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HYPERBOLIC INVERSE MEAN CURVATURE FLOW

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Abstract. We prove the short-time existence of the hyperbolic inverse (mean) curvature flow (with or without the specified forcing term) under the assumption that the initial compact smooth hypersurface of $\mathbb{R}^{n+1}$ ($n \geqslant 2$) is mean convex and star-shaped. Several interesting examples and some hyperbolic evolution equations for geometric quantities of the evolving hypersurfaces are shown. Besides, under different assumptions for the initial velocity, we can get the expansion and the convergence results of a hyperbolic inverse mean curvature flow in the plane $\mathbb{R}^2$, whose evolving curves move normally.

Keywords: evolution equation; hyperbolic inverse mean curvature flow; short time existence

MSC 2010: 58J45, 58J47

1. Introduction

The study of curvature flows has been a hot topic in the research of differential geometry in the past several decades. It is well known that Perelman used the Ricci flow, an intrinsic curvature flow, to successfully solve the 3-dimensional Poincaré conjecture. Among extrinsic curvature flows, an important one is the mean curvature flow (MCF for short), which means a submanifold of a prescribed ambient space moves with a velocity equal to its mean curvature vector. A classical result in the study of MCF due to Huisken (see [11]) says that for a strictly convex, compact hypersurface immersed in $\mathbb{R}^{n+1}$ ($n \geqslant 2$), if it evolves along the MCF, then the evolving hypersurfaces contract to a single point at some finite time, and moreover, after area-preserving rescaling, the rescaled evolving hypersurfaces converge to a round sphere in the C^∞-topology as time tends to infinity. Many improvements have been

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obtained after this classical result. Besides, the theory of MCF also has some interesting applications. For instance, Topping in [21] used curve shortening flow on surfaces, which is the lower dimensional version of MCF, to get isoperimetric inequalities on surfaces. The theory of curve shortening flow can also be used to do the image processing (see, e.g. [4]). The MCF is called the *inward flow*, and conversely, the inverse mean curvature flow (IMCF for short), which means a submanifold of a prescribed ambient space moves in direction of the outward unit normal vector of the submanifold with a speed equal to $1/H$ ($H \neq 0$ denotes the mean curvature), is called the *outward flow*. The IMCF is also a very important extrinsic flow, which has many interesting and important applications. For instance, the evolution of non-star-shaped initial surfaces under the IMCF may have singularities in finite time, but, through defining a notion of weak solution to IMCF equation, Huisken-Ilmanen in [12] proved the Riemannian Penrose inequality by using the method of IMCF (the Riemannian Penrose inequality can also be proved by applying the positive mass theorem, see [1] for details). Using the method of IMCF, Brendle, Hung and Wang in [2] proved a sharp Minkowski inequality for mean convex and star-shaped hypersurfaces in the n-dimensional ($n \geqslant 3$) anti-de Sitter-Schwarzschild manifold, which generalized the related conclusions in the Euclidean space $\mathbb{R}^n$.

The corresponding author Mao has been working on IMCF for several years and has also obtained some interesting results with his collaborators. For instance, Chen and Mao in [5] considered the evolution of a smooth, star-shaped and F-admissible (i.e., F is a 1-homogeneous function of principal curvatures satisfying some suitable conditions) embedded closed hypersurface in the n-dimensional ($n \geqslant 3$) anti-de Sitter-Schwarzschild manifold along its outward normal direction with a speed equal to $1/F$ (clearly, this evolution process is a natural generalization of IMCF, and we call it the *inverse curvature flow*, we write ICF for short). They proved that this ICF exists for all the time and, after rescaling, the evolving hypersurfaces converge to a sphere as time tends to infinity. For warped products $I \times_{\lambda(r)} N^n$, where I is an unbounded connected interval of $\mathbb{R}$ (i.e., the set of real numbers) and N^n is a Riemannian manifold of nonnegative Ricci curvature, under suitable growth assumptions on the warping function $\lambda(r)$, Chen, Mao, Xiang and Xu [6] successfully proved that if an n-dimensional ($n \geqslant 2$) compact $C^{2,\alpha}$-hypersurface with boundary, which meets a given cone in $I \times_{\lambda(r)} N^n$ perpendicularly and is star-shaped with respect to the center of the cone, evolves along the IMCF, then the flow exists for all the time and, after rescaling, the evolving hypersurfaces converge to a piece of the geodesic sphere as time tends to infinity, which generalized the main conclusion in [15]. In fact, except these interesting improvements for IMCF or more general ICF obtained by Mao and his collaborators, recently there are several interesting conclusions also on this topic, which we would like to mention. The IMCF has been investigated deeply in warped cylinders of nonpositive radial curvature by Scheuer (see [19]) and in warped products (with suitable growth assumptions on the warping function) by Zhou (see [23]). The IMCF in complex hyperbolic spaces or quaternionic hyperbolic spaces, whose geometry is richer than that of warped products, has been initially studied by Pipoli (see [16], [17]), and some interesting results have been shown.

We know that the MCF and the IMCF describe the motion of a prescribed submanifold, that is, the velocity d/dt equals some scalar multiple of the unit normal vector of the submanifold. If the velocity d/dt is replaced by the acceleration $\mathrm{d}^2/\mathrm{d}t^2$, what happens? Yau in [22] suggested the curvature flow

$$\frac{\mathrm{d}^2 X}{\mathrm{d}t^2} = H\vec{n}, \tag{1.1}$$

where, as before, H denotes the mean curvature and $\vec{n}$ is the unit inner normal vector of the initial hypersurface $X(\cdot,0)$, and pointed out very little about the global time behavior of the evolving hypersurfaces. The curvature flow (1.1) can be seen as the hyperbolic version of MCF, and that is the reason why it is called the *hyperbolic mean curvature flow* (HMCF for short). In fact, if $\mathcal{M}$ is an n-dimensional ($n \geqslant 2$) smooth compact Riemannnian manifold and $X(\cdot,t)$ is a one-parameter family of smooth hypersurface immersions in $\mathbb{R}^{n+1}$ satisfying (1.1), where $X(\cdot,0)$ is the hypersurface immersion of $\mathcal{M}$ into $\mathbb{R}^{n+1}$, then it is not hard to show that (1.1) is a second-order hyperbolic PDE, which is used to get the short time existence of the flow (see [9], Section 2 for details). Mao in [14] considered a hyperbolic curvature flow whose form is given by (1.1) plus a forcing term in the direction of the position vector, that is,

$$\frac{\partial^2 X}{\partial t^2} = H\vec{n} + c(t)X$$

with $c(t)$ a bounded continuous function w.r.t. the time variable t only, and successfully improved most conclusions in [9] under suitable assumptions.

Based on our research experience on the ICF and the HMCF, it is natural to consider the hyperbolic version of the IMCF.

Let M_0 be a compact, mean convex, star-shaped smooth hypersurface of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ ($n \geqslant 2$), which is given as an embedding

$$X_0\colon\ \mathbb{S}^n \to \mathbb{R}^{n+1},$$

where $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ denotes the unit sphere in $\mathbb{R}^{n+1}$. Define a one-parameter family of smooth hypersurfaces embedded in $\mathbb{R}^{n+1}$ given by

$$X(\cdot,t)\colon\ \mathbb{S}^n \to \mathbb{R}^{n+1}$$

with $X(\cdot,0) = X_0(\cdot)$. We say that it is a solution of the *hyperbolic inverse mean curvature flow* (HIMCF for short) if $X(\cdot,t)$ satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}X(x,t) = H^{-1}(x,t)\vec{\nu}(x,t), \quad \forall\, x \in \mathbb{S}^n,\ t > 0, \tag{1.2}$$

where $H(x,t)$ is the mean curvature of $X(x,t)$, $\vec{\nu}(x,t)$ is the unit outward normal vector on $X(x,t)$. If $X(\cdot,0) = X_0$, $\mathrm{d}X/\mathrm{d}t(\cdot,0) = X_1(x)$ with $X_1(x)$ a smooth vector-valued function on $\mathbb{S}^n$, then one can get the existence of the one-parameter family of smooth hypersurfaces $X(\cdot,t)$ embedded in $\mathbb{R}^{n+1}$ on the time interval $[0,T)$ with $T < \infty$ (see Theorem 2.1 for the precise statement). Besides, under different assumptions for the initial velocity, we separately discuss the expansion and the convergence of a HIMCF in the plane $\mathbb{R}^2$, *whose evolving curves move normally*, in the last section (see Theorem 5.9 for the precise statement).

Remark 1.1. As mentioned before, some interesting conclusions about IMCF or ICF can be generalized from the setting that the ambient space is the Euclidean space to the setting of warped products (see, e.g. [5], [6]). Hence, one might ask the following question:

▷ If we consider the HIMCF or the HICF (see Remark 2.2 (2) below for this notion) in the warped product $I \times_{\lambda(r)} N^n$ with I an unbounded connected interval of $\mathbb{R}$ and N^n a Riemannian manifold of nonnegative Ricci curvature, could we get results similar to this paper under some suitable assumptions on $\lambda(r)$?

2. Local existence and uniqueness

Denote by M_t the evolving hypersurface under the flow (1.2). Since M_0 is star-shaped, M_t should also be star-shaped on $[0,\varepsilon)$ for some sufficiently small $\varepsilon > 0$ by continuity. Let the surface M_t be represented as a graph over $\mathbb{S}^n$, i.e., the embedding vector $x = (x^\alpha)$ now has the components

$$x^{n+1} = u(x,t), \quad x^i = x^i(t),$$

with (x^i) local coordinates of $\mathbb{S}^n$. Furthermore, let $\xi = (\xi^i)$ be a local coordinate system of M_t, which implies the graphic function u can be written as $u = u(x(\xi),t)$. Clearly, the outward unit normal vector in (x,u) has the form

$$\vec{\nu} = v^{-1}(-D_i u, 1),$$

where

$$D_i u = \frac{\partial u}{\partial x^i} := u_i,$$
$$\upsilon = (1 + u^{-2}|Du|^2)^{1/2} = (1 + u^{-2}\sigma^{ij} D_i u D_j u)^{1/2},$$

and where (σ_{ij}) being the metric of $\mathbb{S}^n$ in the coordinates (x^i) and naturally (σ^{ij}) being its inverse. Therefore, now, the Euclidean metric can be written as

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \sigma_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j.$$

Then the evolution equation (1.2) now yields

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = \frac{1}{H\upsilon}, \quad \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\frac{D^i u \cdot u^{-2}}{H\upsilon}. \tag{2.1}$$

On the other hand, by the chain rule, we have

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial u}{\partial t},$$

and
$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = \Big(\frac{\partial^2 u}{\partial x^i \partial x^j}\frac{\mathrm{d}x^j}{\mathrm{d}t} + \frac{\partial^2 u}{\partial x^i \partial t}\Big)\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial u}{\partial x^i}\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \frac{\partial^2 u}{\partial x^i \partial t}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial^2 u}{\partial t^2}.$$
Substituting (2.1) into the above equation yields

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - \frac{\partial u}{\partial x^i}\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} - \Big(\frac{\partial^2 u}{\partial x^i \partial x^j}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t} + 2\frac{\partial^2 u}{\partial x^i \partial t}\frac{\mathrm{d}x^i}{\mathrm{d}t}\Big) \\
&= \frac{1}{H\upsilon} + D_i u \cdot \frac{D^i u \cdot u^{-2}}{H\upsilon} - \Big(\frac{\partial^2 u}{\partial x^i \partial x^j}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t} + 2\frac{\partial^2 u}{\partial x^i \partial t}\frac{\mathrm{d}x^i}{\mathrm{d}t}\Big) \\
&= \frac{\upsilon}{H} - \Big(\frac{\partial^2 u}{\partial x^i \partial x^j}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t} + 2\frac{\partial^2 u}{\partial x^i \partial t}\frac{\mathrm{d}x^i}{\mathrm{d}t}\Big).
\end{aligned}$$

Let $\varphi = \log u$. For a graph M_t over $\mathbb{S}^n$, the metric has the components

$$g_{ij} = u_i u_j + u^2 \sigma_{ij} = u^2(\sigma_{ij} + \varphi_i \varphi_j),$$

and their inverses are
$$g^{ij} = u^{-2}\Big(\sigma^{ij} - \frac{\varphi^i \varphi^j}{\upsilon^2}\Big).$$
Besides, υ can be expressed as

$$\upsilon = (1 + u^{-2}\sigma^{ij} D_i u D_j u)^{1/2} = (1 + \sigma^{ij} D_i \varphi D_j \varphi)^{1/2} = (1 + |D\varphi|^2)^{1/2},$$

and the second fundamental form can be given as

$$
\begin{aligned}
h_{ij} &= -\frac{1}{\upsilon}\Big(u_{ij} - u\sigma_{ij} - \frac{2}{u}u_i u_j\Big) = \frac{u}{\upsilon}\Big(\sigma_{ij} - \frac{u_{ij}}{u} + \frac{2}{u^2}u_i u_j\Big) \\
&= \frac{u}{\upsilon}\Big(\sigma_{ij} - \frac{uu_{ij} - u_i u_j}{u^2} + \frac{u_i u_j}{u^2}\Big) = \frac{u}{\upsilon}\Big(\sigma_{ij} - \varphi_{ij} + \varphi_i\varphi_j\Big).
\end{aligned}
$$

