Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 69 (2019), No. 4, 1039–1052

Persistent URL: http://dml.cz/dmlcz/147912

Czechoslovak Mathematical Journal, 69 (144) (2019), 1039–1052

MULTI-MORREY SPACES FOR NON-DOUBLING MEASURES

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Received January 17, 2018. Published online May 21, 2019.

Abstract. The spaces of multi-Morrey type for positive Radon measures satisfying a growth condition on $\mathbb{R}^d$ are introduced. After defining the spaces, we investigate the multilinear maximal function, the multilinear fractional integral operator and the multilinear Calderón-Zygmund operators, respectively, from multi-Morrey spaces to Morrey spaces.

Keywords: multi-Morrey space; multilinear maximal function; multilinear fractional integral operator; multilinear Calderón-Zygmund operator

MSC 2010: 42B35, 42B25

1. Introduction

In recent years, many results have indicated that the doubling condition is superfluous for most of the classical Calderón-Zygmund theory (see [8], [9], [10]). Considerable attention has been paid to the study of the classical theory of harmonic analysis on Euclidean spaces with non-doubling measures only satisfying the polynomial growth condition (see [2], [3], [13], [14], [15], [16]). To be precise, let μ be a positive Radon measure on $\mathbb{R}^d$ which satisfies the polynomial growth condition that for all $x \in \mathbb{R}^d$ and $r > 0$,

$$\mu(B(x,r)) \leqslant c_0 r^n,$$

where c_0 is a positive constant, $0 < n \leqslant d$, and $B(x,r)$ is the open ball centered at x and having radius r. The analysis associated with such a non-doubling measure μ has proved to play a striking role in solving the long-standing open Painlevé's problem and Vitushkin's conjecture (see [14]).

The research was supported by the Natural Science Foundation of Xinjiang Uygur Autonomous Region (2019D01C334).

DOI: 10.21136/CMJ.2019.0031-18

The main purpose of this paper is to give the definition of multi-Morrey spaces and establish the boundedness of multilinear operators from multi-Morrey spaces to Morrey spaces associated with μ.

To state the main results of this paper, we need first to recall some necessary notation and notion. We use $\mathcal{Q}(\mu)$ to denote the family of all cubes in $\mathbb{R}^d$ satisfying $\mu(Q) > 0$. For $c > 0$, cQ will denote a cube concentric to Q with its sidelength $cl(Q)$.

The Morrey space $M_p^{p_0}$ was defined by Morrey in 1938, see [7]. In 2005, Sawano and Tanaka in [12] gave the definition of Morrey spaces for non-doubling measures. Let $k > 1$ and $1 \leqslant p \leqslant p_0 < \infty$. The Morrey spaces $M_p^{p_0}(k, \mu)$ are defined by the norm

$$\|f\|_{M_p^{p_0}(k,\mu)} := \sup_{Q \in \mathcal{Q}(\mu)} \mu(kQ)^{1/p_0} \Big(\frac{1}{\mu(kQ)} \int_Q |f(x)|^p \, \mathrm{d}x \Big)^{1/p}.$$

It is proved in [12] that the Morrey spaces $M_q^{p_0}(k, \mu)$ are independent of the choices of the constant $k \in (1, \infty)$. Later, Sawano in [11] introduced generalized Morrey spaces for non-doubling measures.

Now, we give the definition of the multi-Morrey norm for the non-doubling measure. In the setting of the Euclidean space, Iida, Sato, Sawano and Tanaka in [4] have proved that the multi-Morrey norm is strictly smaller than the m-fold product of the Morrey norms.

Definition 1.1. Let $k > 1$, $\vec{P} = (p_1, \ldots, p_m)$ with $1 \leqslant p_1, \ldots, p_m \leqslant \infty$ and $0 < p \leqslant p_0 < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$. For some collection of measurable functions $\vec{f} = (f_1, \ldots, f_m)$ on $\mathbb{R}^d$, the multi-Morrey norm is defined by

$$\|\vec{f}\|_{\mathcal{B}_{\vec{P}}^{p_0}(k,\mu)} = \sup_{Q \in \mathcal{Q}(\mu)} \mu(kQ)^{1/p_0} \prod_{i=1}^{m} \Big(\frac{1}{\mu(kQ)} \int_Q |f_i(x)|^{p_i} \, \mathrm{d}\mu(x) \Big)^{1/p_i} < \infty.$$

We define the multi-Morrey space $\mathcal{M}_{\vec{P}}^{p_0}(k, \mu)$ as the set of all measurable functions g on $(\mathbb{R}^d)^m$ which can be written as

$$g(x_1, \ldots, x_m) = \sum_{j=1}^{\infty} \prod_{i=1}^{m} f_i^{(j)}(x_i) \tag{1.1}$$

in the sense of almost everywhere convergence in $(\mathbb{R}^d)^m$, and $(f_1^{(j)}, \ldots, f_m^{(j)})$ satisfying

$$\sum_{j=1}^{\infty} \|(f_1^{(j)}, \ldots, f_m^{(j)})\|_{\mathcal{B}_{\vec{P}}^{p_0}(k,\mu)} < \infty. \tag{1.2}$$

The norm of $g \in \mathcal{M}_{\vec{P}}^{p_0}(k, \mu)$ is defined by

$$\|g\|_{\mathcal{M}_{\vec{P}}^{p_0}(k,\mu)} := \inf \sum_{j=1}^{\infty} \|(f_1^{(j)}, \ldots, f_m^{(j)})\|_{\mathcal{B}_{\vec{P}}^{p_0}(k,\mu)},$$

where the functions in infimum run over all expressions as (1.1).

