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Dawood Hassanzadeh-Lelekaami; Hajar Roshan-Shekalgourabi
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ASSOCIATED PRIMES OF LOCAL COHOMOLOGY MODULES OF GENERALIZED LASKERIAN MODULES

Dawood Hassanzadeh-Lelekaami,
Hajar Roshan-Shekalgourabi, Arak

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Abstract. Let $\mathcal{I}$ be a set of ideals of a commutative Noetherian ring R. We use the notion of $\mathcal{I}$-closure operation which is a semiprime closure operation on submodules of modules to introduce the class of $\mathcal{I}$-Laskerian modules. This enables us to investigate the set of associated prime ideals of certain $\mathcal{I}$-closed submodules of local cohomology modules.

Keywords: associated prime ideals; Grothendieck spectral sequence; local cohomology module; semiprime closure operation

MSC 2010: 13D45, 13A15, 13E99

1. Introduction

Throughout the paper, R is a commutative Noetherian ring with identity and all modules are unitary. Also, $\mathfrak{a}$ is an ideal of R and $V(\mathfrak{a})$ is the set of all prime ideals of R containing $\mathfrak{a}$. Let M be an R-module. For each $i \geqslant 0$, the *ith local cohomology module of M with respect to* $\mathfrak{a}$ is defined as

$$H^i_{\mathfrak{a}}(M) = \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

For the basic properties of local cohomology the reader can refer to [2] of Brodmann and Sharp. An interesting problem in commutative algebra is determining when the set of associated prime ideals of the local cohomology module $H^i_{\mathfrak{a}}(M)$ is finite. Huneke in [7] raised the following conjecture: If M is a finitely generated R-module, then the set of associated primes of $H^i_{\mathfrak{a}}(M)$ is finite for every ideal $\mathfrak{a}$ and every $i \geqslant 0$. Singh in [12] and Katzman in [8] gave counterexamples to this conjecture. However,

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this problem has been studied by many authors and it was shown that this conjecture is true in many situations. In particular, it is shown in [9], Theorem B that if for a finitely generated R-module M and an integer s, the local cohomology modules $H^i_{\mathfrak{a}}(M)$ are finitely generated for all $i < s$, then the set $\mathrm{Ass}_R(H^s_{\mathfrak{a}}(M))$ is finite. A few months later, it was shown in [1], Theorem 2.2 that under this assumptions, $\mathrm{Ass}_R(H^s_{\mathfrak{a}}(M)/N)$ is finite for any finitely generated submodule N of $H^s_{\mathfrak{a}}(M)$.

Throughout the paper, $\mathcal{I}$ denotes an arbitrary set of ideals of R. The notion of $\mathcal{I}$-closure as a semiprime closure operation on submodules of modules was introduced by the first author in [6].

Definition 1.1. Let M be an R-module and P a proper submodule of M.

(1) For a set of ideals $\mathcal{I}$ of R, P is said to be $\mathcal{I}$-prime if $P = (P :_M I)$ for each $I \in \mathcal{I}$ (see [6], Definition 2.1).

(2) Let N be a submodule of M. Recall that by the *closure of* N *with respect to* $\mathcal{I}$ (or $\mathcal{I}$-*closure* of N), we mean the intersection of all $\mathcal{I}$-prime submodules of M containing N, and designate it by $\mathrm{Cl}_{\mathcal{I}}(N)$. Moreover, N is said to be $\mathcal{I}$-*closed* if $N = \mathrm{Cl}_{\mathcal{I}}(N)$ (see [6], Definition 3.1).

Example 1.2. Suppose $x_1, \ldots, x_n$ is an M-sequence, where M is an R-module, and $I_j = (x_1, \ldots, x_j)$ is the ideal of R generated by $x_1, \ldots, x_j$ for each $j \in \{1, \ldots, n-1\}$. If $\mathcal{I}$ is the set of all ideals of R containing $x_1, \ldots, x_n$, then $I_j M$ is an $\mathcal{I}$-prime submodule of M for each $j \in \{1, \ldots, n-1\}$.

