(ii) If $F$ is disjunctive, the proof is analogous to that of (i).

□

**Proposition 5.2.** Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice and $F$ be a binary operation on $L$.

(i) If $F$ is conjunctive and has a left- or a right-neutral element $e \in L$, then $e = 1$;

(ii) If $F$ is disjunctive and has a left- or a right-neutral element $e \in L$, then $e = 0$.

Proof.

(i) Suppose that $F$ has a left-neutral (resp. a right-neutral) element $e \in L$. Since $F$ is conjunctive, it holds that $1 = F(e, 1) \leqslant e$ (resp. $1 = F(1, e) \leqslant e$). Hence, $e = 1$.

(ii) The proof is similar to that of (i).

□

In view of Proposition 5.2, we obtain the following corollary.

**Corollary 5.1.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $F$ be a binary operation on $L$. If $(L, \leqslant, \wedge, \vee)$ has no greatest (resp. no least) element and $F$ is conjunctive (resp. disjunctive), then $F$ has no left- and no right-neutral element.

The following result characterizes the conjunction and the disjunction proprieties of a binary operation on a bounded lattice in terms of its neutral element.

**Proposition 5.3.** Let $(L, \leqslant, \wedge, \vee, 0, 1)$ be a bounded lattice and $F$ be a binary operation on $L$. If $F$ is increasing and having a neutral element $e \in L$, then the following statements hold:

(i) $F$ is conjunctive if and only if $e = 1$;

(ii) $F$ is disjunctive if and only if $e = 0$.

Proof.

(i) Proposition 5.2 guarantees that if $F$ is conjunctive, then $e = 1$. Conversely, suppose that $e = 1$. Let $x, y \in L$, then $x \leqslant e$ and $y \leqslant e$. Since $F$ is increasing, it follows that $F(x, y) \leqslant F(x, e) = x$ and $F(x, y) \leqslant F(e, y) = y$. Hence, $F(x, y) \leqslant x \wedge y$. Thus, $F$ is conjunctive.

(ii) The proof is similar to that of (i).

□

The following propositions list some conditions for the existence and the uniqueness of a left-neutral (resp. a right-neutral) element of a binary operation on a given lattice.

**Proposition 5.4. (Existence and uniqueness)** Let $(L, \leqslant, \wedge, \vee, 1)$ be a lattice with the greatest element $1 \in L$ and $F$ be a binary operation on $L$. If $F$ is idempotent, conjunctive and left-increasing (resp. right-increasing), then 1 is the unique left-neutral (resp. the unique right-neutral) element of $F$.

Proof. Since $F$ is idempotent, conjunctive and left-increasing (resp. right-increasing), it follows that $x = F(x, x) \leqslant F(1, x) \leqslant x$ (resp. $x = F(x, x) \leqslant F(x, 1) \leqslant x$), for any $x \in L$. This implies that $F(1, x) = x$ (resp. $F(x, 1) = x$), for any $x \in L$. Hence, 1 is a left-neutral (resp. a right-neutral) element of $F$. Moreover, the fact that $F$ is conjunctive, it holds from Proposition 5.1 that 1 is the unique left-neutral (resp. the unique right-neutral) element of $F$. □

Dually, we get the following proposition.

**Proposition 5.5. (Existence and uniqueness)** Let $(L, \leqslant, \wedge, \vee, 0)$ be a lattice with the least element $0 \in L$ and $F$ be a binary operation on $L$. If $F$ is idempotent, disjunctive and left-increasing (resp. right-increasing), then 0 is the unique left-neutral (resp. the unique right-neutral) element of $F$.

Proof. The proof is analogous to that of Proposition 5.4. □

**Remark 5.2.** From Propositions 5.4 and 5.5, and Corollary 5.1, we conclude the following links between the different conditions of the above representations of a given lattice based on a binary operation on it.

(i) If $(L, \leqslant, \wedge, \vee, 1)$ is a lattice with the greatest element 1 and $F$ is the meet operation of $L$, then 1 is a neutral element of $F$. In this case, $(L, \leqslant, \wedge, \vee)$ can be represented by using the both Theorems 5.2 and 5.4.

(ii) If $(L, \leqslant, \wedge, \vee)$ has no greatest element and $F$ is the meet operation of $L$, then $F$ has no neutral element. In this case, $(L, \leqslant, \wedge, \vee)$ can be represented only by using Theorem 5.4.

(iii) If $(L, \leqslant, \wedge, \vee, 0)$ is a lattice with the least element 0 and $F$ is the join operation of $L$, then 0 is a neutral element of $F$. In this case, $(L, \leqslant, \wedge, \vee)$ can be represented by using the both Theorems 5.2 and 5.4.

(iv) If $(L, \leqslant, \wedge, \vee)$ has no least element and $F$ is the join operation of $L$, then $F$ has no neutral element. In this case, $(L, \leqslant, \wedge, \vee)$ can be represented only by using Theorem 5.4.

## 6. CONCLUSION AND FUTURE RESEARCH

In this work, we have studied and characterized some properties of a binary operation on a lattice. We have also showed necessary and sufficient conditions that a binary operation on a given lattice coincides with its meet (resp. its join) operation. Furthermore, we have provided some representations of a given lattice in terms of two new lattices constructed by means of a binary operation on that lattice.

To clarify some usefulness of this study, we anticipate that these properties and characterizations of binary operations on an arbitrary lattice are prerequisite to introduce the different notions of $(F, G)$-derivations on a lattice, where $F$ and $G$ are binary operations. Furthermore, the representation theorems are likely to provide us with the proper tools for characterizing a given lattice in terms of the principal $(F, G)$-derivations defined on it. These studies would not be possible without the analysis of the different properties and characterizations of binary operations on a lattice addressed in this paper. Also, to avoid making the precedent studies used the lattice meet and join operations.

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