By the Hölder inequality and the triangle inequality, we have the following result on the structure of $\mathcal{M}^{p_0}_{\vec{P}}$. If (1.2) holds, then $\sum\limits_{j=1}^{\infty}\prod\limits_{i=1}^{m} f_i^{(j)}(x_i)$ converges absolutely almost everywhere in $(\mathbb{R}^d)^m$. As an application, in order to prove the boundedness of the linear operator T from multi-Morrey spaces $\mathcal{M}^{p_0}_{\vec{P}}(k,\mu)$ to Morrey spaces $M^{q_0}_q(k,\mu)$, it suffices to prove that there exists a constant C such that

$$\|T(\vec{f})\|_{M^{q_0}_q(k,\mu)} \leqslant C\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)}$$

holds for all $\vec{f}=(f_1,\dots,f_m)$.

Next, we prove that the definition of multi-Morrey spaces is independent of the choice of the parameter $k>1$. This result covers the ones in [12]. For the sake of convenience, we provide the details.

Proposition 1.2. *Let $k_1,k_2>1$, $\vec{P}=(p_1,\dots,p_m)$ with $1\leqslant p_1,\dots,p_m\leqslant\infty$ and $0<p\leqslant p_0<\infty$ with $1/p=1/p_1+\ldots+1/p_m$. Then $\mathcal{M}^{p_0}_{\vec{P}}(k_1,\mu)$ and $\mathcal{M}^{p_0}_{\vec{P}}(k_2,\mu)$ coincide and their norms are equivalent.*

Proof. Let $k_1\leqslant k_2$. Then the inclusion $\mathcal{M}^{p_0}_{\vec{P}}(k_1,\mu)\subset\mathcal{M}^{p_0}_{\vec{P}}(k_2,\mu)$ is trivial by the definition of the norms. Let us show the reverse inclusion.

For any $Q\in\mathcal{Q}(\mu)$, there exist cubes $Q_1,Q_2,\dots,Q_N$ with the same sidelength such that

$$Q\subset\bigcup_{j=1}^{N}Q_j,\quad k_2Q_j\subset k_1Q\quad\text{and}\quad N\leqslant C_Q\Big(\frac{k_2-1}{k_1-1}\Big)^d.$$

Let $g\in\mathcal{M}^{p_0}_{\vec{P}}(k_2,\mu)$ and let $\sum\limits_{j=1}^{\infty}\prod\limits_{i=1}^{m} f_i^{(j)}(x_i)$ be an admissible expression of g. For any $Q\in\mathcal{Q}(\mu)$, using the fact that $1/p_0\leqslant\sum\limits_{i=1}^{m}1p_i$ and the covering above, we easily obtain

$$\begin{aligned}
&\mu(k_1Q)^{1/p_0}\prod_{i=1}^{m}\Big(\frac{1}{\mu(k_1Q)}\int_Q|f_i^{(j)}(x)|^{p_i}\,\mathrm{d}\mu(x)\Big)^{1/p_i}\\
&\quad\leqslant\sum_{j=1}^{N}\mu(k_1Q)^{1/p_0}\prod_{i=1}^{m}\Big(\frac{1}{\mu(k_1Q)}\int_{Q_j}|f_i^{(j)}(x)|^{p_i}\,\mathrm{d}\mu(x)\Big)^{1/p_i}\\
&\quad\leqslant\sum_{j=1}^{N}\mu(k_2Q_j)^{1/p_0}\prod_{i=1}^{m}\Big(\frac{1}{\mu(k_2Q_j)}\int_{Q_j}|f_i^{(j)}(x)|^{p_i}\,\mathrm{d}\mu(x)\Big)^{1/p_i}\\
&\quad\leqslant\sum_{j=1}^{N}\|(f_1^{(j)},\dots,f_m^{(j)})\|_{\mathcal{B}^{p_0}_{\vec{P}}(k_2,\mu)}\leqslant C_Q\Big(\frac{k_2-1}{k_1-1}\Big)^d\|(f_1^{(j)},\dots,f_m^{(j)})\|_{\mathcal{B}^{p_0}_{\vec{P}}(k_2,\mu)}.
\end{aligned}$$

Then, the desired result follows from the structure of $\mathcal{M}^{p_0}_{\vec{P}}(k,\mu)$. □

2. Boundedness of multilinear maximal function

Let $\varrho > 1$. The maximal function $\mathcal{M}_\varrho$ is defined by

$$M_\varrho(f)(x) := \sup_{x\in Q} \frac{1}{\mu(\varrho Q)} \int_Q |f(y)|\,\mathrm{d}\mu(y).$$

The multilinear maximal function can be defined as follows:

$$\mathcal{M}_{\varrho,m} f(x) := \sup_{x\in Q} \prod_{i=1}^m \frac{1}{\mu(\varrho Q)} \int_Q |f_i(y)|\,\mathrm{d}\mu(y).$$

It is well-known that $\mathcal{M}_\varrho$ is bounded on $L^p(\mu)$, $1 < p \leqslant \infty$. For details we refer to [8]. In [5], $\mathcal{M}_{\varrho,m}$ is used to obtain a precise control on the multilinear singular integral of Calderón-Zygmund type in the setting of the Euclidean space. We present the main theorem in this section.