According to Kirby [10], a nonempty set $\mathcal{I}$ of ideals of R is said to be *closed* if it has the following properties: (1) If $I \subseteq J$ and $I \in \mathcal{I}$, then $J \in \mathcal{I}$. (2) If $\{a_\lambda\}_{\lambda\in\Lambda}$ generate an ideal I of $\mathcal{I}$ and $\{I_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{I}$, then $\sum\limits_{\lambda\in\Lambda} a_\lambda I_\lambda \in \mathcal{I}$. For example, if $\mathfrak{p}$ is a prime ideal of R, then $\{I \trianglelefteq R;\ I \nsubseteq \mathfrak{p}\}$ is closed. The intersection of all closed sets of ideals which contain $\mathcal{I}$ is called the *closure of* $\mathcal{I}$ and is denoted by $\widetilde{\mathcal{I}}$ (for more details see [10], Definition 4).

Lemma 1.3. *If M is an R-module, then for each submodule N of M we have*

$$\mathrm{Cl}_{\mathcal{I}}(N) = \{m \in M\colon\ Jm \subseteq N \text{ for some } J \in \widetilde{\mathcal{I}}\}.$$

P r o o f. See [6], Theorem 3.8. □

Lemma 1.4. *Let R be a commutative ring with identity and $\mathcal{I}$ any set of ideals of R. Then for any submodule N of M,*

(1) *if $J \in \mathcal{I}$ for some ideal $J \subseteq \mathrm{Ann}(M)$, then $\mathrm{Cl}_{\mathcal{I}}(N) = M$, and*

(2) *if $\mathcal{I} = \{R\}$, then $\mathrm{Cl}_{\mathcal{I}}(N) = N$.*

P r o o f. (1) It is a direct consequence of Lemma 1.3. (2) By definition, $\mathcal{I} = \{R\}$ is a closed set of ideals of R. Thus, Lemma 1.3 implies that $\mathrm{Cl}_{\mathcal{I}}(N) = N$. □

Roughly speaking, if $\mathcal{I}$ is big enough, then $\mathrm{Cl}_{\mathcal{I}}(-)$ is the biggest possible closure operation, whereas if $\mathcal{I}$ is small enough, then $\mathrm{Cl}_{\mathcal{I}}(-)$ is the smallest possible closure operation. And then there is a lot of room in between.

In this paper, we first use the notion of $\mathcal{I}$-closure operation to introduce the class of $\mathcal{I}$-Laskerian modules. This class includes all Noetherian modules and also all Artinian modules. Moreover, this class is large enough to contain all weakly Laskerian modules as well as all Laskerian modules. Notice that if we are able to prove a statement about the submodule $\mathrm{Cl}_{\mathcal{I}}(0)$ of M where $\mathcal{I}$ is arbitrary, then by an appropriate choice of $\mathcal{I}$ we can prove that statement for M. For example, it follows from Lemma 1.3 that it is enough for us to take $\mathcal{I}$ such that $\mathrm{Ann}(M) \in \mathcal{I}$. This approach provides a common generalization of some previous results. For instance, if $\mathcal{I}$ is arbitrary and we investigate associated prime ideals of $\mathrm{Cl}_{\mathcal{I}}(0_M)$, then certainly we obtain some results on $\mathrm{Ass}(M)$. As the main result of this paper, we show that if M is an $\mathcal{I}$-Laskerian R-module and t is a natural integer, then there is a finite subset X of $\mathrm{Spec}(R)$ such that the set of associated primes of the $\mathcal{I}$-closure of the zero submodule of $H^t_{\mathfrak{a}}(M)$ is contained in X united with the union of the sets of associated primes of the $\mathcal{I}$-closures of the zero submodule of $\mathrm{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$, where $0 \leqslant i < t$ and $0 \leqslant j \leqslant t^2+1$. This result therefore generalizes the main results of [1] and [3].

2. The main result

Definition 2.1. An R-module M is said to be $\mathcal{I}$-*Laskerian* if the set of associated primes of $\mathrm{Cl}_{\mathcal{I}}(N)/N$ is finite for any submodule N of M.

Example 2.2. An R-module M is said to be *Laskerian* if any submodule of M is an intersection of a finite number of primary submodules. Obviously, any Noetherian module is Laskerian. Also, an R-module M is said to be *weakly Laskerian* if the set of associated primes of any quotient module of M is finite (see [3]). Any Laskerian module is weakly Laskerian. Obviously, $\mathrm{Ass}(\mathrm{Cl}_{\mathcal{I}}(N)/N) \subseteq \mathrm{Ass}(M/N)$ for any submodule N of M. Thus, an $\mathcal{I}$-Laskerian module is a generalization of a weakly Laskerian module. In particular, any Noetherian module is $\mathcal{I}$-Laskerian for all $\mathcal{I}$.

