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ON STABILITY AND THE ŁOJASIEWICZ EXPONENT AT INFINITY OF COERCIVE POLYNOMIALS

Tomáš Bajbar and Sönke Behrends

In this article we analyze the relationship between the growth and stability properties of coercive polynomials. For coercive polynomials we introduce the degree of stable coercivity which measures how stable the coercivity is with respect to small perturbations by other polynomials. We link the degree of stable coercivity to the Łojasiewicz exponent at infinity and we show an explicit relation between them.

Keywords: coercivity, stability of coercivity, Łojasiewicz exponent at infinity

Classification: 26C05

1. INTRODUCTION

In the present work we consider polynomials $f \in \mathbb{R}[X_1, \dots, X_n]$ which are coercive, that is, polynomials having the growth property $f(x) \to +\infty$ whenever $\|x\| \to +\infty$, and we use the Łojasiewicz exponent at infinity $\mathcal{L}_\infty(f)$ to measure how fast f grows for large argument values.

Coercivity of polynomials $f \in \mathbb{R}[X_1, \dots, X_n]$ is an interesting property for various reasons. In polynomial optimization theory it is a recurring question whether a given polynomial $f \in \mathbb{R}[X_1, \dots, X_n]$ attains its infimum over $\mathbb{R}^n$ (see, e. g. [1, 5, 9, 11, 12, 17, 18, 19, 20]). Coercivity of f is a sufficient condition for f having this property, and, thus, it is a natural task to verify or disprove whether f is coercive. Also, since coercivity of f is equivalent to the boundedness of its lower level sets $\{x \in \mathbb{R}^n \mid f(x) \le \alpha\}$ for all $\alpha \in \mathbb{R}$, understanding coercivity can be useful to decide whether a basic semi-algebraic set is bounded. Furthermore, properness of polynomial maps $F : \mathbb{R}^n \to \mathbb{R}^n$ can be characterized by coercivity of the polynomial $\|F\|_2^2$, which, can be used to decide whether F is globally invertible and it directly refers to real versions of the Jacobian conjecture (see, e. g. [3, 6, 8]).

We investigate how coercivity of a polynomial f behaves under perturbations by other polynomials. This gives rise to the degree of stable coercivity $s(f)$, which equals the maximum degree small polynomial perturbations can possess such that they do not affect the coercivity of f. This notion is inspired by the concept of stable boundedness

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of polynomials [14]. We show that any perturbations by polynomials of order stricly smaller than $s(f)$ do not influence the coercivity of f however big they are.

Our first result, Theorem 3.2, gives an explicit relation between $\mathcal{L}_\infty(f)$ and $s(f)$ for arbitrary coercive polynomials stating that $s(f)$ is always equal to the lower integer part of $\mathcal{L}_\infty(f)$. Our second result, Theorem 3.3, is concerned with the special case of coercive polynomials with $\mathcal{L}_\infty(f)$ being maximum possible. For this case, we formulate several equivalent characterizations in terms of $\mathcal{L}_\infty(f)$ and $s(f)$. As an interesting consequence (Corollary 3.4), for coercive polynomials f of degree d, we find that $\mathcal{L}_\infty(f)$ cannot attain values in $(d-2, d)$, and, similarly, $s(f)$ is either less than or equal to $d-2$, or equal to d.

2. DEFINITIONS AND ELEMENTARY PROPERTIES

In this article, $\|\cdot\|$ stands for an arbitrary norm on $\mathbb{R}^n$ unless specified otherwise, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. By $\mathbb{R}[X_1, \ldots, X_n]$ we denote the ring of polynomials in n variables with real coefficients. The degree of f is abbreviated as $\deg(f)$, and f decomposes uniquely into its homogeneous components $f_0, \ldots, f_d \in \mathbb{R}[X_1, \ldots, X_n]$, where every f_i is homogeneous of degree i, or the zero polynomial. By $\mathbb{R}[X_1, \ldots, X_n]_d$ we denote the set of all polynomials $f \in \mathbb{R}[X_1, \ldots, X_n]$ with $\deg(f) \leq d$. We also define the number $\|f\|_\infty$ to be the largest absolute value of the coefficients of f.