Theorem 2.1. *Let $k, \varrho, p_1, \ldots, p_m > 1$, $\vec{P} = (p_1, \ldots, p_m)$ and $1 < p \leqslant p_0 < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$. Then*

$$\|\mathcal{M}_{\varrho,m}(\vec{f})\|_{M_p^{p_0}(k,\mu)} \leqslant C\|\vec{f}\|_{\mathcal{B}_{\vec{P}}^{p_0}(k,\mu)}.$$

Proof. In fact, we need only to prove that

$$\|\mathcal{M}_{\varrho,m}(\vec{f})\|_{M_p^{p_0}(2\varrho(\varrho+7)/(\varrho^2-1),\mu)} \leqslant C\|\vec{f}\|_{\mathcal{B}_{\vec{P}}^{p_0}(2\varrho/(\varrho+1),\mu)}.$$

Fix $Q \in \mathcal{Q}(\mu)$. Let $L = 4l(Q)/(\varrho-1)$, $f_{i,1} := \chi_{((\varrho+7)/(\varrho-1))Q} f_i$ and $f_{i,2} := f_i - f_{i,1}$ with $i = 1, \ldots, m$. For any $x \in Q$, it follows from the definition of $\mathcal{M}_{\varrho,m}$ that

$$\mathcal{M}_{\varrho,m}(f_{1,\sigma(1)}, \ldots, f_{m,\sigma(m)})(x) \leqslant \sup_{\substack{x\in Q'\in\mathcal{Q}(\mu)\\ l(Q')\geqslant L}} \prod_{i=1}^m \frac{1}{\mu(\varrho Q')} \int_{Q'} |f_i(y)|\,\mathrm{d}\mu(y),$$

where $\sigma(1), \ldots, \sigma(m) \in \{1, 2\}$ and $(\sigma(1), \ldots, \sigma(m)) \neq (1, \ldots, 1)$. It follows from $x \in Q \cap Q'$ and $l(Q') \geqslant (4/(\varrho-1))l(Q)$ that $Q \subset \frac{1}{2}(\varrho+1)Q'$. By a standard argument we have

$$\mathcal{M}_{\varrho,m}(f_{1,\sigma(1)}, \ldots, f_{m,\sigma(m)})(x) \leqslant \sup_{Q\subset Q'\in\mathcal{Q}(\mu)} \prod_{i=1}^m \Big(\mu\Big(\frac{2\varrho}{\varrho+1}Q'\Big)\Big)^{-1} \int_{Q'} |f_i(y)|\,\mathrm{d}\mu(y).$$

Thus, we have for $x \in Q$,

$$\mathcal{M}_{\varrho,m}(\vec{f})(x) \leqslant M_{\varrho,m}(f_{1,1},\dots,f_{m,1})(x) \\ + C \sup_{Q\subset Q'\in\mathcal{Q}(\mu)} \prod_{i=1}^{m} \Big(\mu\Big(\frac{2\varrho}{\varrho+1}Q'\Big)\Big)^{-1} \int_{Q'} |f_i(y)|\,\mathrm{d}\mu(y).$$

Let $k = 2\varrho(\varrho+7)/(\varrho^2-1) > 1$. First, by the Hölder inequality and the (p,p) boundedness of M_ϱ, we have

$$\begin{aligned}
&\mu(kQ)^{1/p_0}\Big(\frac{1}{\mu(kQ)}\int_Q |\mathcal{M}_{\varrho,m}(f_{1,1},\dots,f_{m,1})(x)|^{1/p}\,\mathrm{d}\mu(x)\Big)^{1/p}\\
&\leqslant \mu(kQ)^{1/p_0}\prod_{i=1}^{m}\Big(\frac{1}{\mu(kQ)}\int_Q |M_\varrho(f_{i,1})(x)|^{1/p_i}\,\mathrm{d}\mu(x)\Big)^{1/p_i}\\
&\leqslant C\mu\Big(\frac{2\varrho(\varrho+7)}{\varrho^2-1}Q\Big)^{1/p_0}\\
&\quad\times\prod_{i=1}^{m}\bigg(\Big(\mu\Big(\frac{2\varrho(\varrho+7)}{\varrho^2-1}Q\Big)\Big)^{-1}\int_{((\varrho+7)/(\varrho-1))Q}\big|f_i(x)\big|^{1/p_i}\,\mathrm{d}\mu(x)\bigg)^{1/p_i}\\
&\leqslant C\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(2\varrho/(\varrho+1),\mu)}.
\end{aligned}$$

Second, we have again by the Hölder inequality

$$\begin{aligned}
&\mu(kQ)^{1/p_0}\Big(\frac{1}{\mu(kQ)}\int_Q |\mathcal{M}_{\varrho,m}(f_{1,2},\dots,f_{m,2})(x)|^{1/p}\,\mathrm{d}\mu(x)\Big)^{1/p}\\
&\leqslant \mu\Big(\frac{2\varrho}{\varrho+1}Q\Big)^{1/p_0}\sup_{Q\subset Q'\in\mathcal{Q}(\mu)}\prod_{i=1}^{m}\bigg(\Big(\mu\Big(\frac{2\varrho}{\varrho+1}Q'\Big)\Big)^{-1}\int_{Q'}|f_i(y)|^{1/p_i}\,\mathrm{d}\mu(y)\bigg)^{1/p_i}\\
&\leqslant C\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(2\varrho/(\varrho+1),\mu)}.
\end{aligned}$$

Hence, we have

$$\|\mathcal{M}_{\varrho,m}(\vec{f})\|_{M^{p_0}_p(2\varrho(\varrho+7)/(\varrho^2-1),\mu)} \leqslant C\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(2\varrho/(\varrho+1),\mu)}.$$

Theorem 2.1 is therefore proved. □

3. Boundedness of multilinear fractional integral operators

The aim of this section is to investigate the boundedness of the Adams type (see [1]) multilinear fractional integral $I_{\alpha,m}$, $0<\alpha<mn$, which is given by

$$I_{\alpha,m}(\vec{f})(x)=\int_{(\mathbb{R}^n)^m}\frac{f_1(y_1)\dots f_m(y_m)\,\mathrm{d}\mu(\vec{y})}{(|x-y_1|+\dots+|x-y_m|)^{mn-\alpha}},$$

where $\vec{f}=(f_1,\dots,f_m)$.