Therefore, the mean curvature is

$$
\begin{aligned}
H = g^{ij}h_{ij} &= u^{-2}\Big(\sigma_{ij} - \frac{\varphi_i\varphi_j}{\upsilon^2}\Big)\cdot\frac{u}{\upsilon}\Big(\sigma_{ij} - \varphi_{ij} + \varphi_i\varphi_j\Big) \\
&= \frac{1}{u\upsilon}\Big(n - \sigma^{ij}\varphi_{ij} + \sigma^{ij}\varphi_i\varphi_j - \frac{\sigma_{ij}\varphi^i\varphi^j}{\upsilon^2} + \frac{\varphi^i\varphi^j}{\upsilon^2}\varphi_{ij} - \frac{\varphi^i\varphi^j\varphi_i\varphi_j}{\upsilon^2}\Big) \\
&= \frac{1}{u\upsilon}\Big(n + \Big(-\sigma^{ij} + \frac{\varphi^i\varphi^j}{\upsilon^2}\Big)\varphi_{ij}\Big).
\end{aligned}
$$

So, together with (2.1), we can obtain the equation

$$
\frac{\partial^2 u}{\partial t^2} = \frac{u\upsilon^2}{n + (-\sigma^{ij} + \varphi^i\varphi^j/\upsilon^2)\varphi_{ij}} - \Big(\frac{\partial^2 u}{\partial x^i\partial x^j}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} + 2\frac{\partial^2 u}{\partial x^i\partial t}\frac{\mathrm{d}x^i}{\mathrm{d}t}\Big). \tag{2.2}
$$

Note that

$$
\frac{\partial\varphi}{\partial t} = \frac{1}{u}\frac{\partial u}{\partial t},
$$

then together with (2.2), we have

$$
\begin{aligned}
\frac{\partial^2\varphi}{\partial t^2} = \frac{1}{u}\frac{\partial^2 u}{\partial t^2} - \frac{1}{u^2}\Big(\frac{\partial u}{\partial t}\Big)^2 &= \frac{\upsilon^2}{n + (-\sigma^{ij} + \varphi^i\varphi^j/\upsilon^2)\varphi_{ij}} \\
&\quad - \frac{1}{u}\Big(\frac{\partial^2 u}{\partial x^i\partial x^j}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} + 2\frac{\partial^2 u}{\partial x^i\partial t}\frac{\mathrm{d}x^i}{\mathrm{d}t}\Big) - \Big(\frac{\partial\varphi}{\partial t}\Big)^2 \\
= \frac{\upsilon^2}{n + (-\sigma^{ij} + \varphi^i\varphi^j/\upsilon^2)\varphi_{ij}}& \\
- \Big[(\varphi_{ij} + \varphi_i\varphi_j)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} &+ 2(\varphi_{it} + \varphi_i\varphi_t)\frac{\mathrm{d}x^i}{\mathrm{d}t}\Big] - \Big(\frac{\partial\varphi}{\partial t}\Big)^2.
\end{aligned} \tag{2.3}
$$

Let

$$
\begin{aligned}
\phi(x, \varphi_{ij}, \varphi_{it}, \varphi_i, \varphi_t, \varphi) := &\frac{\upsilon^2}{n + (-\sigma^{ij} + \varphi^i\varphi^j/\upsilon^2)\varphi_{ij}} \\
&- \Big[(\varphi_{ij} + \varphi_i\varphi_j)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} + 2(\varphi_{it} + \varphi_i\varphi_t)\frac{\mathrm{d}x^i}{\mathrm{d}t}\Big] - \Big(\frac{\partial\varphi}{\partial t}\Big)^2.
\end{aligned}
$$

Consider the matrix

$$
\Big(\frac{\partial\phi}{\partial\varphi_{ij}}\Big|_{t=0}\Big) = \begin{pmatrix} \frac{1}{H_0^2}g^{11} - \frac{\mathrm{d}x^1}{\mathrm{d}t}\frac{\mathrm{d}x^1}{\mathrm{d}t} & \cdots & \frac{1}{H_0^2}g^{1n} - \frac{\mathrm{d}x^1}{\mathrm{d}t}\frac{\mathrm{d}x^n}{\mathrm{d}t} \\ \vdots & & \vdots \\ \frac{1}{H_0^2}g^{n1} - \frac{\mathrm{d}x^n}{\mathrm{d}t}\frac{\mathrm{d}x^1}{\mathrm{d}t} & \cdots & \frac{1}{H_0^2}g^{nn} - \frac{\mathrm{d}x^n}{\mathrm{d}t}\frac{\mathrm{d}x^n}{\mathrm{d}t} \end{pmatrix},
$$

which, by a suitable linear transformation, becomes

$$\begin{pmatrix} \frac{1}{H_0^2} g^{11} & \cdots & \frac{1}{H_0^2} g^{1n} \\ \vdots & & \vdots \\ \frac{1}{H_0^2} g^{n1} & \cdots & \frac{1}{H_0^2} g^{nn} \end{pmatrix},$$

where H_0 is the mean curvature of the initial hypersurface M_0. Clearly, this matrix is positive definite since M_0 is mean convex, which implies that the evolution equation (2.3) is a second-order uniformly hyperbolic PDE on some small time interval $[0, l)$. By applying the standard theory of second-order linear hyperbolic PDEs (see, e.g. [7], Chapter 7 or [10]), together with the inverse function theorem, we have the following short-time existence.

Theorem 2.1 (Local existence and uniqueness)**.** *If the initial hypersurface M_0 is a compact, mean convex, star-shaped smooth hypersurface of $\mathbb{R}^{n+1}$ ($n \geqslant 2$), which is given as an embedding*

$$X_0 \colon \mathbb{S}^n \to \mathbb{R}^{n+1},$$

then there exists a constant $T_{\max} > 0$ such that the initial value problem (IVP for short)

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}t^2} X(x,t) = H^{-1}(x,t)\vec{\nu}(x,t), \quad \forall\, x \in \mathbb{S}^n,\ t > 0, \\ \dfrac{\mathrm{d}X}{\mathrm{d}t}(x,0) = X_1(x), \\ X(x,0) = X_0(x), \end{cases} \tag{2.4}$$

has a unique smooth solution $X(x,t)$ on $\mathbb{S}^n \times [0, T_{\max})$, where $X_1(x)$ is a smooth vector-valued function on $\mathbb{S}^n$.

Remark 2.2. (1) If the IVP (2.4) is replaced by

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}t^2} X(x,t) = H^{-1}(x,t)\vec{\nu}(x,t) + c(t)X(x,t), \quad \forall\, x \in \mathbb{S}^n,\ t > 0, \\ \dfrac{\mathrm{d}X}{\mathrm{d}t}(x,0) = X_1(x), \\ X(x,0) = X_0(x), \end{cases} \tag{2.5}$$

with $c(t)$ a bounded continuous function w.r.t. to t, and other assumptions are the same as those in Theorem 2.1, then one can also get the local existence and uniqueness of the forced HIMCF (2.5) since the first evolution equation in (2.5) is a second-order hyperbolic PDE by nearly the same argument in this section. Although we only add a forcing term $c(t)X(x,t)$ in the direction of the position vector, the convergent situation of (2.5) will be much different from (2.4), which can be seen from examples shown in Section 3 and Remark 3.3.

(2) Let F be a symmetric, positive, 1-homogeneous function defined on an open cone Γ of $\mathbb{R}^n$ with vertex at the origin, which contains the positive diagonal, i.e., all n-tuples of the form $(\lambda,\dots,\lambda)$, $\lambda>0$. Assume that $F\in C^0(\overline{\Gamma})\cap C^2(\Gamma)$ is monotone, concave, i.e.,

$$\frac{\partial F}{\partial\lambda^i}>0,\quad i=1,2,\dots,n,\ \text{in }\Gamma,$$
$$\frac{\partial^2 F}{\partial\lambda^i\partial\lambda^j}\leqslant 0,$$

and that

$$F=0\quad\text{on }\partial\Gamma.$$

We also use the normalization convention $F(1,\dots,1)=n+1$. Based on Gerhardt's work (see [8]) on the ICF in $\mathbb{R}^{n+1}$, we can consider the IVP

$$\tag{2.6}\begin{cases}\dfrac{\mathrm{d}^2}{\mathrm{d}t^2}X(x,t)=F^{-1}(x,t)\vec{\nu}(x,t), \quad \forall\, x\in\mathbb{S}^n,\ t>0,\\ \dfrac{\mathrm{d}X}{\mathrm{d}t}(x,0)=X_1(x),\\ X(x,0)=X_0(x),\end{cases}$$

where F defined on Γ is a function of principal curvatures described as above, and other assumptions are the same as those in Theorem 2.1. Clearly, the IVP (2.4) is a special case of the IVP (2.6), and the first evolution equation in (2.6) is a hyperbolic version of the ICF, which leads to the fact that we call it the *hyperbolic inverse curvature flow* (HICF for short). We claim that the hyperbolic flow (2.6) also has a unique smooth solution $X(x,t)$ on $\mathbb{S}^n\times[0,T_2)$ with some $T_2>0$. Using the argument in Section 2, together with the first evolution equation of (2.6), one can obtain the evolution equation

$$\tag{2.7}\frac{\partial^2\varphi}{\partial t^2}=\frac{v}{uF}-\Big[(\varphi_{ij}+\varphi_i\varphi_j)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}+2(\varphi_{it}+\varphi_i\varphi_t)\frac{\mathrm{d}x^i}{\mathrm{d}t}\Big]-\Big(\frac{\partial\varphi}{\partial t}\Big)^2.$$

Denote by $\mathcal{M}(\Gamma)$ the class of all real $(n\times n)$-matrices whose eigenvalues belong to Γ. Then one can define a function $\mathcal{F}$ on $\mathcal{M}(\Gamma)$ as

$$\mathcal{F}(a^{ij})=F(\lambda^i),$$

where (λ^i) are the eigenvalues of the matrix (a^{ij}). It has been proven in [3] that the monotonicity and concavity of F now take the form

$$\mathcal{F}_{ij} = \frac{\partial \mathcal{F}}{\partial a^{ij}} \quad \text{is positive definite,} \tag{2.8}$$

and

$$\mathcal{F}_{ij,rs} = \frac{\partial^2 \mathcal{F}}{\partial a^{ij} \partial a^{rs}} \quad \text{is negative semidefinite.} \tag{2.9}$$

Consider the tensor

$$h^i_j = g^{ik} h_{kj} = \frac{1}{uv}\Big[\delta^i_j + \Big(-\sigma^{ik} + \frac{\varphi^i \varphi^k}{v^2}\Big)\varphi_{kj}\Big].$$

Define the symmetric tensor

$$\widehat{h}_{ij} = \frac{1}{2}(\widetilde{\sigma}_{ik} h^k_j + \widetilde{\sigma}_{jk} h^k_i),$$

where

$$\widetilde{\sigma}_{ij} = \sigma_{ij} + \varphi_i \varphi_j.$$

Set

$$\widetilde{h}_{ij} := \frac{u}{v}\widehat{h}_{ij} = v^{-2}(\sigma_{ij} - \varphi_{ij} + \varphi_i \varphi_j),$$

then, together with (2.7), we have

$$\frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{\mathcal{F}(\widetilde{h}_{ij})} - \Big[(\varphi_{ij} + \varphi_i \varphi_j)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} + 2(\varphi_{it} + \varphi_i \varphi_t)\frac{\mathrm{d}x^i}{\mathrm{d}t}\Big] - \Big(\frac{\partial \varphi}{\partial t}\Big)^2,$$

where the nonlinearity $\mathcal{F}$ depends only on $D\varphi$ and $D^2\varphi$.