The following auxiliary result proves useful for our later purposes, we give a proof for completeness.

Observation 2.1. For $f \in \mathbb{R}[X_1, \ldots, X_n]_d$, where $n \in \mathbb{N}$, $d \in \mathbb{N}_0$, and any $q \in [d, +\infty)$, the following estimate holds all $x \in \mathbb{R}^n$:

$$|f(x)| \leq \tbinom{n+d}{d} \cdot \|f\|_\infty \cdot \left(\|x\|_\infty^q + 1\right).$$

P r o o f. Fix $n \in \mathbb{N}$, $d \in \mathbb{N}_0$, $f \in \mathbb{R}[X_1, \ldots, X_n]$ of degree at most d and $q \geq d$. We write f in multi-index notation as $f = \sum_{\alpha \in A(f)} a_\alpha X^\alpha$ with $A(f) \subseteq \mathbb{N}_0^n$, where $a_\alpha \in \mathbb{R}$ for $\alpha \in A(f)$ and $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ for $\alpha \in \mathbb{N}_0^n$.

It is well-known that

$$\dim\{f \in \mathbb{R}[X_1, \ldots, X_n] : \deg(f) \leq d\} = \tbinom{n+d}{d}, \tag{1}$$

see, e. g., [15, Remark 1.2.5]. Hence, $|A(f)| \leq \binom{n+d}{d}$.

Also, for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = \alpha_1 + \ldots + \alpha_n \leq q$, we have

$$|x^\alpha| \leq \|x\|_\infty^{|\alpha|} \leq \max\left(\|x\|_\infty^q, 1\right) \leq \|x\|_\infty^q + 1.$$

The estimates combine to

$$\begin{aligned} |f(x)| = |\sum_{\alpha \in A(f)} a_\alpha x^\alpha| &\leq \|f\|_\infty \sum_{\alpha \in A(f)} |x^\alpha| \leq \|f\|_\infty \sum_{\alpha \in A(f)} \left(\|x\|_\infty^q + 1\right) \\ &= |A(f)| \cdot \|f\|_\infty \cdot \left(\|x\|_\infty^q + 1\right) \leq \tbinom{n+d}{d} \cdot \|f\|_\infty \cdot \left(\|x\|_\infty^q + 1\right). \end{aligned}$$

□

The *Łojasiewicz exponent at infinity* $\mathcal{L}_\infty(f)$ of $f : \mathbb{R}^n \to \mathbb{R}$ is the supremum of all $\nu \in \mathbb{R}$ such that there exists constants $c, M > 0$ with

$$|f(x)| \geq c\,\|x\|^\nu \quad \text{whenever} \quad \|x\| \geq M.$$

If f is a polynomial, it is a well-known result [7, 10, 13, 16] that the Łojasiewicz exponent at infinity is attained, i. e., there are $c, M > 0$ with

$$|f(x)| \geq c\,\|x\|^{\mathcal{L}_\infty(f)} \quad \text{whenever} \quad \|x\| \geq M, \tag{2}$$

and moreover there is $c' > 0$ and a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ with $\|x_k\| \to +\infty$ such that

$$|f(x_k)| \leq c'\|x_k\|^{\mathcal{L}_\infty(f)} \quad \text{for all} \quad k \in \mathbb{N}. \tag{3}$$

Given a coercive polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ we are interested in how stable this coercivity property is under small perturbations of f by other polynomials. This gives rise to the following definition for stability of coercivity which was already analyzed from the viewpoint of the underlying Newton polytopes in [2].