Theorem 3.1. *Let $0<\alpha<mn$, $1<k$, $p_1,\dots,p_m<\infty$, $\vec{P}=(p_1,\dots,p_m)$, $0<p\leqslant p_0<\infty$ and $0<q\leqslant q_0<\infty$. Suppose that*

$$\frac{1}{p}=\frac{1}{p_1}+\dots+\frac{1}{p_m},\quad \frac{1}{q_0}=\frac{1}{p_0}-\frac{\alpha}{n}\quad\text{and}\quad\frac{q}{q_0}=\frac{p}{p_0}.$$

Then

$$\|I_{\alpha,m}(\vec{f})\|_{M_q^{q_0}(k,\mu)}\leqslant C\|\vec{f}\|_{\mathcal{B}_{\vec{P}}^{p_0}(k,\mu)}.$$

By the following lemma, Theorem 3.1 can be deduced immediately.

Lemma 3.2. *Let $0<\alpha<mn$, $1<k$, $p_1,\dots,p_m<\infty$, $\vec{P}=(p_1,\dots,p_m)$ and $0<p\leqslant p_0<n/\alpha$ with $1/p=1/p_1+\dots+1/p_m$, assume that each f_j is measurable. Then*

$$|I_\alpha(\vec{f})(x)|\leqslant C\|\vec{f}\|^{\alpha p/n}_{\mathcal{B}_{\vec{P}}^{p_0}(k,\mu)}\mathcal{M}_{\varrho,m}(\vec{f})(x)^{1-\alpha p/n}.$$

Proof. Take $\varepsilon>0$ which will be determined later on. We separate $I_{\alpha,m}(\vec{f})(x)$ into

$$\mathrm{I}_1:=\int_{|x-y_1|+\dots+|x-y_m|\leqslant\varepsilon}\frac{f_1(y_1)\dots f_m(y_m)\,\mathrm{d}\mu(\vec{y})}{(|x-y_1|+\dots+|x-y_m|)^{mn-\alpha}}$$

and

$$\mathrm{I}_2:=\int_{|x-y_1|+\dots+|x-y_m|>\varepsilon}\frac{f_1(y_1)\dots f_m(y_m)\,\mathrm{d}\mu(\vec{y})}{(|x-y_1|+\dots+|x-y_m|)^{mn-\alpha}}.$$

For I_1, we write

$$\begin{aligned}|\mathrm{I}_1|&=\sum_{j=1}^{\infty}\int_{2^{-j}\varepsilon<|x-y_1|+\dots+|x-y_m|\leqslant 2^{-j+1}\varepsilon}\frac{|f_1(y_1)|\dots|f_m(y_m)|\,\mathrm{d}\mu(\vec{y})}{(|x-y_1|+\dots+|x-y_m|)^{mn-\alpha}}\\&\leqslant\sum_{j=1}^{\infty}\frac{1}{(2^{-j}\varepsilon)^{mn-\alpha}}\prod_{i=1}^{m}\int_{|x-y_i|\leqslant 2^{-j+1}\varepsilon}|f_i(y_i)|\,\mathrm{d}\mu(y_i)\leqslant C\varepsilon^{\alpha}\mathcal{M}_{\varrho,m}(\vec{f})(x).\end{aligned}$$

As to I_2,

$$|\mathrm{I}_2| = \sum_{j=1}^{\infty} \int_{2^{j-1}\varepsilon<|x-y_1|+\ldots+|x-y_m|\leqslant 2^j\varepsilon} \frac{|f_1(y_1)|\ldots|f_m(y_m)|\,\mathrm{d}\mu(\vec{y})}{(|x-y_1|+\ldots+|x-y_m|)^{mn-\alpha}}$$

$$\leqslant C\sum_{j=1}^{\infty} \frac{1}{(k2^j\varepsilon)^{mn-\alpha}} \prod_{i=1}^{m} \int_{|x-y_i|\leqslant 2^j\varepsilon} |f_i(y_i)|\,\mathrm{d}\mu(y_i) \leqslant C\varepsilon^{\alpha-n/p_0}\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)}.$$

Together with the estimates above, we see that

$$|I_\alpha(\vec{f})(x)| \leqslant C(\varepsilon^\alpha \mathcal{M}_{\varrho,m}(\vec{f})(x) + \varepsilon^{\alpha-n/p_0}\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)}).$$

Now take

$$\varepsilon = \left(\frac{\|\vec{f}\|_{\mathcal{M}^{p_0}_{\vec{P}}(k,\mu)}}{\mathcal{M}_{\varrho,m}(\vec{f})(x)}\right)^{p_0/n}.$$

Then

$$\varepsilon^\alpha \mathcal{M}_{\varrho,m}(\vec{f})(x) = \varepsilon^{\alpha-n/p_0}\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)} = \|\vec{f}\|^{\alpha p_0/n}_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)} \mathcal{M}_{\varrho,m}(\vec{f})(x)^{1-\alpha p_0/n}.$$

Thus the lemma is proved. □

Now, we consider the endpoint case $\alpha \geqslant n/p_0$. Lin and Yang in [6] showed that $I_{\alpha,m}$ is a bounded operator from m-fold product of Morrey spaces to $\mathrm{RBMO}(\mu)$ (or the Lipschitz space). We can improve the results as follows.