Now, we do the linearization process. Setting

$$Q(\varphi, D\varphi, D^2\varphi) := \frac{1}{\mathcal{F}(\widetilde{h}_{ij})} - \Big[(\varphi_{ij} + \varphi_i \varphi_j)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} + 2(\varphi_{it} + \varphi_i \varphi_t)\frac{\mathrm{d}x^i}{\mathrm{d}t}\Big] - \Big(\frac{\partial \varphi}{\partial t}\Big)^2,$$

one can obtain

$$Q^{ij} = \frac{\partial Q}{\partial \varphi_{ij}} = -\frac{1}{\mathcal{F}^2(\widetilde{h}_{ij})}\frac{\partial \mathcal{F}}{\partial \widetilde{h}_{ij}}\frac{\partial \widetilde{h}_{ij}}{\partial \varphi_{ij}} - \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = \frac{1}{v^2 \mathcal{F}^2}\frac{\partial F}{\partial \widetilde{h}_{ij}} - \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}.$$

Therefore, we have

$$\frac{\partial^2 \varphi}{\partial t^2} = Q^{ij}\varphi_{ij} - 2\frac{\mathrm{d}x^i}{\mathrm{d}t}\varphi_{it} + I(x, \varphi_i, \varphi_t, \varphi),$$

where the last term $I(x, \varphi_i, \varphi_t, \varphi)$ depends only on x, φ_i, φ_t, φ. The coefficient matrix of the terms involving second-order derivatives of φ in the above evolution equation is

$$\begin{pmatrix} -1 & -\dfrac{\mathrm{d}x^1}{\mathrm{d}t} & \cdots & -\dfrac{\mathrm{d}x^n}{\mathrm{d}t} \\ -\dfrac{\mathrm{d}x^1}{\mathrm{d}t} & \dfrac{1}{v^2\mathcal{F}^2}\dfrac{\partial\mathcal{F}}{\partial\widetilde{h}_{11}} - \dfrac{\mathrm{d}x^1}{\mathrm{d}t}\dfrac{\mathrm{d}x^1}{\mathrm{d}t} & \cdots & \dfrac{1}{v^2\mathcal{F}^2}\dfrac{\partial\mathcal{F}}{\partial\widetilde{h}_{1n}} - \dfrac{\mathrm{d}x^1}{\mathrm{d}t}\dfrac{\mathrm{d}x^n}{\mathrm{d}t} \\ \vdots & \vdots & & \vdots \\ -\dfrac{\mathrm{d}x^n}{\mathrm{d}t} & \dfrac{1}{v^2\mathcal{F}^2}\dfrac{\partial\mathcal{F}}{\partial\widetilde{h}_{n1}} - \dfrac{\mathrm{d}x^n}{\mathrm{d}t}\dfrac{\mathrm{d}x^1}{\mathrm{d}t} & \cdots & \dfrac{1}{v^2\mathcal{F}^2}\dfrac{\partial\mathcal{F}}{\partial\widetilde{h}_{nn}} - \dfrac{\mathrm{d}x^n}{\mathrm{d}t}\dfrac{\mathrm{d}x^n}{\mathrm{d}t} \end{pmatrix}$$

which, by a suitable linear transformation, becomes

$$\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & \dfrac{1}{v^2\mathcal{F}^2}\dfrac{\partial\mathcal{F}}{\partial\widetilde{h}_{11}} & \cdots & \dfrac{1}{v^2\mathcal{F}^2}\dfrac{\partial\mathcal{F}}{\partial\widetilde{h}_{1n}} \\ \vdots & \vdots & & \vdots \\ 0 & \dfrac{1}{v^2\mathcal{F}^2}\dfrac{\partial\mathcal{F}}{\partial\widetilde{h}_{n1}} & \cdots & \dfrac{1}{v^2\mathcal{F}^2}\dfrac{\partial\mathcal{F}}{\partial\widetilde{h}_{nn}} \end{pmatrix}, \tag{2.10}$$

which, by (2.8) and (2.9), implies that the matrix (2.10) is negative definite. So, the equation is a second-order uniformly hyperbolic PDE. Our claim follows by the standard theory of second-order linear hyperbolic PDEs.

(3) Although we can also get the short-time existence of the IVP (2.6), in this paper we mainly discuss the IVP (2.4) since if the initial hypersurface M_0 is more special (e.g., sphere, cylinder), the evolution equation of the flow, which in general is a second-order hyperbolic PDE, degenerates into a second-order ordinary differential equation (ODE for short) and then the convergent situation of the evolving hypersurfaces can be easily known by directly checking the explicit solution to the ODE (for details, see examples shown in Section 3).

3. Examples

In order to possibly understand the convergence of HIMCF (2.4) well, we would like to consider the following interesting examples in this section.

Example 3.1. Consider a family of spheres (or circles) in $\mathbb{R}^{n+1}$ $(n \geqslant 1)$

$$\begin{aligned} X(x,t) = r(t)(&\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \ldots, \\ &\sin\theta_1\sin\theta_2\sin\theta_3\ldots\sin\theta_{n-1}\cos\theta_n, \sin\theta_1\sin\theta_2\sin\theta_3\ldots\sin\theta_{n-1}\sin\theta_n), \end{aligned}$$

where $\theta_1 \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$, $\theta_\beta \in [0, 2\pi]$ for $\beta = 2, 3, \ldots, n$. The mean curvature H of the evolving sphere (or the curvature k of the evolving curve) is

$$H = \frac{n}{r}, \quad n \geqslant 2 \quad (\text{or } k = \frac{1}{r},\ n = 1).$$

In this setting, the HIMCF (2.4) becomes

$$\begin{cases} r_{tt} = \dfrac{r}{n}, \\ r(0) = r_0 > 0, \quad r_t(0) = r_1, \end{cases} \tag{3.1}$$

for some constant r_1. Solving (3.1) directly yields

$$r(t) = \frac{1}{2}(r_0 + \sqrt{n} r_1) \mathrm{e}^{t\sqrt{n}/n} + \frac{1}{2}(r_0 - \sqrt{n} r_1) \mathrm{e}^{-t\sqrt{n}/n}$$

on $[0, T_{\max})$ for some $0 < T_{\max} \leqslant \infty$, and then we have:

- ▷ If $r_0 + \sqrt{n} r_1 > 0$, then $T_{\max} = \infty$ (i.e., the flow exists for all the time). Moreover, if furthermore, $r_0 - \sqrt{n} r_1 \leqslant 0$, the evolving spheres (or circles) *expand* exponentially to the infinity, and if furthermore, $r_0 - \sqrt{n} r_1 > 0$, then the evolving spheres (or circles) *converge* first for a while and then *expand* exponentially to the infinity.
- ▷ If $r_0 + \sqrt{n} r_1 = 0$, then $r(t) = \sqrt{n} r_0 \mathrm{e}^{-t\sqrt{n}/n}$, which implies $T_{\max} = \infty$ and the evolving spheres (or circles) *converge* to a single point as time tends to infinity.
- ▷ If $r_0 + \sqrt{n} r_1 < 0$, then $T_{\max} = \sqrt{n}/n \ln((\sqrt{n} r_1 - r_0)/(\sqrt{n} r_1 + r_0))$ and the evolving spheres (or circles) *converge* to a single point as $t \to T_{\max}$.

From the above argument, at least we can get an impression that the convergent situation of the HIMCF (2.4) is much complicated and has close relation with the initial data.

Example 3.2. Now, we would like to consider the cylinder solution for the HIMCF (2.4) in $\mathbb{R}^3$ which has the form

$$X(x, t) = (r(t) \cos\alpha, r(t) \sin\alpha, \varrho),$$

where $\alpha \in [0, 2\pi]$, $\varrho \in [0, \varrho_0]$ for some $\varrho_0 > 0$. The mean curvature is

$$H = \frac{1}{r}.$$

Besides, the outward unit normal vector of each $X(\cdot, t)$ is $\vec{v} = (\cos\alpha, \sin\alpha, 0)$. Therefore, in this setting, the HIMCF (2.4) becomes

$$\begin{cases} r_{tt} = r, \\ r(0) = r_0 > 0, \quad r_t(0) = r_1, \end{cases} \tag{3.2}$$

with $X_1(x) = (r_1 \cos\alpha, r_1 \sin\alpha, \varrho)$ for some constant r_1. Solving (3.2) directly yields

$$r(t) = \frac{1}{2}(r_0 + r_1)\mathrm{e}^t + \frac{1}{2}(r_0 - r_1)\mathrm{e}^{-t}$$

on $[0, T_{\max})$ for some $0 < T_{\max} \leqslant \infty$. It is not difficult to know that:

- ▷ If $r_0 + r_1 > 0$, then $T_{\max} = \infty$ (i.e., the flow exists for all the time). Moreover, if furthermore, $r_0 - r_1 \leqslant 0$, the evolving cylinders *expand* exponentially to the infinity, and if furthermore, $r_0 - r_1 > 0$, then the evolving cylinders *converge* first for a while and then *expand* exponentially to the infinity.
- ▷ If $r_0 + r_1 = 0$, then $r(t) = r_0\mathrm{e}^{-t}$, which implies $T_{\max} = \infty$ and the evolving cylinders *converge* to a *straight line* as time tends to infinity.
- ▷ If $r_0 + r_1 < 0$, then $T_{\max} = \ln((r_1 - r_0)/(r_0 + r_1))$ and the evolving cylinders *converge* to a *straight line* as $t \to T_{\max}$.

Of course, as shown in Example 3.1, one can also consider the high-dimensional case of Example 3.2, i.e., the generalized cylinder solutions to the HIMCF (2.4). However, through a simple calculation, one can easily find that, similarly to the sphere case, there is no obvious difference between Example 3.2 and its high-dimensional version.

Remark 3.3. If the HIMCF (2.4) is replaced by the forced HIMCF (2.5) in examples shown above, then the convergent situation will be more complicated. For instance, if the replacement has been made in Example 3.1 with $n = 2$, then (3.1) will become

$$\begin{cases} r_{tt} = \frac{1}{2}r + c(t)r, \\ r(0) = r_0 > 0, \quad r_t(0) = r_1. \end{cases}$$

Denote the solution to the above IVP by $r(t)$. Since $c(t)$ is bounded continuous, there exist c^-, c^+ such that $c^- \leqslant c(t) \leqslant c^+$. Consider the IVPs

$$\begin{cases} r_{tt} = \frac{1}{2}r + c^- r, \\ r(0) = r_0 > 0, \quad r_t(0) = r_1, \end{cases}$$

and

$$\begin{cases} r_{tt} = \frac{1}{2}r + c^+ r, \\ r(0) = r_0 > 0, \quad r_t(0) = r_1, \end{cases}$$

whose solutions are denoted by $r^-(t)$ and $r^+(t)$, respectively. Clearly, $r^-(t) \leqslant r(t) \leqslant r^+(t)$. So, the convergent situation of $r(t)$ deeply depends on that of $r^-(t)$, $r^+(t)$ which is not simple. This is because one has to discuss the sign of $(c^- + \frac{1}{2})$, $(c^+ + \frac{1}{2})$, which leads to the fact that the convergent situation of $r(t)$ here will be more complicated than that described in Example 3.1.

From the evolution equation for the HIMCF (2.4) we can derive evolution equations for some geometric quantities of the hypersurface $X(\cdot,t)$, and these equations will play an important role in the future study of the HIMCF.

Lemma 4.1. *Under the HIMCF* (2.4), *we have*

$$\frac{\partial^2 g_{ij}}{\partial t^2}=2H^{-1}h_{ij}+2\Big\langle\frac{\partial X_i}{\partial t},\frac{\partial X_j}{\partial t}\Big\rangle,$$

where in this section $\langle,\rangle$ *denotes the inner product corresponding to the metric* $\mathrm{d}s^2=\mathrm{d}r^2+r^2\sigma_{ij}\mathrm{d}x^i\mathrm{d}x^j$ *in* $\mathbb{R}^{n+1}$.

P r o o f. By direct computation we have

$$\begin{aligned}\frac{\partial^2}{\partial t^2}g_{ij}&=\frac{\partial^2}{\partial t^2}\Big\langle\frac{\partial X}{\partial x^i},\frac{\partial X}{\partial x^j}\Big\rangle=\frac{\partial}{\partial t}\Big(\Big\langle\frac{\partial^2 X}{\partial x^i\partial t},\frac{\partial X}{\partial x^j}\Big\rangle+\Big\langle\frac{\partial X}{\partial x^i},\frac{\partial^2 X}{\partial x^j\partial t}\Big\rangle\Big)\\&=\Big\langle\frac{\partial^3 X}{\partial t^2\partial x^i},\frac{\partial X}{\partial x^j}\Big\rangle+\Big\langle\frac{\partial^3 X}{\partial t^2\partial x^j},\frac{\partial X}{\partial x^i}\Big\rangle+2\Big\langle\frac{\partial^2 X}{\partial t\partial x^i},\frac{\partial^2 X}{\partial t\partial x^j}\Big\rangle\\&=2\Big\langle\frac{\partial}{\partial x^i}(H^{-1}\vec{\nu}),\frac{\partial X}{\partial x^j}\Big\rangle+2\Big\langle\frac{\partial^2 X}{\partial t\partial x^i},\frac{\partial^2 X}{\partial t\partial x^j}\Big\rangle\\&=2H^{-1}\Big\langle h_{ik}g^{kl}\frac{\partial X}{\partial x^l},\frac{\partial X}{\partial x^j}\Big\rangle+2\Big\langle\frac{\partial^2 X}{\partial t\partial x^i},\frac{\partial^2 X}{\partial t\partial x^j}\Big\rangle\\&=2H^{-1}h_{ij}+2\Big\langle\frac{\partial X_i}{\partial t},\frac{\partial X_j}{\partial t}\Big\rangle,\end{aligned}$$

which completes the proof of Lemma 4.1. □

Lemma 4.2. *Under the HIMCF* (2.4), *we have*

$$\begin{aligned}\frac{\partial^2\vec{\nu}}{\partial t^2}&=H^{-2}g^{ij}X_j\frac{\partial H}{\partial x^i}-g^{ij}\Big\langle\vec{\nu},\frac{\partial X_i}{\partial t}\Big\rangle\frac{\partial X_j}{\partial t}\\&\quad+g^{ij}g^{kl}\Big\langle\vec{\nu},\frac{\partial X_i}{\partial t}\Big\rangle\Big(\Big\langle\frac{\partial X_j}{\partial t},X_l\Big\rangle+2\Big\langle X_j,\frac{\partial X_l}{\partial t}\Big\rangle\Big)X_k.\end{aligned}$$