Definition 2.2. (Stable coercivity) A polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ is called q-stably coercive for $q \in \mathbb{N}_0$, if there exists an $\varepsilon > 0$ such that for all $g \in \mathbb{R}[X_1, \ldots, X_n]$ with $\deg g \leq q$ and $\|g\|_\infty \leq \varepsilon$ it holds that $f + g$ is coercive. The degree of stable coercivity $s(f)$ of f is the largest q such that f is q-stably coercive.

We also introduce the following stronger notion for the stability of coercivity.

Definition 2.3. (Strongly stable coercivity) A polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ is called strongly q-stable coercive for $q \in \mathbb{N}_0$, if for all $g \in \mathbb{R}[x]$ with $\deg g \leq q$ it holds that $f + g$ is coercive. The degree of strongly stable coercivity $\tilde{s}(f)$ of f is the largest q such that f is strongly q-stable coercive.

3. MAIN RESULT

In this section we show how the degree of stable and strongly stable coercivity are tied to the Łojasiewicz exponent at infinity (Theorem 3.2). In case of a positive definite leading form, a stronger characterization is available (Theorem 3.3). We use the following estimate in the proof of both.

Lemma 3.1. Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be coercive. Then the following inequalities are fulfilled:

$$\tilde{s}(f) \leq s(f) \leq \mathcal{L}_\infty(f) \leq \tilde{s}(f) + 1.$$

Proof. The first inequality $\tilde{s}(f) \leq s(f)$ follows obviously from the Definitions 2.2 and 2.3. To see $s(f) \leq \mathcal{L}_\infty(f)$, for $q := s(f)$ we introduce polynomials

$$f_{c,\sigma} := f - c \cdot \left(\sum_{j=1}^{n} \sigma_j X_j\right)^q,$$

parametrized by $c \in \mathbb{R}$ and $\sigma \in \Sigma := \{-1,1\}^n$. As $s(f) = q$, for every $\sigma \in \Sigma$ there is $\varepsilon_\sigma > 0$ such that $f_{c,\sigma}$ is coercive whenever $c \in [-\varepsilon_\sigma, \varepsilon_\sigma]$. Let $\hat{\varepsilon} := \min_{\sigma\in\Sigma} \varepsilon_\sigma$ and fix $\hat{c} \in (0, \hat{\varepsilon})$. Hence $f_{\hat{c},\sigma}$ is coercive for all $\sigma \in \Sigma$ and thus also bounded from below. Boundedness from below means for every σ there is $k_\sigma \geq 0$ with

$$f(x) \geq \hat{c}\left(\sum_{j=1}^n \sigma_j x_j\right)^q - k_\sigma, \quad x \in \mathbb{R}^n,\ \sigma \in \Sigma.$$

Put $\hat{k} := \max_{\sigma\in\Sigma} k_\sigma$. Then for $x \in \mathbb{R}^n$

$$f(x) \geq \hat{c} \cdot \max_{\sigma\in\Sigma}\left(\sum_{j=1}^n \sigma_j x_j\right)^q - \hat{k} = \hat{c} \cdot \left(\sum_{j=1}^n |x_j|\right)^q - \hat{k} = \hat{c} \cdot \|x\|_1^q - \hat{k},$$

hence $\mathcal{L}_\infty(f) \geq q = s(f)$.

Now we proceed to prove the third inequality $\mathcal{L}_\infty(f) \leq \tilde{s}(f)+1$. Assume the contrary: Let now $q := \tilde{s}(f)$ and suppose $\mathcal{L}_\infty(f) > q+1$. We have arrived at a contradiction if we may show that for any $g \in \mathbb{R}[X_1,\ldots,X_n]$ of degree at most $q+1$, $f+g$ is coercive, as in this case $\tilde{s}(f) \geq q+1 = \tilde{s}(f)+1$. To this end fix an arbitrary $g \in \mathbb{R}[X_1,\ldots,X_n]$ with $\deg(q) \leq q+1$. Now choose $c_1 > \binom{n+d}{d} \cdot \|g\|_\infty$. As $\mathcal{L}_\infty(f) > q+1$, continuity of f implies the existence of $c_2 \in \mathbb{R}$ such that $f(x) \geq c_1\|x\|_\infty^{q+1} - c_2$ holds for $x \in \mathbb{R}^n$, and hence, by Observation 2.1,