Theorem 3.3. *Let $1 < k$, $p_1,\ldots,p_m < \infty$, $0 < \alpha = n/p_0 < 1$ and $\vec{P} = (p_1,\ldots,p_m)$ with $1/p_1 + \ldots + 1/p_m \geqslant 1/p_0$, assume that μ satisfies the growth condition. Then*

$$\|I_{\alpha,m}(\vec{f})\|_{\mathrm{RBMO}(\mu)} \leqslant C\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)}.$$

Theorem 3.4. *Let $1 < k, p_1,\ldots,p_m < \infty, 0 < \alpha - n/p_0 < 1$ and $\vec{P} = (p_1,\ldots,p_m)$ with $1/p_1 + \ldots + 1/p_m \geqslant 1/p_0$, assume that μ satisfies the growth condition. Then*

$$\|I_{\alpha,m}(\vec{f})\|_{\mathrm{Lip}_{\alpha-n/p_0}(\mu)} \leqslant C\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)}.$$

The proof of Theorems 3.3 and 3.4 will be given below, we begin with recalling the definition of $\mathrm{RBMO}(\mu)$ and the Lipschitz spaces $\mathrm{Lip}_\alpha(\mu)$.

Definition 3.5. Let $k \in (1, \infty)$. Set

$$K_{Q,R} = 1 + \sum_{j=1}^{N_{Q,R}} \frac{\mu(2^j Q)}{(l(2^j Q))^n},$$

where $N_{Q,R}$ is the smallest positive integer j such that $l(2^j Q) \geqslant l(R)$. Given a cube $Q \subset \mathbb{R}^d$, let N be the smallest integer not less then 0 such that $2^N Q$ is doubling. We denote this cube by $\widetilde{Q}$.

A function $f \in L^1_{\text{loc}}(\mu)$ is said to belong to the space $\text{RBMO}(\mu)$ if there exists a positive constant C such that

$$\sup_Q \frac{1}{\mu(kQ)} \int_Q |f(x) - m_{\widetilde{Q}}(f)| \, \mathrm{d}\mu(x) \leqslant C \tag{3.1}$$

and that for any two doubling cubes $Q \subset R$,

$$|m_Q(f) - m_R(f)| \leqslant C K_{Q,R}. \tag{3.2}$$

The minimal constant C in (3.1) and (3.2) is defined to be the norm of f in $\text{RBMO}(\mu)$ and denoted by $\|f\|^p_{\text{RBMO}}(\mu)$.

In [13], Tolsa proved that the definition of the space $\text{RBMO}(\mu)$ does not depend on the constant $k > 1$. He also obtained that this space also satisfies a John-Nirenberg inequality and its predual is an atomic space H^1.

The following Lipschitz space is a special case of the Lipschitz spaces introduced by García-Cuerva and Gatto in [2]. We also give the definition in the setting of nonhomogeneous space in [17].

Definition 3.6. Let $\beta \in (0, \infty)$. A locally integrable function b is said to belong to the Lipschitz space $\text{Lip}_\alpha(\mu)$ if there exists a positive constant C such that

$$|b(x) - b(y)| \leqslant C|x - y|^\beta$$

for μ-almost all x and y in the support of μ. The minimal constant C is defined by $\|b\|_{\text{Lip}_\alpha(\mu)}$.

Proof of Theorem 3.3. For simplicity, we assume that $m = 2$. For any cube Q, set $\Omega_Q = \{(y_1, y_2)\colon\ |x - y_1| + |x - y_2| < \frac{4}{3} l(Q)\}$ and

$$C_Q^\infty := \frac{1}{\mu(Q)} \int_Q \int_{(\mathbb{R}^d)^2 \setminus \Omega_Q} \frac{f_1(y_1) f_2(y_2) \, \mathrm{d}\mu(y_1) \, \mathrm{d}\mu(y_2)}{(|x - y_1| + |x - y_2|)^{2n-\alpha}} \, \mathrm{d}\mu(x).$$

We first verify the inequality

$$\frac{1}{\mu(\frac{3}{2}Q)}\int_Q |I_{\alpha,2}(f_1,f_2)(x)-C_Q^\infty|\,\mathrm{d}\mu(x)\leqslant C\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(\mu)}. \tag{3.3}$$

For (3.3), we write

$$\begin{aligned}
&\frac{1}{\mu(\frac{3}{2}Q)}\int_Q |I_{\alpha,2}(f_1,f_2)(x)-C_Q^\infty|\,\mathrm{d}\mu(x)\\
&\quad\leqslant \frac{1}{\mu(\frac{3}{2}Q)}\int_Q\int_{\Omega_Q}\frac{|f_1(y_1)||f_2(y_2)|\,\mathrm{d}\mu(y_1)\,\mathrm{d}\mu(y_2)}{(|x-y_1|+|x-y_2|)^{2n-\alpha}}\,\mathrm{d}\mu(x)\\
&\qquad+\frac{1}{\mu(\frac{3}{2}Q)}\int_Q\left|\int_{(\mathbb{R}^d)^2\setminus\Omega_Q}\frac{f_1(y_1)f_2(y_2)\,\mathrm{d}\mu(y_1)\,\mathrm{d}\mu(y_2)}{(|x-y_1|+|x-y_2|)^{2n-\alpha}}-C_Q^\infty\right|\mathrm{d}\mu(x)\\
&\quad=: \mathrm{II}_1+\mathrm{II}_2.
\end{aligned}$$