P r o o f. First, we have

$$\frac{\partial\vec{\nu}}{\partial t}=\Big\langle\frac{\partial\vec{\nu}}{\partial t},\frac{\partial X}{\partial x^i}\Big\rangle g^{ij}\frac{\partial X}{\partial x^j}=-\Big\langle\vec{\nu},\frac{\partial^2 X}{\partial t\partial x^i}\Big\rangle g^{ij}\frac{\partial X}{\partial x^j}.$$

Then, by direct computation, it follows that

$$
\begin{aligned}
\frac{\partial^2 \vec{\nu}}{\partial t^2} = & -\Big\langle \frac{\partial \vec{\nu}}{\partial t}, \frac{\partial^2 X}{\partial t \partial x^i} \Big\rangle g^{ij} \frac{\partial X}{\partial x^j} - \Big\langle \vec{\nu}, \frac{\partial^3 X}{\partial t^2 \partial x^i} \Big\rangle g^{ij} \frac{\partial X}{\partial x^j} - \Big\langle \vec{\nu}, \frac{\partial^2 X}{\partial t \partial x^i} \Big\rangle \frac{\partial g^{ij}}{\partial t} \frac{\partial X}{\partial x^j} \\
& - \Big\langle \vec{\nu}, \frac{\partial^2 X}{\partial t \partial x^i} \Big\rangle g^{ij} \frac{\partial^2 X}{\partial t \partial x^j} \\
= & \, g^{ij} g^{kl} \Big\langle \vec{\nu}, \frac{\partial^2 X}{\partial t \partial x^k} \Big\rangle \Big\langle \frac{\partial X}{\partial x^l}, \frac{\partial^2 X}{\partial t \partial x^i} \Big\rangle \frac{\partial X}{\partial x^j} - \Big\langle \vec{\nu}, \frac{\partial}{\partial x^i} (H^{-1} \vec{\nu}) \Big\rangle g^{ij} \frac{\partial X}{\partial x^j} \\
& + \Big\langle \vec{\nu}, \frac{\partial^2 X}{\partial t \partial x^i} \Big\rangle g^{ik} g^{jl} \frac{\partial g_{kl}}{\partial t} \frac{\partial X}{\partial x^j} - \Big\langle \vec{\nu}, \frac{\partial^2 X}{\partial t \partial x^i} \Big\rangle g^{ij} \frac{\partial^2 X}{\partial t \partial x^j} \\
= & \, H^{-2} g^{ij} \frac{\partial H}{\partial x^i} \frac{\partial X}{\partial x^j} - g^{ij} \Big\langle \vec{\nu}, \frac{\partial^2 X}{\partial t \partial x^i} \Big\rangle \frac{\partial^2 X}{\partial t \partial x^j} + g^{ij} g^{kl} \Big\langle \vec{\nu}, \frac{\partial^2 X}{\partial t \partial x^i} \Big\rangle \\
& \times \Big(\Big\langle \frac{\partial^2 X}{\partial t \partial x^j}, \frac{\partial X}{\partial x^l} \Big\rangle + 2 \Big\langle \frac{\partial X}{\partial x^j}, \frac{\partial^2 X}{\partial t \partial x^l} \Big\rangle \Big) \frac{\partial X}{\partial x^k} \\
= & \, H^{-2} g^{ij} \frac{\partial H}{\partial x^i} \frac{\partial X}{\partial x^j} - g^{ij} \Big\langle \vec{\nu}, \frac{\partial X_i}{\partial t} \Big\rangle \frac{\partial X_j}{\partial t} + g^{ij} g^{kl} \Big\langle \vec{\nu}, \frac{\partial X_i}{\partial t} \Big\rangle \\
& \times \Big(\Big\langle \frac{\partial X_j}{\partial t}, X_l \Big\rangle + 2 \Big\langle X_j, \frac{\partial X_l}{\partial t} \Big\rangle \Big) X_k,
\end{aligned}
$$

which completes the proof of Lemma 4.2. □

Before we derive the evolution equation for the second fundamental form h_{ij}, we need to recall the following facts:

(4.1)	$X_{ij} = -h_{ij} \vec{\nu}$,	Gauss formula,
(4.2)	$\vec{\nu}_i = h_{ij} g^{jk} X_k$,	Weingarten formula,
(4.3)	$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}$,	Gauss equation,
(4.4)	$h_{ij,k} = h_{ik,j}$,	Codazzi equation,

where R_{ijkl} denote the components of the Riemannian curvature tensor on M_t, a comma means the start of covariant differentiation except specifications. *Here, clearly, $X_{ij} = X_{i,j}$, and we write X_{ij} just for convenience. This convenient writing for $X_{i,j}$ will be used in the sequel of this section also.* These formulae can be found in Zhu [24]. Using the Gauss equation (4.3) and the Codazzi equation (4.4), together with the Ricci identity, one can obtain the following relation

$$
\begin{aligned}
h_{rs,ij} = h_{ir,sj} &= h_{ir,js} + R_{krsj} g^{kl} h_{li} + R_{kisj} g^{kl} h_{lr} \\
&= h_{ij,rs} + (h_{ks} h_{rj} - h_{kj} h_{rs}) h_i^k + (h_{ks} h_{ij} - h_{kj} h_{is}) h_r^k.
\end{aligned}
\tag{4.5}
$$

Using the relation (4.5), the following fact can be obtained directly.

Lemma 4.3. *For any hypersurface $X(\cdot,t)$ in $\mathbb{R}^{n+1}$, the following identities hold:*

$$\Delta h_{ij} = H_{ij} + H h_{il} g^{lm} h_{mj} - |A|^2 h_{ij},$$
$$\Delta |A|^2 = 2 g^{ik} g^{jl} h_{kl} H_{ij} + 2|\nabla A|^2 + 2H \operatorname{tr}(A^3) - 2|A|^4,$$

where Δ, ∇ are the Laplace and the gradient operators on the hypersurface, respectively,

$$|A|^2 = g^{ij} g^{kl} h_{ik} h_{jl}, \quad \operatorname{tr}(A^3) = g^{ij} g^{kl} g^{mn} h_{ik} h_{lm} h_{nj}.$$

The proof of Lemma 4.3 can also be found in [24], Lemma 2.3.

Lemma 4.4. *Under the HIMCF* (2.4), *we have*

$$\frac{\partial^2 h_{ij}}{\partial t^2} = H^{-2}\Delta h_{ij} + H^{-2}|A|^2 h_{ij} - 2H^{-3} H_j H_i + g^{kl} h_{ij} \Big\langle \vec{\nu}, \frac{\partial X_k}{\partial t} \Big\rangle \cdot \Big\langle \vec{\nu}, \frac{\partial X_l}{\partial t} \Big\rangle.$$

Proof. By (4.1), we have

$$\begin{aligned}
-\frac{\partial^2 h_{ij}}{\partial t^2} &= \frac{\partial}{\partial t}\Big(\Big\langle X_{ij}, \frac{\partial \vec{\nu}}{\partial t} \Big\rangle + \Big\langle \frac{\partial X_{ij}}{\partial t}, \vec{\nu} \Big\rangle \Big) \\
&= 2\Big\langle \frac{\partial X_{ij}}{\partial t}, \frac{\partial \vec{\nu}}{\partial t} \Big\rangle + \Big\langle X_{ij}, \frac{\partial^2 \vec{\nu}}{\partial t^2} \Big\rangle + \Big\langle \Big(\frac{\partial^2 X}{\partial t^2} \Big)_{ij}, \vec{\nu} \Big\rangle \\
&= 2\Big\langle \frac{\partial X_{ij}}{\partial t}, \frac{\partial \vec{\nu}}{\partial t} \Big\rangle + \Big\langle X_{ij}, \frac{\partial^2 \vec{\nu}}{\partial t^2} \Big\rangle + \langle (H^{-1}\vec{\nu})_{ij}, \vec{\nu} \rangle \\
&= 2\frac{\partial}{\partial t}\Big\langle X_{ij}, \frac{\partial \vec{\nu}}{\partial t} \Big\rangle - \Big\langle X_{ij}, \frac{\partial^2 \vec{\nu}}{\partial t^2} \Big\rangle + (H^{-1})_{ij} + H^{-1}\langle (\vec{\nu})_{ij}, \vec{\nu} \rangle.
\end{aligned}$$

Applying Lemma 4.2 and substituting the fact

$$\frac{\partial \vec{\nu}}{\partial t} = \Big\langle \frac{\partial \vec{\nu}}{\partial t}, \frac{\partial X}{\partial x^i} \Big\rangle g^{ij} \frac{\partial X}{\partial x^j} = -\Big\langle \vec{\nu}, \frac{\partial^2 X}{\partial t \partial x^i} \Big\rangle g^{ij} \frac{\partial X}{\partial x^j} = -\Big\langle \vec{\nu}, \frac{\partial X_i}{\partial t} \Big\rangle g^{ij} X_j \tag{4.6}$$

and the Gauss formula (4.1) into the above equality result in

$$\begin{aligned}
\frac{\partial^2 h_{ij}}{\partial t^2} &= \Big\langle X_{ij}, H^{-2} g^{kl} X_k \frac{\partial H}{\partial x^l} \Big\rangle - g^{kl} \Big\langle \vec{\nu}, \frac{\partial X_k}{\partial t} \Big\rangle \Big\langle \frac{\partial X_l}{\partial t}, X_{ij} \Big\rangle \\
&\quad + g^{rs} g^{kl} \Big\langle \vec{\nu}, \frac{\partial X_r}{\partial t} \Big\rangle \Big(\Big\langle \frac{\partial X_s}{\partial t}, X_l \Big\rangle + 2\Big\langle X_s, \frac{\partial X_l}{\partial t} \Big\rangle \Big) \Big\langle X_k, X_{ij} \Big\rangle \\
&\quad + -2H^{-3} H_j H_i + H^{-2} H_{ij} - H^{-1} \langle (\vec{\nu})_{ij}, \vec{\nu} \rangle.
\end{aligned} \tag{4.7}$$

From the Weingarten formula (4.2), one has

$$\vec{\nu}_{ij} = h_{il,j} g^{lk} X_k + h_{il} g^{lk} X_{k,j} = h_{il,j} g^{lk} X_k - h_{il} g^{lk} h_{kj} \vec{\nu}.$$

Substituting this fact into (4.7) yields

$$\frac{\partial^2 h_{ij}}{\partial t^2} = H^{-2} H_{ij} - 2H^{-3} H_j H_i + H^{-1} h_{il} g^{lk} h_{kj} + g^{kl} h_{ij} \Big\langle \vec{\nu}, \frac{\partial X_k}{\partial t} \Big\rangle \cdot \Big\langle \vec{\nu}, \frac{\partial X_l}{\partial t} \Big\rangle,$$

which, together with Lemma 4.3, implies the conclusion of Lemma 4.4. $\square$

Lemma 4.5. *Under the HIMCF* (2.4), *we have*

$$\frac{\partial^2 H}{\partial t^2} = H^{-2}\Delta H - 2H^{-3}|\nabla H|^2 - H^{-1}|A|^2 + Hg^{kl}\Big\langle \vec{\nu}, \frac{\partial X_k}{\partial t}\Big\rangle \cdot \Big\langle \vec{\nu}, \frac{\partial X_l}{\partial t}\Big\rangle$$
$$- 2g^{ik}g^{jl}h_{ij}\Big\langle \frac{\partial X_k}{\partial t}, \frac{\partial X_l}{\partial t}\Big\rangle + 2g^{ik}g^{jp}g^{lq}h_{ij}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{kl}}{\partial t} - 2g^{ik}g^{jl}\frac{\partial g_{kl}}{\partial t}\frac{\partial h_{ij}}{\partial t}.$$

P r o o f. First, we have

$$\frac{\partial^2 H}{\partial t^2} = \frac{\partial}{\partial t}\Big(\frac{\partial g^{ij}}{\partial t}h_{ij} + g^{ij}\frac{\partial h_{ij}}{\partial t}\Big) = 2\frac{\partial g^{ij}}{\partial t}\frac{\partial h_{ij}}{\partial t} + \frac{\partial^2 g^{ij}}{\partial t^2}h_{ij} + g^{ij}\frac{\partial^2 h_{ij}}{\partial t^2}. \tag{4.8}$$

On the other hand, by direct calculation, one has

$$\frac{\partial g^{ij}}{\partial t} = -g^{ik}g^{jl}\frac{\partial g_{kl}}{\partial t}$$

and

$$\frac{\partial^2 g^{ij}}{\partial t^2} = 2g^{ik}g^{jp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{kl}}{\partial t} - g^{ik}g^{jl}\frac{\partial^2 g_{kl}}{\partial t^2}.$$