$$\begin{aligned} f(x)+g(x) &\geq f(x) - |g(x)| \geq c_1\|x\|_\infty^{q+1} - c_2 - \binom{n+d}{d} \cdot \|g\|_\infty \left(\|x\|_\infty^{q+1}+1\right) \\ &= c_1' \cdot \|x\|_\infty^{q+1} - c_2', \quad x \in \mathbb{R}^n, \end{aligned}$$

for some appropriately chosen $c_1' > 0$, $c_2' \in \mathbb{R}$. Thus $f+g$ is coercive. □

We show now how the integer part of the Łojasiewicz exponent at infinity and our notions of stability are related to each other.

Theorem 3.2. Let $f \in \mathbb{R}[X_1,\ldots,X_n]$ be coercive.

1. If $\mathcal{L}_\infty(f)$ is integer, then

$$\tilde{s}(f)+1 = s(f) = \mathcal{L}_\infty(f).$$

2. If $\mathcal{L}_\infty(f)$ is fractional, then

$$\tilde{s}(f) = s(f) = \lfloor \mathcal{L}_\infty(f) \rfloor.$$

P r o o f. In order to prove (1), we show $\tilde{s}(f)+1 = \mathcal{L}_\infty(f)$ first. By integrality of $\tilde{s}(f)$, $\mathcal{L}_\infty(f)$ and by the property $\mathcal{L}_\infty(f) \in [\tilde{s}(f), \tilde{s}(f)+1]$ holding due to Lemma 3.1, it is enough to show that $\tilde{s}(f) < \mathcal{L}_\infty(f)$. Suppose the contrary, that is $\tilde{s}(f) = \mathcal{L}_\infty(f) =: q$. Now for $c > 0$ and $\sigma \in \Sigma := \{-1,1\}^n$, define

$$f_{c,\sigma} := f - c \cdot \left(\sum_{j=1}^n \sigma_j X_j\right)^q \in \mathbb{R}[X_1,\ldots,X_n].$$

By definition of $\tilde{s}(f)$, the polynomial $f_{c,\sigma}$ is coercive and hence bounded from below for all $c > 0$ and $\sigma \in \Sigma$. That is, for every $c > 0$ and $\sigma \in \Sigma$, there exists $k_{c,\sigma} \geq 0$ such that

$$f(x) \geq c \cdot \left(\sum_{j=1}^{n} \sigma_j x_j \right)^q - k_{c,\sigma}, \quad x \in \mathbb{R}^n,\ c > 0,\ \sigma \in \Sigma,$$

and hence with $k_c := \max_{\sigma \in \Sigma} k_{c,\sigma}$, we have for all $x \in \mathbb{R}^n$ and $c > 0$ the property

$$f(x) \geq c \cdot \max_{\sigma \in \Sigma} \left(\sum_{j=1}^{n} \sigma_j x_j \right)^q - k_c = c \cdot \left(\sum_{j=1}^{n} |x_j| \right)^q - k_c = c \cdot \|x\|_1^q - k_c.$$

This, however, contradicts (3), so we may conclude that $\tilde{s}(f) + 1 = \mathcal{L}_\infty(f)$.