There exist $\widetilde{p}_0,\widetilde{p}_1,\widetilde{p}_2\in(1,\infty)$ such that $\widetilde{p}_i\leqslant p_i$ for $i=1,2$ and $1/\widetilde{p}_1+1/\widetilde{p}_2=1/\widetilde{p}_0-\alpha/n>0$. For II_1, from $1/p_0=\alpha/n$, the Kolmogorov inequality and the boundedness of $I_{\alpha,2}$ from $L^{\widetilde{p}_1}(\mu)\times L^{\widetilde{p}_2}(\mu)$ to $L^{\widetilde{p}_0}(\mu)$, it follows that

$$\begin{aligned}
\mathrm{II}_1&\leqslant\frac{1}{\mu(\frac{3}{2}Q)}\int_Q |I_{\alpha,2}(|f_1\chi_{4/3Q}|,|f_2\chi_{4/3Q}|)(x)|\,\mathrm{d}\mu(x)\\
&\leqslant C\frac{(\mu(Q)^{1-1/\widetilde{p}_0})}{\mu(\frac{3}{2}Q)}\|f_1\chi_{4/3Q}\|_{L^{\widetilde{p}_1}(\mu)}\|f_2\chi_{4/3Q}\|_{L^{\widetilde{p}_2}(\mu)}\leqslant C\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(9/8,\mu)}.
\end{aligned}$$

Let

$$\Omega_{Q,j}:=\Big\{(y_1,y_2)\colon\ 2^{j-1}\tfrac{4}{3}l(Q)<|x-y_1|+|x-y_2|\leqslant 2^j\tfrac{4}{3}l(Q)\Big\}.$$

Notice that for $(y_1,y_2)\in(\mathbb{R}^d)^2\setminus\Omega_Q$ and $x,x'\in Q$, it is easy to see that

$$\begin{aligned}
&\Big|\frac{1}{(|x-y_1|+|x-y_2|)^{2n-\alpha}}-\frac{1}{(|x'-y_1|+|x'-y_2|)^{2n-\alpha}}\Big|\\
&\quad\leqslant\frac{C|x-x'|}{(|x-y_1|+\ldots+|x-y_m|)^{2n-\alpha+1}}\leqslant\frac{Cl(Q)}{(|x-y_1|+\ldots+|x-y_m|)^{2n-\alpha+1}}.
\end{aligned}$$

Then we have the following estimate for II_2:

$$\begin{aligned}
\mathrm{II}_2&\leqslant\frac{1}{\mu(\frac{3}{2}Q)}\frac{1}{\mu(Q)}\int_Q\int_Q\int_{(\mathbb{R}^d)^2\setminus\Omega_Q}\Big|\frac{f_1(y_1)f_2(y_2)}{(|x-y_1|+|x-y_2|)^{2n-\alpha}}\\
&\qquad-\int_{(\mathbb{R}^d)^2\setminus\Omega_Q}\frac{f_1(y_1)f_2(y_2)}{(|x'-y_1|+|x'-y_2|)^{2n-\alpha}}\Big|\,\mathrm{d}\mu(y_1)\,\mathrm{d}\mu(y_2)\,\mathrm{d}\mu(x')\,\mathrm{d}\mu(x)\\
&\leqslant\frac{Cl(Q)}{\mu(\frac{3}{2}Q)}\int_Q\int_{(\mathbb{R}^d)^2\setminus\Omega_Q}\frac{|f_1(y_1)||f_2(y_2)|\,\mathrm{d}\mu(y_1)\,\mathrm{d}\mu(y_2)}{(|x-y_1|+|x-y_2|)^{2n-\alpha+1}}\,\mathrm{d}\mu(x)
\end{aligned}$$

$$\leqslant \frac{Cl(Q)}{\mu(\frac{3}{2}Q)} \sum_{j=1}^{\infty} \int_Q \int_{\Omega_{Q,j}} \frac{|f_1(y_1)||f_2(y_2)| \, \mathrm{d}\mu(y_1) \, \mathrm{d}\mu(y_2)}{(|x-y_1|+|x-y_2|)^{2n-\alpha+1}} \, \mathrm{d}\mu(x)$$

$$\leqslant \sum_{j=1}^{\infty} \frac{Cl(Q)\mu(Q)}{\mu(\frac{3}{2}Q)(2^j \frac{4}{3} l(Q))^{2n-\alpha+1}} \int_{2^j 4/3Q} \int_{2^j 4/3Q} |f_1(y_1)||f_2(y_2)| \, \mathrm{d}\mu(y_1) \, \mathrm{d}\mu(y_2)$$

$$\leqslant C\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(9/8,\mu)}.$$

Thus, (3.3) holds. Next, we need to estimate the most right-hand side as follows. For any cube $Q \subset R$,

$$|C_R^\infty - C_Q^\infty| \leqslant CK_{Q,R}\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(9/8,\mu)}. \tag{3.4}$$

Similarly to II_2, we have

$$|C_R^\infty - C_Q^\infty| \leqslant \frac{C}{\mu(\frac{3}{2}Q)\mu(Q)}$$

$$\times \int_Q \int_Q \int_{((\mathbb{R}^d)^2\setminus\Omega_Q)\setminus((\mathbb{R}^d)^2\setminus\Omega_R)} \frac{|f_1(y_1)||f_2(y_2)||x-x'| \, \mathrm{d}\mu(y_1)\mu(\mathrm{d}y_2)}{(|x-y_1|+|x-y_2|)^{2n-\alpha+1}} \, \mathrm{d}\mu(x') \, \mathrm{d}\mu(x)$$

$$\leqslant C \sum_{j=1}^{N_{Q,R}} \frac{l(Q)}{\mu(\frac{3}{2}Q)} \int_Q \int_{\Omega_{Q,j}} \frac{|f_1(y_1)||f_2(y_2)| \, \mathrm{d}\mu(y_1)\mu(\mathrm{d}y_2)}{(|x-y_1|+|x-y_2|)^{2n-\alpha+1}} \, \mathrm{d}\mu(x)$$

$$\leqslant CK_{Q,R}\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(9/8,\mu)}.$$