In the sequel, the notation $\Gamma_{\mathcal{I}}(M)$ represents the $\mathcal{I}$-closure of the zero submodule of M, i.e., $\Gamma_{\mathcal{I}}(M) = \mathrm{Cl}_{\mathcal{I}}(0_M)$. According to [6], Lemma 3.10, case (3) we have $\mathrm{Cl}_{\mathcal{I}}(N)/N = \mathrm{Cl}_{\mathcal{I}}(0_{M/N}) = \Gamma_{\mathcal{I}}(M/N)$. Thus, M is $\mathcal{I}$-Laskerian if $\mathrm{Ass}_R(\Gamma_{\mathcal{I}}(M/N))$ is finite. For the purpose of applications throughout the paper, in the following lemma we list some basic properties of $\Gamma_{\mathcal{I}}(-)$.

Lemma 2.3. *Let $\mathcal{C}(R)$ be the category of all R-modules and R-homomorphisms.*

(1) *$\Gamma_{\mathcal{I}}(-)$ is a covariant, R-linear and left exact functor from $\mathcal{C}(R)$ to itself.*

(2) *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Then M is $\mathcal{I}$-Laskerian if and only if both M' and M'' are $\mathcal{I}$-Laskerian. Thus any subquotient of an $\mathcal{I}$-Laskerian module, as well as any finite direct sum of $\mathcal{I}$-Laskerian modules, is $\mathcal{I}$-Laskerian.*

(3) *Let M and N be two R-modules. If M is $\mathcal{I}$-Laskerian and N is finitely generated, then $\operatorname{Ext}^i_R(N,M)$ and $\operatorname{Tor}^R_i(N,M)$ are $\mathcal{I}$-Laskerian for all $i \geqslant 0$.*

(4) *Let M be an R-module such that $\operatorname{Supp}_R(M)$ is finite. Then M is $\mathcal{I}$-Laskerian. In particular, any Artinian R-module is $\mathcal{I}$-Laskerian.*

P r o o f. (1) For each R-module M, $\Gamma_{\mathcal{I}}(M) = \operatorname{Cl}_{\mathcal{I}}(0_M)$ is a submodule of M and for a homomorphism $f\colon M \to N$ of R-modules we have $f(\Gamma_{\mathcal{I}}(M)) \subseteq \Gamma_{\mathcal{I}}(N)$, and so there is a mapping $\Gamma_{\mathcal{I}}(f)\colon \Gamma_{\mathcal{I}}(M) \to \Gamma_{\mathcal{I}}(N)$ which agrees with f on each element of $\Gamma_{\mathcal{I}}(M)$. It is clear that, if $g\colon M \to N$ and $h\colon N \to L$ are further homomorphisms of R-modules and $r \in R$, then $\Gamma_{\mathcal{I}}(h \circ f) = \Gamma_{\mathcal{I}}(h) \circ \Gamma_{\mathcal{I}}(f)$, $\Gamma_{\mathcal{I}}(f+g) = \Gamma_{\mathcal{I}}(f) + \Gamma_{\mathcal{I}}(g)$, $\Gamma_{\mathcal{I}}(rf) = r\Gamma_{\mathcal{I}}(f)$ and $\Gamma_{\mathcal{I}}(\mathrm{Id}_M) = \mathrm{Id}_{\Gamma_{\mathcal{I}}(M)}$. Thus, with these assignments, $\Gamma_{\mathcal{I}}(-)$ becomes a covariant, R-linear functor from $\mathcal{C}(R)$ to itself.

Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be a short exact sequence of R-modules and R-homomorphisms. We must show that

$$0 \longrightarrow \Gamma_{\mathcal{I}}(L) \xrightarrow{\Gamma_{\mathcal{I}}(f)} \Gamma_{\mathcal{I}}(M) \xrightarrow{\Gamma_{\mathcal{I}}(g)} \Gamma_{\mathcal{I}}(N)$$

is still exact. It is clear that $\Gamma_{\mathcal{I}}(f)$ is a monomorphism and it follows immediately from the last paragraph that $\Gamma_{\mathcal{I}}(g) \circ \Gamma_{\mathcal{I}}(f) = 0$, so that $\operatorname{Im}(\Gamma_{\mathcal{I}}(f)) \subseteq \operatorname{Ker}(\Gamma_{\mathcal{I}}(g))$. To prove the reverse inclusion, let $m \in \operatorname{Ker}(\Gamma_{\mathcal{I}}(g))$. Thus $m \in \Gamma_{\mathcal{I}}(M)$ is such that $g(m) = 0$ and $Im = 0$ for some $I \in \widetilde{\mathcal{I}}$ by Lemma 1.3. Now there exists $l \in L$ such that $f(l) = m$. For each $r \in I$, we have $f(rl) = rf(l) = rm = 0$, so that $rl = 0$ because f is a monomorphism. Hence $Il = 0$. By Lemma 1.3, this implies that $l \in \Gamma_{\mathcal{I}}(L)$, as required.

(2) We may assume that M' is a submodule of M and $M'' = M/M'$. If M is $\mathcal{I}$-Laskerian, it is easy to see that M' and M'' are $\mathcal{I}$-Laskerian. Now, suppose that M' and M/M' are $\mathcal{I}$-Laskerian. Let N be an arbitrary submodule of M. Then by part (1), the exact sequence

$$0 \to \frac{M'+N}{N} \to \frac{M}{N} \to \frac{M}{M'+N} \to 0$$

induces the exact sequence

$$0 \to \Gamma_{\mathcal{I}}\Big(\frac{M'}{M' \cap N}\Big) \to \Gamma_{\mathcal{I}}\Big(\frac{M}{N}\Big) \to \Gamma_{\mathcal{I}}\Big(\frac{M}{M'+N}\Big).$$

Now, since

$$\mathrm{Ass}_R\,\Gamma_{\mathcal{I}}(M/N) \subseteq \mathrm{Ass}_R\,\Gamma_{\mathcal{I}}(M'/M'\cap N) \cup \mathrm{Ass}_R\,\Gamma_{\mathcal{I}}(M/M'+N)$$

and the sets $\mathrm{Ass}_R\,\Gamma_{\mathcal{I}}(M'/M'\cap N)$ and $\mathrm{Ass}_R\,\Gamma_{\mathcal{I}}(M/M'+N)$ are finite, it follows that the set $\mathrm{Ass}_R\,\Gamma_{\mathcal{I}}(M/N)$ is finite, and so M is $\mathcal{I}$-Laskerian.

(3) We only prove the assertion for Ext modules, as the proof for Tor modules is similar. Because R is a Noetherian ring and N is finitely generated, it follows that N possesses a free resolution

$$F_\bullet\colon\ \ldots F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0\ ,$$

consisting of finitely generated free modules. If $F_i = \bigoplus^{t} R$ for some integer t, then $\mathrm{Ext}^i_R(N,M) = H^i(\mathrm{Hom}_R(F_\bullet, M))$ is a subquotient of $\bigoplus^{t} M$. Therefore, it follows from part (2) that $\mathrm{Ext}^i_R(N,M)$ is $\mathcal{I}$-Laskerian for all $i \geqslant 0$.

(4) It is well-known that for every submodule N of M we have

$$\mathrm{Ass}(\mathrm{Cl}_{\mathcal{I}}(N)/N) \subseteq \mathrm{Supp}(M/N) \subseteq \mathrm{Supp}(M).$$

□

Example 2.4. An R-module M is said to be *minimax* if M has a finitely generated submodule N such that M/N is Artinian (see [13]). By Lemma 2.3, it follows that any minimax R-module is $\mathcal{I}$-Laskerian for every $\mathcal{I}$.

Let $\mathfrak{a}$ be an ideal of R. Then, by [11], Theorem 10.47, there is a Grothendieck spectral sequence with $E_2^{p,q} := \mathrm{Ext}^p_R(R/\mathfrak{a}, H^q_{\mathfrak{a}}(M)) \underset{p}{\Longrightarrow} \mathrm{Ext}^{p+q}_R(R/\mathfrak{a}, M)$. Let $E_\infty := \{E_\infty^{p,q}\}$ be the limit term of this spectral sequence. In the sequel, we show that if M is $\mathcal{I}$-Laskerian, then $E_\infty^{p,q}$ is $\mathcal{I}$-Laskerian for all p, q with $0 \leqslant p \leqslant q$.