Substituting the above two equalities into (4.8) and applying Lemmas 4.1 and 4.4, one can obtain

$$\begin{aligned}
\frac{\partial^2 H}{\partial t^2} = & - 2g^{ik}g^{jl}\frac{\partial g_{kl}}{\partial t}\frac{\partial h_{ij}}{\partial t} \\
& + h_{ij}\Big[2g^{ik}g^{jp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{kl}}{\partial t} - g^{ik}g^{jl}\Big(2H^{-1}h_{kl} + 2\Big\langle \frac{\partial X_k}{\partial t}, \frac{\partial X_l}{\partial t}\Big\rangle\Big)\Big] \\
& + g^{ij}\Big(H^{-2}\Delta h_{ij} + H^{-2}|A|^2 h_{ij} - 2H^{-3}H_jH_i + g^{kl}h_{ij}\Big\langle \vec{\nu}, \frac{\partial X_k}{\partial t}\Big\rangle \cdot \Big\langle \vec{\nu}, \frac{\partial X_l}{\partial t}\Big\rangle\Big) \\
= {} & H^{-2}\Delta H - 2H^{-3}|\nabla H|^2 - H^{-1}|A|^2 + Hg^{kl}\Big\langle \vec{\nu}, \frac{\partial X_k}{\partial t}\Big\rangle\Big\langle \vec{\nu}, \frac{\partial X_l}{\partial t}\Big\rangle \\
& - 2g^{ik}g^{jl}h_{ij}\Big\langle \frac{\partial X_k}{\partial t}, \frac{\partial X_l}{\partial t}\Big\rangle + 2g^{ik}g^{jp}g^{lq}h_{ij}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{kl}}{\partial t} - 2g^{ik}g^{jl}\frac{\partial g_{kl}}{\partial t}\frac{\partial h_{ij}}{\partial t},
\end{aligned}$$

which completes the proof of Lemma 4.5. □

Finally, the evolution equation for the norm of the second fundamental form can be derived as follows:

Lemma 4.6. *Under the HIMCF* (2.4), *we have*

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}|A|^2 = {} & H^{-2}\Delta|A|^2 - 2H^{-2}|\nabla A|^2 \\
& - 4H^{-1}\operatorname{tr}(A^3) + 2H^{-2}|A|^4 - 4H^{-3}g^{ij}g^{kl}h_{jl}H_kH_i \\
& + 2g^{im}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{mn}}{\partial t}h_{ik}h_{jl}(2g^{jp}g^{nq}g^{kl} + g^{jn}g^{kp}g^{lq}) + 2g^{ij}g^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t} \\
& - 8g^{im}g^{jn}g^{kl}h_{jl}\frac{\partial g_{mn}}{\partial t}\frac{\partial h_{ik}}{\partial t} + 2|A|^2g^{mn}\Big\langle \vec{\nu}, \frac{\partial X_m}{\partial t}\Big\rangle \cdot \Big\langle \vec{\nu}, \frac{\partial X_n}{\partial t}\Big\rangle \\
& - 4g^{im}g^{jn}g^{kl}h_{ik}h_{jl}\Big\langle \frac{\partial X_m}{\partial t}, \frac{\partial X_n}{\partial t}\Big\rangle.
\end{aligned}$$

P r o o f. By direct computation we have

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}|A|^2 &= \frac{\partial^2}{\partial t^2}(g^{ij}g^{kl}h_{ik}h_{jl}) \\
&= 2\frac{\partial^2 g^{ij}}{\partial t^2}g^{kl}h_{ik}h_{jl} + 2\frac{\partial g^{ij}}{\partial t}\frac{\partial g^{kl}}{\partial t}h_{ik}h_{jl} + 8\frac{\partial g^{ij}}{\partial t}g^{kl}\frac{\partial h_{ik}}{\partial t}h_{jl} \\
&\quad + 2g^{ij}g^{kl}\frac{\partial^2 h_{ik}}{\partial t^2}h_{jl} + 2g^{ij}g^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t} \\
&= 2\Big(2g^{im}g^{jp}g^{nq}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{mn}}{\partial t} - g^{im}g^{jn}\frac{\partial^2 g_{mn}}{\partial t^2}\Big)g^{kl}h_{ik}h_{jl} \\
&\quad + 2g^{ij}g^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t} + 2g^{im}g^{jn}g^{kp}g^{lq}h_{ik}h_{jl}\frac{\partial g_{mn}}{\partial t}\frac{\partial g_{pq}}{\partial t} \\
&\quad - 8g^{im}g^{jn}g^{kl}h_{jl}\frac{\partial g_{mn}}{\partial t}\frac{\partial h_{ik}}{\partial t} + 2g^{ij}g^{kl}\frac{\partial^2 h_{ik}}{\partial t^2}h_{jl}.
\end{aligned}
\tag{4.9}
$$

Applying Lemmas 4.1, 4.3, and 4.4 directly to (4.9) yields

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}|A|^2 &= 4g^{im}g^{jp}g^{nq}g^{kl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{mn}}{\partial t}h_{ik}h_{jl} + 2g^{ij}g^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t} \\
&\quad + 2g^{im}g^{jn}g^{kp}g^{lq}h_{ik}h_{jl}\frac{\partial g_{mn}}{\partial t}\frac{\partial g_{pq}}{\partial t} - 8g^{im}g^{jn}g^{kl}h_{jl}\frac{\partial g_{mn}}{\partial t}\frac{\partial h_{ik}}{\partial t} \\
&\quad - 2g^{im}g^{jn}\Big(2H^{-1}h_{mn} + 2\Big\langle\frac{\partial X_m}{\partial t}, \frac{\partial X_n}{\partial t}\Big\rangle\Big)g^{kl}h_{ik}h_{jl} + 2g^{ij}g^{kl}h_{jl} \\
&\quad \times \Big(H^{-2}H_{ik} - 2H^{-3}H_kH_i + H^{-1}h_{in}g^{mn}h_{mk} \\
&\qquad\qquad + g^{mn}h_{ik}\Big\langle\vec{\nu}, \frac{\partial X_m}{\partial t}\Big\rangle \cdot \Big\langle\vec{\nu}, \frac{\partial X_n}{\partial t}\Big\rangle\Big) \\
&= 2g^{im}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{mn}}{\partial t}h_{ik}h_{jl}(2g^{jp}g^{nq}g^{kl} + g^{jn}g^{kp}g^{lq}) + 2g^{ij}g^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t} \\
&\quad - 8g^{im}g^{jn}g^{kl}h_{jl}\frac{\partial g_{mn}}{\partial t}\frac{\partial h_{ik}}{\partial t} + 2|A|^2g^{mn}\Big\langle\vec{\nu}, \frac{\partial X_m}{\partial t}\Big\rangle \cdot \Big\langle\vec{\nu}, \frac{\partial X_n}{\partial t}\Big\rangle \\
&\quad - 2H^{-1}\operatorname{tr}(A^3) - 4g^{im}g^{jn}g^{kl}h_{ik}h_{jl}\Big\langle\frac{\partial X_m}{\partial t}, \frac{\partial X_n}{\partial t}\Big\rangle \\
&\quad + 2H^{-2}g^{ij}g^{kl}h_{jl}H_{ik} - 4H^{-3}g^{ij}g^{kl}h_{jl}H_kH_i \\
&= 2g^{im}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{mn}}{\partial t}h_{ik}h_{jl}(2g^{jp}g^{nq}g^{kl} + g^{jn}g^{kp}g^{lq}) + 2g^{ij}g^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t} \\
&\quad - 8g^{im}g^{jn}g^{kl}h_{jl}\frac{\partial g_{mn}}{\partial t}\frac{\partial h_{ik}}{\partial t} + 2|A|^2g^{mn}\Big\langle\vec{\nu}, \frac{\partial X_m}{\partial t}\Big\rangle \cdot \Big\langle\vec{\nu}, \frac{\partial X_n}{\partial t}\Big\rangle \\
&\quad - 2H^{-1}\operatorname{tr}(A^3) - 4g^{im}g^{jn}g^{kl}h_{ik}h_{jl}\Big\langle\frac{\partial X_m}{\partial t}, \frac{\partial X_n}{\partial t}\Big\rangle \\
&\quad - 4H^{-3}g^{ij}g^{kl}h_{jl}H_kH_i + H^{-2}(\Delta|A|^2 - 2|\nabla A|^2 - 2H\operatorname{tr}(A^3) + 2|A|^4)
\end{aligned}
$$

$$
\begin{aligned}
&= H^{-2}\Delta|A|^2 - 2H^{-2}|\nabla A|^2 - 4H^{-1}\operatorname{tr}(A^3) + 2H^{-2}|A|^4 \\
&\quad - 4H^{-3}g^{ij}g^{kl}h_{jl}H_kH_i + 2g^{im}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{mn}}{\partial t}h_{ik}h_{jl}(2g^{jp}g^{nq}g^{kl} + g^{jn}g^{kp}g^{lq}) \\
&\quad + 2g^{ij}g^{kl}\frac{\partial h_{ik}}{\partial t}\frac{\partial h_{jl}}{\partial t} - 8g^{im}g^{jn}g^{kl}h_{jl}\frac{\partial g_{mn}}{\partial t}\frac{\partial h_{ik}}{\partial t} + 2|A|^2g^{mn}\Big\langle \vec{\nu}, \frac{\partial X_m}{\partial t}\Big\rangle \\
&\quad \times \Big\langle \vec{\nu}, \frac{\partial X_n}{\partial t}\Big\rangle - 4g^{im}g^{jn}g^{kl}h_{ik}h_{jl}\Big\langle \frac{\partial X_m}{\partial t}, \frac{\partial X_n}{\partial t}\Big\rangle,
\end{aligned}
$$

which completes the proof of Lemma 4.6. □

As we can see from *complicated* evolution equations in this section, it is difficult to get gradient estimates and higher-order estimates for the mean curvature and the second fundamental forms, which leads to the result that so far we cannot say anything about the convergence of the HIMCF (2.4) and also the hyperbolic flows (2.5), (2.6). However, for the lower dimensional case (i.e., the HIMCF in the plane $\mathbb{R}^2$), we can get the expanding and convergent conclusions, which will be shown clearly in the following section.

5. HIMCF in the plane $\mathbb{R}^2$

5.1. The short-time existence. Consider a family of closed convex plane curves $F\colon \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ which satisfy IVP

$$
\begin{cases}
\dfrac{\partial^2}{\partial t^2}F(u,t) = k^{-1}(u,t)\vec{\nu}(u,t) - \nabla\varrho(u,t), \quad \forall\, u \in \mathbb{S}^1,\ t \in [0,T), \\
\dfrac{\partial F}{\partial t}(\cdot,0) = f(u)\vec{\nu}_0, \\
F(\cdot,0) = F_0,
\end{cases} \tag{5.1}
$$

where $k(u,t)$ and $\vec{\nu}$ are the curvature and the unit outward normal vector of the plane curve $F(u,t)$, respectively, $f(u) \in C^\infty(\mathbb{S}^1)$ is the initial normal velocity, and $\vec{\nu}_0$ is the unit outward normal vector of the smooth *strictly convex* plane curve $F_0(u)$. Besides, $\nabla\varrho$ is defined by

$$
\nabla\varrho := \Big\langle \frac{\partial^2 F}{\partial s\partial t}, \frac{\partial F}{\partial t}\Big\rangle \vec{T}(u,t),
$$

where, by abuse of a notation, $\langle,\rangle$ denotes the standard Euclidean metric in $\mathbb{R}^2$, and $\vec{T}$, s are the unit tangent vector of $F(u,t)$ and the arc-length parameter, respectively.

Now, we would like to show that the HIMCF (5.1) is a normal flow. However, before that, we need the following definition which was mentioned in [13], [14].

Definition 5.1. A curve $F\colon \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ evolves normally if and only if its tangential velocity vanishes.

Lemma 5.2. *The hyperbolic curvature flow* (5.1) *is a normal flow.*

P r o o f. By direct computation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial u}\Big\rangle = \Big\langle \frac{\partial^2 F}{\partial t^2}, \frac{\partial F}{\partial u}\Big\rangle + \Big\langle \frac{\partial F}{\partial t}, \frac{\partial^2 F}{\partial t \partial u}\Big\rangle = \Big\langle -\nabla\varrho, \frac{\partial F}{\partial u}\Big\rangle + \Big\langle \frac{\partial F}{\partial t}, \frac{\partial^2 F}{\partial t \partial u}\Big\rangle$$
$$= -\Big\langle \frac{\partial^2 F}{\partial s \partial t}, \frac{\partial F}{\partial t}\Big\rangle \cdot \Big\langle \frac{\partial F}{\partial s}, \frac{\partial F}{\partial u}\Big\rangle + \Big\langle \frac{\partial F}{\partial t}, \frac{\partial^2 F}{\partial t \partial u}\Big\rangle = 0,$$

which, together with the fact that the initial velocity of the IVP (5.1) is normal, implies the conclusion of Lemma 5.2. □

By the IVP (5.1) and Lemma 5.2, it is easy to get

$$\begin{cases} \dfrac{\partial F}{\partial t}(u,t) = \sigma(u,t)\vec{\nu}, \\ F(u,0) = F_0(u), \end{cases} \tag{5.2}$$

where $\sigma(u,t) = f(u) + \int_0^t k^{-1}(u,\xi)\,\mathrm{d}\xi$. So, we have

$$\frac{\partial \sigma}{\partial t} = k^{-1}(u,t), \quad \sigma\frac{\partial \sigma}{\partial s} = \Big\langle \frac{\partial^2 F}{\partial s \partial t}, \frac{\partial F}{\partial t}\Big\rangle,$$

where $s = s(\cdot,t)$ is the arc-length parameter of the curve $F(\cdot,t)\colon \mathbb{S}^1 \to \mathbb{R}^2$. By the arc-length formula, we have

$$\frac{\partial}{\partial s} = \frac{1}{\sqrt{(\partial x/\partial u)^2 + (\partial y/\partial u)^2}}\frac{\partial}{\partial u} = \frac{1}{|\partial F/\partial u|}\frac{\partial}{\partial u} := \frac{1}{v}\frac{\partial}{\partial u},$$

where (x,y) are the Cartesian coordinates, and

$$v = \sqrt{\Big(\frac{\partial x}{\partial u}\Big)^2 + \Big(\frac{\partial y}{\partial u}\Big)^2} = \Big|\frac{\partial F}{\partial u}\Big|.$$