It follows from (3.3), (3.4) and $K_{Q,\widetilde{Q}} \leqslant C$ that

$$\Big(\mu\Big(\frac{3}{2}Q\Big)\Big)^{-1} \int_Q |I_{\alpha,2}(\vec{f})(x) - m_{\widetilde{Q}}(I_{\alpha,2}(\vec{f}))| \, \mathrm{d}\mu(x)$$

$$\leqslant \Big(\mu\Big(\frac{3}{2}Q\Big)\Big)^{-1} \int_Q |I_{\alpha,2}(\vec{f})(x) - C_Q^\infty| \, \mathrm{d}\mu(x) + |C_Q^\infty - C_{\widetilde{Q}}^\infty|$$

$$+ \frac{1}{\mu(\widetilde{Q})} \int_{\widetilde{Q}} |I_{\alpha,2}(\vec{f})(x) - C_{\widetilde{Q}}^\infty| \, \mathrm{d}\mu(x) \leqslant C\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(9/8,\mu)}.$$

and for any doubling cube $Q \subset R$,

$$|m_Q(I_{\alpha,2}(\vec{f})) - m_R(I_{\alpha,2}(\vec{f}))| \leqslant |m_Q(I_{\alpha,2}(\vec{f})) - C_Q^\infty| + |C_Q^\infty - C_R^\infty|$$

$$+ |C_R^\infty - m_R(I_{\alpha,2}(\vec{f}))| \leqslant CK_{Q,R}\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(9/8,\mu)},$$

then

$$\|I_{\alpha,m}(\vec{f})\|_{\mathrm{RBMO}(\mu)} \leqslant C\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)}.$$

The theorem is thus proved. □

P r o o f of Theorem 3.4. Let $x \neq y$, $r = |x-y|$ and

$$\Omega := \{(y_1,\dots,y_m)\colon\ |x-y_1|+\dots+|x-y_m| \leqslant 2r\}.$$

Notice that, for $(y_1,\dots,y_m) \in (\mathbb{R}^n)^m \setminus \Omega$ and $x, y \in B(x,2r)$, it is easy to see that

$$\begin{aligned}
|I_\alpha(\vec{f})(x) - I_\alpha(\vec{f})(y)| &\leqslant \int_\Omega \frac{|f_1(y_1)|\dots|f_m(y_m)|\,\mathrm{d}\mu(\vec{y})}{(|x-y_1|+\dots+|x-y_m|)^{mn-\alpha}} \\
&\quad + \int_\Omega \frac{|f_1(y_1)|\dots|f_m(y_m)|\,\mathrm{d}\mu(\vec{y})}{(|y-y_1|+\dots+|y-y_m|)^{mn-\alpha}} \\
&\quad + C|x-y| \int_{(\mathbb{R}^n)^m\setminus\Omega} \frac{|f_1(y_1)|\dots|f_m(y_m)|\,\mathrm{d}\mu(\vec{y})}{(|x-y_1|+\dots+|x-y_m|)^{mn-\alpha+1}} \\
&=: \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3.
\end{aligned}$$

For III_1, since $0 < \alpha - n/p < 1$, there exists α_i $(i = 1,\dots,m)$ such that $\alpha_1 + \dots + \alpha_m = \alpha$ and $0 < \alpha_i - n/p_i < 1$; then

$$\begin{aligned}
\mathrm{III}_1 &\leqslant \prod_{i=1}^m \int_{B(x,2r)} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha_i}}\,\mathrm{d}\mu(y_i) \\
&\leqslant C\prod_{i=1}^m \Big(\int_{B(x,2r)} |f_i(y_i)|^{p_i}\,\mathrm{d}\mu(y_i)\Big)^{1/p_i} \Big(\int_{B(x,2r)} \frac{1}{|x-y_i|^{(n-\alpha)p_i'}}\,\mathrm{d}\mu(y_i)\Big)^{1/p_i'} \\
&\leqslant C\prod_{i=1}^m r^{\alpha_i-n}\mu(B(x,2r))^{1/p_i'} \Big(\int_{B(x,2r)} |f_i(y_i)|^{p_i}\,\mathrm{d}\mu(y_i)\Big)^{1/p_i} \\
&\leqslant Cr^{\alpha-n/p_0}\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)}.
\end{aligned}$$

From the fact that $B(x,2r) \subset B(y,3r)$, we obtain

$$\mathrm{III}_2 \leqslant Cr^{\alpha-n/p_0}\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)}.$$

Similarly to II_2 in Theorem 3.3, we have

$$\mathrm{III}_3 \leqslant Cr^{\alpha-n/p_0}\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)}.$$

Together with the estimate above, this yields that for any $x \neq y$,

$$|I_\alpha(\vec{f})(x) - I_\alpha(\vec{f})(y)| \leqslant C|x-y|^{\alpha-n/p_0}\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)}.$$

Thus, we obtain the desired result. $\square$

4. Boundedness of Multilinear Caldron-Zygmund operators

In this section we consider multilinear singular integral operators. Let μ and n be as above. Recall that the multilinear singular integral operator T is a bounded operator which satisfies

$$\|T(\vec{f})\|_{L^p} \leqslant C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}}$$

for some $1 < p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$, and comes with a kernel K with the following conditions.