Lemma 2.5. *Let $\mathfrak{a}$ be an ideal of R. If M is an $\mathcal{I}$-Laskerian module, then $E_\infty^{p,q}$ is $\mathcal{I}$-Laskerian for all p, q with $0 \leqslant p \leqslant q$. In particular, $\mathrm{Ass}(\Gamma_{\mathcal{I}}(E_\infty^{p,q}))$ is finite for all p, q with $0 \leqslant p \leqslant q$.*

Proof. By [11], Theorem 10.47, the Grothendieck spectral sequence $E_2^{p,q} := \mathrm{Ext}^p_R(R/\mathfrak{a}, H^q_{\mathfrak{a}}(M))$ converges to $H^{p+q} := \mathrm{Ext}^{p+q}_R(R/\mathfrak{a}, M)$, hence, there is a finite filtration

$$0 = \varphi^{q+1}H^q \subseteq \varphi^q H^q \subseteq \ldots \subseteq \varphi^1 H^q \subseteq \varphi^0 H^q = H^q$$

of H^q such that $E_\infty^{p,q} \cong \varphi^p H^q/\varphi^{p+1}H^q$ for all $p = 0, 1, \ldots, q$. Since M is $\mathcal{I}$-Laskerian, by Lemma 2.3, it turns out that $\mathrm{Ext}^q_R(R/\mathfrak{a}, M)$ is also $\mathcal{I}$-Laskerian. Hence, any subquotient of H^q is $\mathcal{I}$-Laskerian, as desired. □

Theorem 2.6. *Let $\mathfrak{a}$ be an ideal of R. Let M be an $\mathcal{I}$-Laskerian R-module and let t be a natural integer. Then there exists a finite subset X of* $\operatorname{Spec}(R)$ *such that*

$$\operatorname{Ass}_R(\Gamma_{\mathcal{I}}(\operatorname{Ext}^l_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)))) \subseteq \bigcup_{\substack{0\leqslant i<t\\ 0\leqslant j\leqslant t^2+1}} \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M)))) \cup X$$

for $l = 0, 1$.

P r o o f. Consider the Grothendieck spectral sequence

$$E_2^{p,q} := \operatorname{Ext}^p_R(R/\mathfrak{a}, H^q_{\mathfrak{a}}(M)) \underset{p}{\Longrightarrow} \operatorname{Ext}^{p+q}_R(R/\mathfrak{a}, M).$$

We define the subset X of $\operatorname{Spec}(R)$ as

$$X := \bigcup_{0\leqslant j\leqslant t^2+1} \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(\operatorname{Ext}^j_R(R/\mathfrak{a}, H^0_{\mathfrak{a}}(M)))) \cup \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_\infty^{0,t})) \cup \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_\infty^{1,t})).$$

By Lemmas 2.3 and 2.5, X is finite. First, we consider the case $l = 0$. So, we must prove that

$$\operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_2^{0,t})) \subseteq \bigcup_{\substack{0\leqslant i<t\\ 0\leqslant j\leqslant t^2+1}} \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_2^{j,i})) \cup X.$$

From the choice of X, it is clear that we may assume M is $\mathfrak{a}$-torsion-free. By Lemma 2.3, the exact sequence

$$0 \longrightarrow \operatorname{Ker}(d_i^{0,t}) \longrightarrow E_i^{0,t} \xrightarrow{d_i^{0,t}} E_i^{i,t-i+1}$$

induces the exact sequence

$$0 \longrightarrow \Gamma_{\mathcal{I}}(\operatorname{Ker}(d_i^{0,t})) \longrightarrow \Gamma_{\mathcal{I}}(E_i^{0,t}) \xrightarrow{\Gamma_{\mathcal{I}}(d_i^{0,t})} \Gamma_{\mathcal{I}}(E_i^{i,t-i+1}) .$$

Thus, we infer that

$$\operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_i^{0,t})) \subseteq \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(\operatorname{Ker}(d_i^{0,t}))) \cup \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_i^{i,t-i+1}))$$

for all $i \geqslant 2$. Since $E_i^{-i,t+i-1} = 0$, $\operatorname{Ker}(d_i^{0,t}) = E_{i+1}^{0,t}$. Hence

$$\operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_i^{0,t})) \subseteq \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_{i+1}^{0,t})) \cup \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_i^{i,t-i+1})) \tag{2.1}$$

for all $i \geqslant 2$. Let $n > t$ be an integer, and consider the sequence

$$E_n^{-n,t+n-1} \xrightarrow{d_n^{-n,t+n-1}} E_n^{0,t} \xrightarrow{d_n^{0,t}} E_n^{n,t-n+1} .$$