For the orthogonal field $\{\vec{\nu}, \vec{T}\}$ of $\mathbb{R}^2$, by the Frenet formula, we have

$$\frac{\partial \vec{T}}{\partial s} = -k\vec{\nu}, \quad \frac{\partial \vec{\nu}}{\partial s} = k\vec{T}. \tag{5.3}$$

Denote by θ the unit inner normal angle for a convex closed curve $F\colon \mathbb{S}^1 \to \mathbb{R}^2$. Then we have

$$\vec{\nu} = (\cos\theta, \sin\theta), \quad \vec{T} = (-\sin\theta, \cos\theta).$$

Together with (5.3), we have

$$\frac{\partial \vec{T}}{\partial s} = \frac{\partial \vec{T}}{\partial \theta}\frac{\partial \theta}{\partial s} = -\vec{\nu}\frac{\partial \theta}{\partial s} = -k\vec{\nu},$$

which implies $\partial\theta/\partial s = k$. Moreover,

$$\frac{\partial \vec{\nu}}{\partial t} = \frac{\partial \vec{\nu}}{\partial \theta}\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial t}\vec{T}, \quad \frac{\partial \vec{T}}{\partial t} = \frac{\partial \vec{T}}{\partial \theta}\frac{\partial \theta}{\partial t} = -\frac{\partial \theta}{\partial t}\vec{\nu}. \tag{5.4}$$

Lemma 5.3. *The derivative of υ with respect to t is $\partial\upsilon/\partial t = k\sigma\upsilon$.*

P r o o f. By direct computation, we have

$$\begin{aligned}\frac{\partial}{\partial t}(\upsilon^2) &= \frac{\partial}{\partial t}\Big\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u}\Big\rangle = 2\Big\langle \frac{\partial F}{\partial u}, \frac{\partial^2 F}{\partial t \partial u}\Big\rangle = 2\Big\langle \Big|\frac{\partial F}{\partial u}\Big|\vec{T}, \frac{\partial}{\partial u}(\sigma\vec{\nu})\Big\rangle \\ &= 2\Big\langle \upsilon\vec{T}, \sigma\frac{\partial \vec{\nu}}{\partial u}\Big\rangle = 2\Big\langle \upsilon\vec{T}, \sigma\frac{\partial \vec{\nu}}{\partial s}\frac{\partial s}{\partial u}\Big\rangle = 2\Big\langle \upsilon\vec{T}, \sigma k\vec{T}\upsilon\Big\rangle = 2\upsilon^2 k\sigma,\end{aligned}$$

which implies the conclusion of Lemma 5.3. $\square$

By Lemma 5.3, we can obtain

$$\frac{\partial^2}{\partial t\partial s} = \frac{\partial}{\partial t}\Big(\frac{1}{\upsilon}\frac{\partial}{\partial u}\Big) = -\frac{1}{\upsilon^2}\frac{\partial \upsilon}{\partial t}\frac{\partial}{\partial u} + \frac{1}{\upsilon}\frac{\partial}{\partial u}\frac{\partial}{\partial t} = -k\sigma\frac{\partial}{\partial s} + \frac{\partial^2}{\partial s\partial t}.$$

Therefore, together with (5.2), we have

$$\frac{\partial \vec{T}}{\partial t} = \frac{\partial^2 F}{\partial t\partial s} = -k\sigma\frac{\partial F}{\partial s} + \frac{\partial^2 F}{\partial s\partial t} = -k\sigma\vec{T} + \frac{\partial}{\partial s}(\sigma\vec{\nu}) = \frac{\partial \sigma}{\partial s}\vec{\nu},$$

which, combined with (5.4), yields

$$\frac{\partial \sigma}{\partial s} = -\frac{\partial \theta}{\partial t}, \quad \frac{\partial \vec{\nu}}{\partial t} = -\frac{\partial \sigma}{\partial s}\vec{T}.$$

Assume $F\colon \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ is a family of convex curves satisfying the flow (5.1). We can use the normal angle to parameterize the evolving curve $F(\cdot,t)$, which will give the notion of the support function used to get the short-time existence of the flow. Set

$$\widetilde{F}(\theta,\tau) = F(u(\theta,\tau), t(\theta,\tau)),$$

where $t(\theta,\tau) = \tau$. By the chain rule, we have

$$0 = \frac{\partial \theta}{\partial \tau} = \frac{\partial \theta}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial \theta}{\partial t},$$

and then

$$\frac{\partial\theta}{\partial t} = -\frac{\partial\theta}{\partial u}\frac{\partial u}{\partial\tau} = -\frac{\partial\theta}{\partial s}\frac{\partial s}{\partial u}\frac{\partial u}{\partial\tau} = -kv\frac{\partial u}{\partial\tau}.$$

Therefore, direct calculation yields

$$\frac{\partial\vec{T}}{\partial\tau} = \frac{\partial\vec{T}}{\partial u}\frac{\partial u}{\partial\tau} + \frac{\partial\vec{T}}{\partial t} = \frac{\partial\vec{T}}{\partial s}\frac{\partial s}{\partial u}\frac{\partial u}{\partial\tau} - \frac{\partial\theta}{\partial t}\vec{\nu} = -\Big(kv\frac{\partial u}{\partial\tau} + \frac{\partial\theta}{\partial t}\Big)\vec{\nu} = 0$$

and

$$\frac{\partial\vec{\nu}}{\partial\tau} = \frac{\partial\vec{\nu}}{\partial u}\frac{\partial u}{\partial\tau} + \frac{\partial\vec{\nu}}{\partial t} = \frac{\partial\vec{\nu}}{\partial s}\frac{\partial s}{\partial u}\frac{\partial u}{\partial\tau} + \frac{\partial\theta}{\partial t}\vec{T} = \Big(kv\frac{\partial u}{\partial\tau} + \frac{\partial\theta}{\partial t}\Big)\vec{T} = 0,$$

which implies $\vec{\nu}$ and $\vec{T}$ are independent of the parameter τ.

Define the support function of the evolving curve $\widetilde{F}(\theta,\tau) = (x(\theta,\tau), y(\theta,\tau))$ as

$$S(\theta,\tau) = \langle\widetilde{F}(\theta,\tau),\vec{\nu}\rangle = x(\theta,\tau)\cos\theta + y(\theta,\tau)\sin\theta.$$

Then we have

$$S_\theta(\theta,\tau) = -x(\theta,\tau)\sin\theta + y(\theta,\tau)\cos\theta = \langle\widetilde{F}(\theta,\tau),\vec{T}\rangle$$

and

$$\begin{cases} x(\theta,\tau) = S\cos\theta - S_\theta\sin\theta, \\ y(\theta,\tau) = S\sin\theta + S_\theta\cos\theta. \end{cases}$$

By direct computation, we have

$$\begin{aligned} S_{\theta\theta} + S &= \langle\widetilde{F}_\theta(\theta,\tau),\vec{T}\rangle + \langle\widetilde{F}(\theta,\tau),-\vec{\nu}\rangle + \langle\widetilde{F}(\theta,\tau),\vec{\nu}\rangle \\ &= \langle\widetilde{F}_\theta(\theta,\tau),\vec{T}\rangle = \Big\langle\frac{\partial F}{\partial u}\frac{\partial u}{\partial s}\frac{\partial s}{\partial\theta},\vec{T}\Big\rangle = \frac{1}{k}. \end{aligned}$$

The above expression makes sense, since the evolving curve is strictly convex.

On the other hand, since $\vec{\nu}$ and $\vec{T}$ are independent of the parameter τ, together with (5.2) and the definition of the support function S we can get

$$S_\tau = \Big\langle\frac{\partial\widetilde{F}}{\partial\tau},\vec{\nu}\Big\rangle = \Big\langle\frac{\partial F}{\partial u}\frac{\partial u}{\partial\tau} + \frac{\partial F}{\partial t},\vec{\nu}\Big\rangle = \Big\langle\frac{\partial F}{\partial t},\vec{\nu}\Big\rangle = \widetilde{\sigma}(\theta,\tau) \tag{5.5}$$

and

$$\begin{aligned} S_{\tau\tau} &= \Big\langle\frac{\partial^2\widetilde{F}}{\partial\tau^2},\vec{\nu}\Big\rangle = \Big\langle\frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial\tau^2} + \frac{\partial^2 F}{\partial u^2}\Big(\frac{\partial u}{\partial\tau}\Big)^2 + 2\frac{\partial^2 F}{\partial u\partial t}\frac{\partial u}{\partial\tau} + \frac{\partial^2 F}{\partial t^2},\vec{\nu}\Big\rangle \\ &= \Big\langle\frac{\partial^2 F}{\partial u^2}\Big(\frac{\partial u}{\partial\tau}\Big)^2 + \frac{\partial^2 F}{\partial u\partial t}\frac{\partial u}{\partial\tau},\vec{\nu}\Big\rangle + \Big\langle\frac{\partial^2 F}{\partial u\partial t}\frac{\partial u}{\partial\tau} + \frac{\partial^2 F}{\partial t^2},\vec{\nu}\Big\rangle \\ &= \frac{\partial u}{\partial\tau}\Big\langle\Big(\frac{\partial F}{\partial u}\Big)_\tau,\vec{\nu}\Big\rangle + \Big\langle\frac{\partial^2 F}{\partial u\partial t}\frac{\partial u}{\partial\tau} + \frac{\partial^2 F}{\partial t^2},\vec{\nu}\Big\rangle \\ &= \Big\langle\frac{\partial^2 F}{\partial u\partial t}\frac{\partial u}{\partial\tau} + \frac{\partial^2 F}{\partial t^2},\vec{\nu}\Big\rangle = \Big\langle\frac{\partial^2 F}{\partial u\partial t}\frac{\partial u}{\partial\tau},\vec{\nu}\Big\rangle + k^{-1}. \end{aligned}$$

Since $F\colon \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ is a normal flow (see Lemma 5.2), which implies

$$\Big\langle \frac{\partial F}{\partial t}, \vec{T} \Big\rangle (u,t) = 0$$

for all $t \in [0,T)$, we have

$$S_{\tau\theta} = \frac{\partial}{\partial \tau}\Big\langle \widetilde{F}, \vec{T} \Big\rangle = \Big\langle \frac{\partial F}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial F}{\partial t}, \vec{T} \Big\rangle = \upsilon \frac{\partial u}{\partial \tau}$$

and

$$S_{\theta\tau} = \frac{\partial}{\partial \theta}\Big\langle \frac{\partial F}{\partial t}, \vec{\nu} \Big\rangle = \Big\langle \frac{\partial^2 F}{\partial u \partial t}\frac{\partial u}{\partial \theta}, \vec{\nu} \Big\rangle = \Big\langle \frac{\partial^2 F}{\partial u \partial t}\frac{\partial u}{\partial s}\frac{\partial s}{\partial \theta}, \vec{\nu} \Big\rangle = \frac{1}{k\upsilon}\Big\langle \frac{\partial^2 F}{\partial u \partial t}, \vec{\nu} \Big\rangle$$

by straightforward computation. Hence, $S(\theta,\tau)$ satisfies

$$S_{\tau\tau} = \Big\langle \frac{\partial^2 F}{\partial u \partial t}\frac{\partial u}{\partial \tau}, \vec{\nu} \Big\rangle + k^{-1} = k\upsilon S_{\theta\tau}\frac{\partial u}{\partial \tau} + k^{-1} = kS_{\theta\tau}^2 + k^{-1},$$

which is equivalent to

$$S_{\tau\tau} = \frac{S_{\theta\tau}^2}{S_{\theta\theta} + S} + (S_{\theta\theta} + S) \quad \forall\, (\theta,\tau) \in \mathbb{S}^1 \times [0,T).$$

Together with (5.1), we know that

$$\begin{cases} SS_{\tau\tau} + S_{\tau\tau}S_{\theta\theta} - S_{\theta\tau}^2 - (S_{\theta\theta} + S)^2 = 0, \\ S(\theta,0) = h(\theta), \\ S_\tau(\theta,0) = \widetilde{f}(\theta), \end{cases} \tag{5.6}$$

where $h(\theta)$ and $\widetilde{f}(\theta)$ are the support functions of the initial curve $F_0(u(\theta))$ and the initial velocity of this initial curve, respectively.