(1) The function K satisfies the size condition

$$|K(y_0, y_1, \ldots, y_m)| \leqslant C \Big(\sum_{l=1}^{m} |y_0 - y_l| \Big)^{-mn}.$$

(2) The function K satisfies the regularity condition

$$|K(y_0, \ldots, y_i, \ldots, y_m) - K(y_0, \ldots, y_i', \ldots, y_m)| \leqslant \frac{C|y_i - y_i'|^{\varepsilon}}{\Big(\sum\limits_{l=1}^{m} |y_0 - y_l| \Big)^{-mn+\varepsilon}}$$

for some $\varepsilon > 0$ and all $i = 0, \ldots, m$, whenever $|y_i - y_i'| \leqslant \frac{1}{2} \max\limits_{0 \leqslant j \leqslant m} |y_i - y_j|$.

(3) If $x \notin \bigcap\limits_{i=1}^{m} \operatorname{supp}(f_i)$, then

$$T(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m) \, \mathrm{d}\mu(\vec{y}).$$

Theorem 4.1. *Let $1 < k, p_1, \ldots, p_m < \infty$, $\vec{P} = (p_1, \ldots, p_m)$ and $0 < p \leqslant p_0 < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$. Then*

$$\|T(\vec{f})\|_{M_p^{p_0}(k,\mu)} \leqslant C \|\vec{f}\|_{\mathcal{B}_{\vec{P}}^{p_0}(k,\mu)}.$$

Proof. Fix a cube $Q := Q(c_0, l(Q)) \in \mathcal{Q}(\mu)$. For $x \in Q$ and $(y_1, \ldots, y_m) \in (\mathbb{R}^n)^m \setminus (2Q \times \ldots \times 2Q)$, we have $|c_0 - y_1| + \ldots + |c_0 - y_m| > l(Q)$ and

$$|x - y_1| + \ldots + |x - y_m| \approx |c_0 - y_1| + \ldots + |c_0 - y_m|.$$

Therefore,

$$\begin{aligned} |T(\vec{f})(x)| &\leqslant |T(f_1\chi_{2Q}, \ldots, f_m\chi_{2Q})(x)| \\ &\quad + C \int_{|c_0 - y_1| + \ldots + |c_0 - y_m| > l(Q)} \frac{|f_1(y_1)| \ldots |f_m(y_m)| \, \mathrm{d}\mu(\vec{y})}{(|c_0 - y_1| + \ldots + |c_0 - y_m|)^{mn}}. \end{aligned}$$

First, since T is a bounded operator from $L^{p_1}(\mu)\times\ldots\times L^{p_m}(\mu)$ to $L^p(\mu)$,

$$\mu(2kQ)^{1/p_0}\biggl(\frac{1}{\mu(2kQ)}\int_Q|T(f_1\chi_{2Q},\ldots,f_m\chi_{2Q})(x)|^p\,\mathrm{d}\mu(x)\biggr)^{1/p}$$
$$\leqslant C\mu(2kQ)^{1/p_0}\prod_{i=1}^m\biggl(\frac{1}{\mu(2kQ)}\int_{2Q}|f_i(x)|^{p_i}\,\mathrm{d}\mu(x)\biggr)^{1/p_i}\leqslant C\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)}.$$

Second, let

$$\Omega_{Q,j}:=\{(y_1,\ldots,y_m)\colon\ 2^{j-1}l(Q)<|c_0-y_1|+\ldots+|c_0-y_m|\leqslant 2^jl(Q)\}.$$

Then $\Omega_{Q,j}\subset 2^{j+1}Q\times\ldots\times 2^{j+1}Q$ and

$$\bigcup_{j=1}^\infty\Omega_{Q,j}:=\{(y_1,\ldots,y_m)\colon\ l(Q)<|c_0-y_1|+\ldots+|c_0-y_m|\},$$

which yields

$$\mu(2kQ)^{1/p_0}\int_{|c_0-y_1|+\ldots+|c_0-y_m|>l(Q)}\frac{|f_1(y_1)|\ldots|f_m(y_m)|\,\mathrm{d}\mu(\vec{y})}{(|c_0-y_1|+\ldots+|c_0-y_m|)^{mn}}$$
$$\leqslant\mu(2kQ)^{1/p_0}\sum_{j=1}^\infty\int_{\Omega_{Q,j}}\frac{|f_1(y_1)|\ldots|f_m(y_m)|\,\mathrm{d}\mu(\vec{y})}{(|c_0-y_1|+\ldots+|c_0-y_m|)^{mn}}$$
$$\leqslant C\mu(2kQ)^{1/p_0}\sum_{j=1}^\infty\prod_{i=1}^m\frac{(k(2^{j+1}l(Q))^n)^{-1/p_0}}{(k(2^{j+1}l(Q))^n)^{1-1/p_0}}\int_{2^{j+1}Q}|f_i(y_i)|\,\mathrm{d}\mu(y)_i$$
$$\leqslant C\sum_{j=1}^\infty 2^{-n(j+1)/p_0}\frac{\mu(2kQ)^{1/p_0}}{(kl(Q))^{n/p_0}}$$
$$\times\prod_{i=1}^m\frac{1}{\mu(k2^{j+1}Q)^{1/p_i-1/p_0}}\biggl(\int_{2^{j+1}Q}|f_i(y_i)|^{p_i}\,\mathrm{d}\mu(y)_i\biggr)^{1/p_i}$$
$$\leqslant C\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)}.$$

This implies that

$$\|T(\vec{f})\|_{M^{p_0}_p(2k,\mu)}\leqslant C\|\vec{f}\|_{\mathcal{B}^{p_0}_{\vec{P}}(k,\mu)},$$

which yields the desired result. □

Acknowledgments. We would like to thank the anonymous referee for his/her comments.

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