Since M is $\mathfrak{a}$-torsion-free, $E_n^{n,0} = 0$. Note that for each $i \geqslant 2$, the module $E_i^{p,q}$ is a subquotient of $E_2^{p,q}$. Also, $E_n^{i,j} = 0$ if either $i < 0$ or $j < 0$. Thus, we have $\operatorname{Ker}(d_n^{0,t}) = E_n^{0,t}$ and $\operatorname{Im}(d_n^{-n,t+n-1}) = 0$, and so

$$E_{n+1}^{0,t} = \operatorname{Ker}(d_n^{0,t})/\operatorname{Im}(d^{-n,t+n-1}) \cong E_n^{0,t}. \tag{2.2}$$

Using (2.2) successively for all $n > t$, we get $E_{t+1}^{0,t} \cong E_{t+2}^{0,t} \cong \ldots = E_\infty^{0,t}$. Now, by iterating (2.1) for all $i = 2, \ldots, t$, we deduce that

$$\operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_2^{0,t})) \subseteq \bigcup_{i=2}^{t} \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_i^{i,t-i+1})) \cup X. \tag{2.3}$$

It is enough (for the case $l = 0$) to show that

$$\operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_i^{i,t-i+1})) \subseteq \bigcup_{k=1}^{t} \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_2^{ki,t-ki+k}))$$

for all $i = 3, \ldots, t$. By Lemma 2.3, the exact sequence

$$0 \longrightarrow \operatorname{Ker}(d_i^{ki,t-ki+k}) \longrightarrow E_i^{ki,t-ki+k} \xrightarrow{d_i^{ki,t-ki+k}} E_i^{(k+1)i,t-(k+1)i+k+1}$$

induces the exact sequence

$$0 \longrightarrow \Gamma_{\mathcal{I}}(\operatorname{Ker}(d_i^{ki,t-ki+k})) \longrightarrow \Gamma_{\mathcal{I}}(E_i^{ki,t-ki+k}) \xrightarrow{\Gamma_{\mathcal{I}}(d_i^{ki,t-ki+k})} \Gamma_{\mathcal{I}}(E_i^{(k+1)i,t-(k+1)i+k+1}).$$

Since

$$\Gamma_{\mathcal{I}}(\operatorname{Ker}(d_i^{ki,t-ki+k})) \subseteq \Gamma_{\mathcal{I}}(\operatorname{Ker}(d_2^{ki,t-ki+k})) \subseteq \Gamma_{\mathcal{I}}(E_2^{ki,t-ki+k})$$

and $E_i^{p,q} = 0$ for all $q \leqslant 0$, by using the above exact sequence successively for $k = 1, 2, \ldots, t$, we deduce that

$$\operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_i^{i,t-i+1})) \subseteq \bigcup_{k=1}^{t} \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_2^{ki,t-ki+k})). \tag{2.4}$$

Therefore, it follows from (2.3) and (2.4) that

$$\operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_2^{0,t})) \subseteq \bigcup_{\substack{0\leqslant i<t \\ 0\leqslant j\leqslant t^2+1}} \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_2^{j,i})) \cup X.$$

By an argument similar to case $l = 0$ we can show that

$$\operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_2^{1,t})) \subseteq \bigcup_{\substack{0\leqslant i<t \\ 0\leqslant j\leqslant t^2+1}} \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(E_2^{j,i})) \cup X.$$

This completes the proof. □

In [5], Hartshorne defines an R-module M to be $\mathfrak{a}$-*cofinite* if $\operatorname{Supp}_R(M) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all $i \geqslant 0$. He asks when the local cohomology modules of a finitely generated module are $\mathfrak{a}$-cofinite. An R-module M is said to be $\mathfrak{a}$-*weakly cofinite* if $\operatorname{Supp}_R(M) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all $i \geqslant 0$ (see [4]). If M is weakly Laskerian, then M is $\mathfrak{a}$-weakly cofinite. In particular, if either M is finitely generated or Artinian, then M is $\mathfrak{a}$-weakly cofinite. Every $\mathfrak{a}$-cofinite module is $\mathfrak{a}$-weakly cofinite. As a direct consequence of Theorem 2.6 and Lemma 1.4, case (1), we obtain the following extension of [3], Corollary 2.6 and [1], Theorem 2.2.