Similarly to the high-dimensional case mentioned in Section 2, here we would like to get the short-time existence of the IVP (5.6) by the linearization method and the standard theory of second-order linear hyperbolic PDEs. Let

$$Q(S_{\theta\theta}, S_{\theta\tau}, S) := \frac{S_{\theta\tau}^2}{S_{\theta\theta} + S} + (S_{\theta\theta} + S),$$

then we have

$$S_{\tau\tau} = \frac{\partial Q}{\partial S_{\theta\theta}}S_{\theta\theta} + \frac{\partial Q}{\partial S_{\theta\tau}}S_{\theta\tau} + \frac{\partial Q}{\partial S}S, \tag{5.7}$$

where

$$\frac{\partial Q}{\partial S_{\theta\theta}} = 1 - \frac{S_{\theta\tau}^2}{(S_{\theta\theta}+S)^2}, \quad \frac{\partial Q}{\partial S_{\theta\tau}} = \frac{2S_{\theta\tau}}{S_{\theta\theta}+S}.$$

Considering the coefficient matrix of the terms in (5.7) involving the second-order derivatives of S, we have

$$\begin{pmatrix} -1 & \dfrac{S_{\theta\tau}}{S_{\theta\theta}+S} \\ \dfrac{S_{\theta\tau}}{S_{\theta\theta}+S} & 1 - \dfrac{S_{\theta\tau}^2}{(S_{\theta\theta}+S)^2} \end{pmatrix}$$

which, by a suitable linear transformation, yields

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So equation (5.7) is a second-order hyperbolic PDE. By the standard theory of second-order linear hyperbolic PDEs (see, e.g., [7], Chapter 7 or [10]), we have the following result.

Theorem 5.4 (Local existence and uniqueness). *Assume that F_0 is a smooth strictly convex closed plane curve. Then there exist a positive $T_{\max} > 0$ and a family of strictly convex closed curves $F(u,t)$ satisfying the IVP* (5.1) *on $\mathbb{S}^1 \times [0, T_{\max})$, provided $f(u)$ is a smooth function on $\mathbb{S}^1$.*

5.2. Expansion and convergence. As in Section 3, we would like to understand further and then try to get more evolution information about the hyperbolic flow (5.1) through an interesting example. In fact, let $F(u,t)$ be a family of round circles in $\mathbb{R}^2$ with the radius $r(t)$ centered at the origin, i.e.,

$$F(u,t) = r(t)(\cos\theta, \sin\theta).$$

Then Example 3.1 (when $n = 1$) describes the convergence or expansion of the evolving curves $F(u,t)$ under the flow (5.1). From this example, we know that although the initial curve is so special (i.e., circles), the evolution of the flow (5.1) is complicated, which deeply depends on the initial values of the flow. However, for a general initial curve $F(u,0)$, it is very difficult to accurately describe the evolution of the HIMCF (5.1) as time tends to the maximal existence time (i.e., as $t \to T_{\max}$). Fortunately, using the containment principle we will derive (see Proposition 5.6 below), we can overcome this difficulty.

In order to get the containment principle, we need to use the maximum principle for a strip (see Lemma 5.5 below) which has been shown in [18]. However, in order to state the conclusion of Lemma 5.5 clearly, we need to introduce some preliminaries first, which has been mentioned in [13]. Consider the general second-order operator L,

$$L[\omega] := a\omega_{\theta\theta} + 2b\omega_{\theta t} + c\omega_{tt} + d\omega_\theta + e\omega_t \tag{5.8}$$

where a, b, c are twice continuously differentiable functions, d, e are continuously differentiable functions of θ and t. If $b^2 - ac > 0$ at a point (θ, t), then the operator L is said to be hyperbolic at this point. It is hyperbolic in a domain D if it is hyperbolic at each point of D, and uniformly hyperbolic in a domain D if there exists a constant μ such that $b^2 - ac \geqslant \mu > 0$ in D.

Assume that ω and the conormal derivative

$$\frac{\partial\omega}{\partial\nu} \triangleq -b\frac{\partial\omega}{\partial\theta} - c\frac{\partial\omega}{\partial t}$$

are given at $t = 0$. The adjoint operator L^* associated with L can be defined by

$$\begin{aligned}
L^*[\omega] &\triangleq (a\omega)_{\theta\theta} + 2(b\omega)_{\theta t} + (c\omega)_{tt} - (d\omega)_\theta - (e\omega)_t \\
&= a\omega_{\theta\theta} + 2b\omega_{\theta t} + c\omega_{tt} + (2a_\theta + 2b_t - d)\omega_\theta + (2b_\theta + 2c_t - e)\omega_t \\
&\quad + (a_{\theta\theta} + 2b_{\theta t} + c_{tt} - d_\theta - e_t)\omega.
\end{aligned}$$

Set

$$\begin{aligned}
K_+(\theta,t) := {}& \big(\sqrt{b^2 - ac}\big)_\theta + \frac{b}{c}\big(\sqrt{b^2 - ac}\big)_\theta + \frac{1}{c}(b_\theta + c_t - e)\sqrt{b^2 - ac} \\
&+ \Big[-\frac{1}{2c}(b^2 - ac)_\theta + a_\theta + b_t - d - \frac{b}{c}(b_\theta + c_t - e)\Big],
\end{aligned}$$

and

$$\begin{aligned}
K_-(\theta,t) := {}& \big(\sqrt{b^2 - ac}\big)_\theta + \frac{b}{c}\big(\sqrt{b^2 - ac}\big)_\theta + \frac{1}{c}(b_\theta + c_t - e)\sqrt{b^2 - ac} \\
&- \Big[-\frac{1}{2c}(b^2 - ac)_\theta + a_\theta + b_t - d - \frac{b}{c}(b_\theta + c_t - e)\Big].
\end{aligned}$$

As shown in [13], pages 502–503, we know that for

$$l(\theta, t) := 1 + \alpha t - \beta t^2 \tag{5.9}$$

with α, β sufficiently large such that

$$\begin{cases}
2\sqrt{b^2 - ac}(\alpha - 2\beta t) + (1 + \alpha t - \beta t^2)K_+ \geqslant 0 \\
2\sqrt{b^2 - ac}(\alpha - 2\beta t) + (1 + \alpha t - \beta t^2)K_- \geqslant 0 \\
-2c\beta + (2b_\theta + 2c_t - e)(\alpha - 2\beta t) \\
\qquad + (a_{\theta\theta} + 2b_{\theta t} + c_{tt} - d_\theta - e_t + g)(1 + \alpha t - \beta t^2) \geqslant 0
\end{cases} \tag{5.10}$$

and $l(\theta, t) > 0$ on a sufficiently small strip $0 \leqslant t \leqslant t_0$, the hyperbolic operator defined by (5.8) satisfies

$$\begin{cases} 2\sqrt{b^2 - ac}\Big[l_t - \dfrac{1}{c}\big(\sqrt{b^2 - ac} - b\big)l_\theta\Big] + lK_+ \geqslant 0 \\ 2\sqrt{b^2 - ac}\Big[l_t + \dfrac{1}{c}\big(\sqrt{b^2 - ac} - b\big)l_\theta\Big] + lK_- \geqslant 0 \\ (L^* + g)[\omega] \geqslant 0 \end{cases}$$

on the same strip $0 \leqslant t \leqslant t_0$. It is easy to check that with l defined by (5.9), the condition on the conormal derivative becomes

$$\frac{\partial \omega}{\partial \nu} + (b_\theta + c_t - e + c\alpha)\omega \leqslant 0$$

at $t = 0$. Besides, if we select a constant M so large that

$$M \geqslant -(b_\theta + c_t - e + c\alpha) \quad \text{on } \Gamma_0, \tag{5.11}$$

then the following maximum principle for the strip adjacent to the θ-axis can be obtained.

Lemma 5.5. *Suppose that the coefficients of the operator L given by* (5.8) *are bounded and have bounded first and second derivatives. Let D be an admissible domain. If t_0 and M are selected in accordance with* (5.10) *and* (5.11), *then any function ω which satisfies*

$$\begin{cases} (L + g)[\omega] \geqslant 0 & \text{in } D, \\ \dfrac{\partial \omega}{\partial \nu} - M\omega \leqslant 0 & \text{on } \Gamma_0, \\ \omega \leqslant 0 & \text{on } \Gamma_0, \end{cases}$$

also satisfies $\omega \leqslant 0$ in the part of D which lies in the strip $0 \leqslant t \leqslant t_0$. The constants t_0 and M depend only on lower bounds for $-c$ and $\sqrt{b^2 - ac}$ and on bounds for the coefficients of L and their derivatives.

Proposition 5.6 (Containment principle)**.** *Let F_1 and $F_2\colon \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be two convex solutions of* (5.1). *Suppose that $F_2(u, 0)$ lies in the domain enclosed by $F_1(u, 0)$, and $f_2(u) \leqslant f_1(u)$. Then $F_2(u, t)$ is contained in the domain enclosed by $F_1(u, t)$ for all $t \in [0, T)$.*

Proof. Let $S_1(\theta, t)$ and $S_2(\theta, t)$ be the support functions of $F_1(u, t)$ and $F_2(u, t)$, respectively. Then $S_1(\theta, t)$ and $S_2(\theta, t)$ satisfy the same equation (5.6) with $S_2(\theta, 0) \leqslant S_1(\theta, 0)$ and $S_{2t}(\theta, 0) \leqslant S_{1t}(\theta, 0)$.

Let

$$\omega(\theta,t) := S_2(\theta,t) - S_1(\theta,t).$$

Then we have

$$\begin{aligned}
\omega_{tt} = S_{2tt} - S_{1tt} &= \frac{S_{2\theta t}^2 + k_2^{-2}}{S_2 + S_{2\theta\theta}} - \frac{S_{1\theta t}^2 + k_1^{-2}}{S_1 + S_{1\theta\theta}} \\
&= k_1k_2\Big(\frac{1}{k_1k_2} - S_{1\theta t}S_{2\theta t}\Big)\omega_{\theta\theta} + (k_1S_{1\theta t} + k_2S_{2\theta t})\omega_{\theta t} + k_1k_2\Big(\frac{1}{k_1k_2} - S_{1\theta t}S_{2\theta t}\Big)\omega,
\end{aligned}$$

which implies that ω satisfies the system

$$\tag{5.12} \begin{cases}
\omega_{tt} = k_1k_2\Big(\dfrac{1}{k_1k_2} - S_{1\theta t}S_{2\theta t}\Big)\omega_{\theta\theta} + (k_1S_{1\theta t} + k_2S_{2\theta t})\omega_{\theta t} \\
\qquad\qquad + k_1k_2\Big(\dfrac{1}{k_1k_2} - S_{1\theta t}S_{2\theta t}\Big)\omega, \\
\omega_t(\theta,0) = f_2(\theta) - f_1(\theta) = \omega_1(\theta), \\
\omega(\theta,0) = h_2(\theta) - h_1(\theta) = \omega_0(\theta).
\end{cases}$$

Define the operator L by

$$L[\omega] := k_1k_2\Big(\frac{1}{k_1k_2} - S_{1\theta t}S_{2\theta t}\Big)\omega_{\theta\theta} + (k_1S_{1\theta t} + k_2S_{2\theta t})\omega_{\theta t} - \omega_{tt},$$

we know that

$$a = k_1k_2\Big(\frac{1}{k_1k_2} - S_{1\theta t}S_{2\theta t}\Big), \quad b = \frac{1}{2}(k_1S_{1\theta t} + k_2S_{2\theta t}), \quad c = -1$$

are twice continuously differentiable functions of θ and t. By direct computation, we have

$$\begin{aligned}
b^2 - ac &= \frac{1}{4}(k_1S_{1\theta t} + k_2S_{2\theta t})^2 - k_1k_2\Big(\frac{1}{k_1k_2} - S_{1\theta t}S_{2\theta t}\Big)\cdot(-1) \\
&= \frac{1}{4}(k_1S_{1\theta t} - k_2S_{2\theta t})^2 + 1 > 0.
\end{aligned}$$

Hence, the operator L is uniformly hyperbolic in $\mathbb{S}^1 \times [0,T)$. By Lemma 5.5, we deduce that $S_2(\theta,t) \leqslant S_1(\theta,t)$ for all $t \in [0,T)$, which completes the proof. $\square$

Proposition 5.7 (Preserving convexity). *Let $k_0(\theta)$ be the curvature function of F_0 and*

$$\delta = \min_{\theta\in[0,2\pi]} k_0(\theta) > 0.$$

Then for a C^4-solution S of (5.6) we have

$$k(\theta,t) \geqslant \delta$$

for all $t \in [0,T_{\max})$, where $[0,T_{\max})$ is the maximal time interval for the solution $F(\cdot,t)$ of (5.1).