For the second equality $s(f) = \mathcal{L}_\infty(f)$, put $q := \mathcal{L}_\infty(f)$. By (2) as well as coercivity and continuity of f, there are constants $c_1, c_2 > 0$ such that

$$f(x) \geq c_1 \|x\|_\infty^q - c_2 \quad \text{for} \quad x \in \mathbb{R}^n.$$

Define $\varepsilon := \frac{c_1}{2} \cdot \binom{n+q}{q}^{-1}$. Now for any $g \in \mathbb{R}[X_1, \ldots, X_n]$ with $\deg(g) \leq q$ and $\|g\|_\infty \leq \varepsilon$ and all $x \in \mathbb{R}^n$, we have from Observation 2.1

$$\begin{aligned} f(x) + g(x) &\geq f(x) - |g(x)| \\ &\geq c_1 \|x\|_\infty^q - c_2 - \varepsilon \cdot \tbinom{n+q}{q} \left(\|x\|_\infty^q + 1 \right) \\ &= \frac{c_1}{2} \|x\|_\infty^q - c_2 - \frac{c_1}{2}. \end{aligned}$$

To summarize, $f + g$ is coercive whenever $\deg g \leq q$ and $\|g\|_\infty \leq \varepsilon$, that is, f is q-stably coercive, or $s(f) \geq q = \mathcal{L}_\infty(f)$. With $\mathcal{L}_\infty(f) \geq s(f)$ from Lemma 3.1, the claim follows.

Statement (2) follows at once from Lemma 3.1. □

Our next result shows that more characterizations are available for a maximal Łojasiewicz exponent at infinity.

Theorem 3.3. Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ of degree $d \geq 2$ be coercive. Then, the following assertions are equivalent:

1. $f_d(x) > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$.
2. There exists $\delta > 0$ such that $f_d(x) \geq \delta \|x\|^d$ for all $x \in \mathbb{R}^n$.
3. $\mathcal{L}_\infty(f) = d$.
4. $\mathcal{L}_\infty(f) > d - 2$.
5. $s(f) = d$.
6. $s(f) \geq d - 1$.
7. $\tilde{s}(f) = d - 1$.

8. $\tilde{s}(f) \geq d-2$.

Proof. Let $\mathbb{S} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ denote the unit sphere. We start with "(1) $\Rightarrow$ (2)". For $x = 0$ the assertion is trivial. For nonzero $x \in \mathbb{R}^n$ one obtains

$$f_d(x) = \|x\|^d f_d\left(\frac{x}{\|x\|}\right) \geq \|x\|^d \inf_{y \in \mathbb{S}^{n-1}} f_d(y).$$

The infimum is positive by compactness of the sphere. Now for "(2) $\Rightarrow$ (3)", let $c_j = \inf_{y\in\mathbb{S}^{n-1}} f_j(y)$ for $j = 0, \ldots, d-1$ and put $c_d = \delta$. Then by homogeneity of the f_j,

$$f(x) = \sum_{j=0}^{d} f_j(x) \geq \sum_{j=0}^{d} c_j \|x\|^j,$$

hence $\mathcal{L}_\infty(f) \geq d$. We show $\mathcal{L}_\infty(f) \leq d$ by contradiction. Suppose that $\mathcal{L}_\infty(f) > d' > d$. Hence $|f(x)| \geq c\|x\|_\infty^{d'}$ for large $\|x\|_\infty$. Using Observation 2.1 and $q := d$, we can also find an appropriate $C > 0$ with $|f(x)| \leq C\|x\|_\infty^d$ for large $\|x\|_\infty$. This yields

$$c\|x\|_\infty^{d'} \leq |f(x)| \leq C\|x\|_\infty^d$$

for large $\|x\|_\infty$, which is impossible since $d' > d$, and $\mathcal{L}_\infty(f) \leq d$ follows.