Corollary 2.7. *Let $\mathfrak{a}$ be an ideal of R and M an $\mathcal{I}$-Laskerian module. Let $t \in \mathbb{N}_0$ be an integer such that $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite for all $i < t$. Then, the sets of associated primes of $\Gamma_{\mathcal{I}}(H^t_{\mathfrak{a}}(M))$ and of $\Gamma_{\mathcal{I}}(\operatorname{Ext}^1_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)))$ are finite.*

Proof. Let $t = 0$. Then $H^0_{\mathfrak{a}}(M)$ is a submodule of the $\mathcal{I}$-Laskerian module M. Thus, the assertion follows from Lemma 2.3. Now assume that $t > 0$, and let $i < t$ be an integer. Since $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite, for any $j \geqslant 0$ we have that $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is weakly Laskerian, and so $\operatorname{Ass}_R(\Gamma_{\mathcal{I}}(\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))))$ is finite. We infer from $\operatorname{Supp}_R(H^t_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{a})$ that

$$\begin{aligned} \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)))) &= \operatorname{Ass}_R(\operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathcal{I}}(H^t_{\mathfrak{a}}(M)))) \\ &= \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(H^t_{\mathfrak{a}}(M))) \cap V(\mathfrak{a}) = \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(H^t_{\mathfrak{a}}(M))). \end{aligned} \tag{2.5}$$

Consequently, the result follows from Theorem 2.6. □

As we mentioned, Corollary 2.7 is a generalization of [3], Corollary 2.6 and [1], Theorem 2.2, since it clearly implies these results by Lemma 1.4, case (1) when one chooses an $\mathcal{I}$ with $\operatorname{Ann}(M) \in \mathcal{I}$ (e.g. $\mathcal{I}$ = all ideals). Also, we point out that according to Lemma 1.4, {all ideals}-Laskerian is the same as weakly Laskerian, whereas the $\{R\}$-Laskerian property is trivially satisfied by all R-modules.

Let M be a finitely generated R-module. It is shown in [9], Theorem B, case (β) that if t is an integer such that $\operatorname{Supp}_R(H^i_{\mathfrak{a}}(M))$ is finite for all $i < t$, then the set of associated primes of $H^t_{\mathfrak{a}}(M)$ is finite. By Lemma 2.3, case (4) any R-module with finite support is $\mathcal{I}$-Laskerian. Hence, the next corollary generalizes [9], Theorem B, case (β).

Corollary 2.8. *Let $\mathfrak{a}$ be an ideal of R and M an $\mathcal{I}$-Laskerian module. Let $t \in \mathbb{N}_0$ be an integer such that $H^i_{\mathfrak{a}}(M)$ is an $\mathcal{I}$-Laskerian module for all $i < t$. Then, the sets of associated primes of $\Gamma_{\mathcal{I}}(H^t_{\mathfrak{a}}(M))$ and of $\Gamma_{\mathcal{I}}(\operatorname{Ext}^1_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)))$ are finite.*

P r o o f. It follows from Lemma 2.3, case (3) and Theorem 2.6 that the sets of associated primes of $\Gamma_{\mathcal{I}}(\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)))$ and $\Gamma_{\mathcal{I}}(\operatorname{Ext}^1_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)))$ are finite. Moreover, in the light of Equation (2.5) we have

$$\operatorname{Ass}_R(\Gamma_{\mathcal{I}}(H^t_{\mathfrak{a}}(M))) = \operatorname{Ass}_R(\Gamma_{\mathcal{I}}(\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)))).$$

This completes the proof. □

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Authors' address: D a w o o d H a s s a n z a d e h - L e l e k a a m i (corresponding author), H a j a r R o s h a n - S h e k a l g o u r a b i, Department of Basic Sciences, Arak University of Technology, P. O. Box 38135-1177, Arak, Iran, e-mail: `lelekaami@gmail.com`, `Dhmath@arakut.ac.ir`; `Hrsmath@gmail.com`, `roshan@arakut.ac.ir`.