The implication "(3) $\Rightarrow$ (4)" is trivial. The implication "(4) $\Rightarrow$ (1)" holds as follows: Suppose $\mathcal{L}_\infty(f) > d-2$ but $f_d(\tilde{x}) = 0$ for some $\tilde{x} \in \mathbb{R}^n$ with $\tilde{x} \neq 0$. Now f is coercive by assumption. Let us show that this implies $f_{d-1}(\tilde{x}) = 0$. Indeed, we find that for all $\lambda \in \mathbb{R}$ it holds

$$f(\lambda\tilde{x}) = \sum_{j=0}^{d} f_j(\lambda\tilde{x}) = \sum_{j=0}^{d-1} \lambda^j f_j(\tilde{x}),$$

which, as a function of λ is unbounded from below unless $f_{d-1}(\tilde{x}) = 0$. In fact, this holds since $d-1$ is odd. Hence

$$|f(\lambda\tilde{x})| \leq \sum_{j=0}^{d-2} |f_j(\lambda\tilde{x})| = \sum_{j=0}^{d-2} |\lambda|^j |f_j(\tilde{x})|,$$

implying $\mathcal{L}_\infty(f) \leq d-2$, a contradiction, so (1) through (4) are equivalent.

To see "(2) $\Rightarrow$ (5)", let $g \in \mathbb{R}[X_1, \ldots, X_n]$ of degree d be given, and let $c' = \max_{x\in\mathbb{S}^{n-1}} g_d(x)$. Then $|g_d(x)| \leq c'\|x\|^d$ by homogeneity, so for $\varepsilon \in [-\frac{\delta}{2c'}, \frac{\delta}{2c'}]$,

$$f_d(x) + \varepsilon g_d(x) \geq f_d(x) - |\varepsilon g_d(x)| \geq \delta\|x\|^d - \frac{\delta}{2}\|x\|^d = \frac{\delta}{2}\|x\|^d,$$

hence $f + \varepsilon g$ is still coercive, and we conclude $s(f) = d$.

We show now that (5). implies (6) and (7). The first implication is trivial. To see "(5) $\Rightarrow$ (7)", note that Lemma 3.1 implies $\tilde{s}(f) \geq d-1$. As $\tilde{s}(f) \geq d$ is not possible for a degree d polynomial, $\tilde{s}(f) = d-1$. Since both (6) and (7) imply (8) trivially, all equivalences are shown once "(8) $\Rightarrow$ (4)" holds.

So suppose $\tilde{s}(f) \geq d-2$. By assumption, f is coercive, so d must be even. The function $g(x) = \|x\|_2^{d-2}$ is a polynomial of degree $d-2$. The assumption $\tilde{s}(f) \geq d-2$ implies that $f - c_1 g$ is coercive for all $c_1 > 0$. Hence there is M, depending on c_1, such that

$$f(x) - c_1 \|x\|_2^{d-2} \geq 0$$

holds for all $x \in \mathbb{R}^n$ whenever $\|x\| \geq M$. Now (3) forces $\mathcal{L}_\infty(f) > d-2$, which finishes the proof. □

The latter theorem yields the following interesting consequence.

Corollary 3.4. Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ of degree d be coercive. Then

1. $\mathcal{L}_\infty(f) \in (0, d-2] \cup \{d\}$,
2. $s(f) \in \{0, \ldots, d-2, d\}$,
3. $\tilde{s}(f) \in \{0, \ldots, d-1\}$.

Proof. For coercive f, Property (3) yields $\mathcal{L}_\infty(f) > 0$. We have already seen in the proof of Theorem 3.3 that $\mathcal{L}_\infty(f) \leq d$. By Statements (3) and (4) of Theorem 3.3, $\mathcal{L}_\infty(f) \in [0, d-2] \cup \{d\}$, and Assertion 1 follows. Assertions 2 and 3 follow immediately from Theorem 3.3. □

An open question which arises in this context is, whether for coercive polynomials f, further restrictions on $\mathcal{L}_\infty(f)$ and also for $s(f)$ are possible. By varying n (see [10]) or d (see [4]), it is possible to construct coercive polynomials with $\mathcal{L}_\infty(f)$ positive but arbitrarily close to zero. Thus, another open question is whether for fixing both n and d, for a coercive polynomial f, the number $\mathcal{L}_\infty(f)$ can approach zero by only varying the coefficients of f.

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