P r o o f. Since the initial curve is strictly convex, by Theorem 5.4 we know that the solution of (5.6) remains strictly convex on some short time interval $[0, T)$ with some $T \leqslant T_{\max}$ and its support function satisfies

$$S_{tt} = kS_{\theta t}^2 + k^{-1}$$

for all $(\theta, t) \in \mathbb{S}^1 \times [0, T)$. Taking derivative with respect to t, we have

$$k_t = \Big(\frac{1}{S + S_{\theta\theta}}\Big)_t = -\frac{1}{(S + S_{\theta\theta})^2}(S_t + S_{\theta\theta t}) = -k^2(S_t + S_{\theta\theta t}).$$

Together with the fact $S_t = \widetilde{\sigma}$, it is easy to know $k_t = -k^2(\widetilde{\sigma} + \widetilde{\sigma}_{\theta\theta})$. Therefore, we can obtain

$$\begin{aligned}
S_t + S_{\theta\theta t} &= -(S + S_{\theta\theta})^2 k_t = -\frac{1}{k^2}k_t,\\
S_{\theta t} + S_{\theta\theta\theta t} &= \Big(-\frac{1}{k^2}k_t\Big)_\theta = \frac{2}{k^3}k_t k_\theta - \frac{1}{k^2}k_{\theta t},
\end{aligned}$$

and

$$\begin{aligned}
k_{tt} &= \Big(-\frac{1}{(S + S_{\theta\theta})^2}(S_t + S_{\theta\theta t})\Big)_t\\
&= \frac{2}{(S + S_{\theta\theta})^3}(S_t + S_{\theta\theta t})^2 - \frac{1}{(S + S_{\theta\theta})^2}(S_{tt} + S_{\theta\theta tt})\\
&= 2k^3\Big(-\frac{1}{k^2}k_t\Big)^2 - k^2[(S_{tt})_{\theta\theta} + S_{tt}]\\
&= \frac{2}{k}k_t^2 - k^2[(kS_{\theta t}^2 - k + k + k^{-1})_{\theta\theta} + (kS_{\theta t}^2 - k + k + k^{-1})]\\
&= \frac{2}{k}k_t^2 - k^2[((S_{\theta t}^2 - 1)k)_{\theta\theta} + (S_{\theta t}^2 - 1)k + (k + k^{-1})_{\theta\theta} + (k + k^{-1})]\\
&= \frac{2}{k}k_t^2 - k^2[((S_{\theta t}^2 - 1)_\theta k + (S_{\theta t}^2 - 1)k_\theta)_\theta + (S_{\theta t}^2 - 1)k + (k + k^{-1})_{\theta\theta} + (k + k^{-1})]\\
&= \frac{2}{k}k_t^2 - k^2[(S_{\theta t}^2 - 1)_{\theta\theta}k + 2(S_{\theta t}^2 - 1)_\theta k_\theta + (S_{\theta t}^2 - 1)k_{\theta\theta} + (S_{\theta t}^2 - 1)k]\\
&\quad - k^2[(k + k^{-1})_{\theta\theta} + (k + k^{-1})]\\
&= \frac{2}{k}k_t^2 - k^2(S_{\theta t}^2 - 1)(k + k_{\theta\theta}) - k^2[(2S_{\theta t}S_{\theta\theta t})_\theta k + 4S_{\theta t}S_{\theta\theta t}k_\theta]\\
&\quad - k^2\Big[k_{\theta\theta} - \frac{1}{k^2}k_{\theta\theta} + \frac{2}{k^3}k_\theta^2 + (k + k^{-1})\Big]\\
&= \frac{2}{k}k_t^2 - k^2(S_{\theta t}^2 - 1)(k + k_{\theta\theta}) - k^2[2(S_{\theta\theta t}^2 + S_{\theta t}S_{\theta\theta\theta t})k + 4k_\theta S_{\theta t}(S_{\theta\theta} + S - S)_t]\\
&\quad - k^2\Big[\Big(1 - \frac{1}{k^2}\Big)k_{\theta\theta} + \frac{2}{k^3}k_\theta^2 + (k + k^{-1})\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{k}k_t^2 - k^2(S_{\theta t}^2 - 1)(k + k_{\theta\theta}) - k^2 \\
&\quad \times \Big[2((S_{\theta\theta t} + S_t)^2 - 2S_{\theta\theta t}S_t - S_t^2 + S_{\theta t}(S_{\theta\theta} + S)_{\theta t} - S_{\theta t}^2)k + 4k_\theta S_{\theta t}\Big(\frac{1}{k} - S\Big)_t\Big] \\
&\quad - k^2\Big[\Big(1 - \frac{1}{k^2}\Big)k_{\theta\theta} + \frac{2}{k^3}k_\theta^2 + (k + k^{-1})\Big] \\
&= \frac{2}{k}k_t^2 - k^2(S_{\theta t}^2 - 1)(k + k_{\theta\theta}) - k^2 \\
&\quad \times \Big[2\Big((S_{\theta\theta t} + S_t)^2 - 2(S_{\theta\theta t} + S_t)S_t + S_t^2 + S_{\theta t}\Big(\frac{1}{k}\Big)_{\theta t} - S_{\theta t}^2\Big)k - 4k_\theta S_{\theta t}\frac{1}{k^2}k_t \\
&\qquad - 4k_\theta S_{\theta t}S_t\Big] - k^2\Big[\Big(1 - \frac{1}{k^2}\Big)k_{\theta\theta} + \frac{2}{k^3}k_\theta^2 + (k + k^{-1})\Big] \\
&= \frac{2}{k}k_t^2 - k^2(S_{\theta t}^2 - 1)(k + k_{\theta\theta}) - 2k^3 \\
&\quad \times \Big[\Big(-\frac{1}{k^2}k_t\Big)^2 - 2\Big(-\frac{1}{k^2}k_t\Big)S_t + S_t^2 - S_{\theta t}^2 + S_{\theta t}\Big(\frac{2}{k^3}k_t k_\theta - \frac{1}{k^2}k_{\theta t}\Big)\Big] \\
&\quad + 4k^2\Big(k_\theta S_{\theta t}\frac{1}{k^2}k_t + k_\theta S_{\theta t}S_t\Big) - k^2\Big[\Big(1 - \frac{1}{k^2}\Big)k_{\theta\theta} + \frac{2}{k^3}k_\theta^2 + (k + k^{-1})\Big] \\
&= k^2(1 - S_{\theta t}^2)(k + k_{\theta\theta}) - 4kS_t k_t - 2k^3S_t^2 + 2k^3S_{\theta t}^2 + 2kS_{\theta t}k_{\theta t} + 4k^2S_{\theta t}S_t k_\theta \\
&\quad - k^2\Big[\Big(1 - \frac{1}{k^2}\Big)k_{\theta\theta} + \frac{2}{k^3}k_\theta^2 + (k + k^{-1})\Big] \\
&= k^2\Big(\frac{1}{k^2} - S_{\theta t}^2\Big)k_{\theta\theta} + 2kS_{\theta t}k_{\theta t} + 4k^2S_{\theta t}S_t k_\theta - \frac{2}{k}k_\theta^2 \\
&\quad - 4kS_t k_t + k^3(S_{\theta t}^2 - 2S_t^2 - k^{-2}).
\end{aligned}
$$

So, the curvature k satisfies the equation

$$
k_{tt} = k^2\Big(\frac{1}{k^2} - S_{\theta t}^2\Big)k_{\theta\theta} + 2kS_{\theta t}k_{\theta t} + 4k^2S_{\theta t}S_t k_\theta - \frac{2}{k^3}k_\theta^2 - 4kS_t k_t + k^3(S_{\theta t}^2 - 2S_t^2 - k^{-2}).
$$

Define the operator L as

$$
L[k] := k^2\Big(\frac{1}{k^2} - S_{\theta t}^2\Big)k_{\theta\theta} + 2kS_{\theta t}k_{\theta t} - k_{tt} + 4k^2S_{\theta t}S_t k_\theta - \frac{2}{k^3}k_\theta^2 - 4kS_t k_t.
$$

We know that

$$
a = k^2\Big(\frac{1}{k^2} - S_{\theta t}^2\Big), \quad b = kS_{\theta t}, \quad c = -1
$$

are twice continuously differentiable functions of θ and t. So we have

$$
b^2 - ac = (kS_{\theta t})^2 - k^2\Big(\frac{1}{k^2} - S_{\theta t}^2\Big)\cdot(-1) = 1 > 0,
$$

which implies that the operator L is hyperbolic in $\mathbb{S}^1 \times [0, T)$.

Determining a function $k(\theta,t)$ which satisfies the system

$$\begin{cases} (L+\widetilde{h})[k]=0 & \text{in } \mathbb{S}^1\times[0,T), \\ k(\theta,0)=k_0(\theta) & \text{on } \Gamma_0, \\ 0\leqslant \dfrac{\partial k}{\partial \nu}:=-bk_\theta-ck_t:=\beta(\theta) & \text{on } \Gamma_0, \end{cases}$$

where the operator $\widetilde{h}$ is defined as $\widetilde{h}[k]:=k^3(S_{\theta t}^2-2S_t^2-k^{-2})$. It is easy to check that the function $\widetilde{k}(\theta,t)=\min\limits_{\theta\in[0,2\pi]}\{k_0(\theta)\}=\delta$ satisfies

$$\begin{cases} (L+\widetilde{h})[\widetilde{k}]=0 & \text{in } \mathbb{S}^1\times[0,T), \\ \widetilde{k}(\theta,0)\leqslant k_0(\theta) & \text{on } \Gamma_0, \\ \dfrac{\partial \widetilde{k}}{\partial \vec{\nu}}-M\widetilde{k}\leqslant \beta(\theta)-Mk_0(\theta) & \text{on } \Gamma_0, \end{cases}$$

where Γ_0 is the initial domain, and M is the constant determined by (5.11). Applying Lemma 5.5 to $\widetilde{k}-k$ yields

$$\widetilde{k}\leqslant k(\theta,t) \quad \text{in } \mathbb{S}^1\times[0,t_0).$$

with $t_0\leqslant T$. Hence, we know that the solution $F(\cdot,t)$ remains convex on $[0,T_{\max})$ and its curvature function $k(\theta,t)$ has a uniformly positive lower bound $\delta=\min\limits_{\mathbb{S}^1}k_0(\theta)$ on $\mathbb{S}^1\times[0,T_{\max})$, which completes the proof. □

We need the following evolution equations of the arc-length of evolving curves.

Lemma 5.8. *The arc-length $\mathcal{L}(t)$ of the closed curve $F(u,t)$ satisfies*

$$\frac{\mathrm{d}\mathcal{L}(t)}{\mathrm{d}t}=\int_0^{2\pi}\widetilde{\sigma}(\theta,t)\,\mathrm{d}\theta,$$

and

$$\frac{\mathrm{d}^2\mathcal{L}(t)}{\mathrm{d}t^2}=\int_0^{2\pi}\Big[k\Big(\frac{\partial\widetilde{\sigma}}{\partial\theta}\Big)^2+k^{-1}\Big]\,\mathrm{d}\theta.$$

Proof. Since

$$\mathcal{L}(t)=\int_0^{2\pi}\upsilon(\theta,t)\,\mathrm{d}\theta,$$

$S_t(\theta,t)=\widetilde{\sigma}(\theta,t)$ (i.e., the equality (5.5)) and $\partial\upsilon/\partial t=k\upsilon\widetilde{\sigma}$, by direct calculation, we have

$$\frac{\mathrm{d}\mathcal{L}(t)}{\mathrm{d}t}=\int_0^{2\pi}\frac{\partial\upsilon}{\partial t}\,\mathrm{d}\theta=\int_0^{2\pi}k\upsilon\widetilde{\sigma}\,\mathrm{d}\theta=\int_0^{2\pi}\widetilde{\sigma}(\theta,t)\,\mathrm{d}\theta,$$

and

$$
\begin{aligned}
\frac{\mathrm{d}^2\mathcal{L}(t)}{\mathrm{d}t^2} &= \int_0^{2\pi} \frac{\partial}{\partial t}\widetilde{\sigma}(\theta,t)\,\mathrm{d}\theta = \int_0^{2\pi} S_{tt}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} (kS_{\theta t}^2 + k^{-1})\,\mathrm{d}\theta = \int_0^{2\pi} \left[k\Big(\frac{\partial}{\partial\theta}S_t\Big)^2 + k^{-1}\right]\mathrm{d}\theta \\
&= \int_0^{2\pi} \left[k\Big(\frac{\partial\widetilde{\sigma}}{\partial\theta}\Big)^2 + k^{-1}\right]\mathrm{d}\theta,
\end{aligned}
$$

which completes the proof of Lemma 5.8. $\square$

From Example 3.1 (when $n = 1$), we know that the behavior of evolving plane curves of HIMCF (5.1) is complicated. However, using Propositions 5.6 and 5.7, Lemma 5.8, we can get the following conclusion about the asymptotic behavior of the hyperbolic flow (5.1).

Theorem 5.9. *Suppose that F_0 is a smooth strictly convex closed plane curve with the curvature function $k_0(\theta)$ whose minimum and maximum are given by $\delta = \min_{\mathbb{S}^1} k_0(\theta) > 0$ and $\zeta := \max_{\mathbb{S}^1} k_0(\theta)$, respectively. Then there exists a family of strictly convex closed plane curves $F(\cdot,t)$ satisfying the IVP (5.1) on the time interval $[0,T_{\max})$ with $0 < T_{\max} \leqslant \infty$. Moreover, we have:*

(I) *If $\zeta^{-1} + \min_{u\in\mathbb{S}^1} f(u) > 0$, then $T_{\max} = \infty$, i.e., the flow exists for all the time.*

(II) *If $\delta^{-1} + \max_{u\in\mathbb{S}^1} f(u) < 0$, then $T_{\max} < \infty$. Moreover, if furthermore $\delta^{-1}T_{\max} + \max_{u\in\mathbb{S}^1} f(u) < 0$, then as $t \to T_{\max}$, one of the following conclusions must be true:*

- ▷ *the solution $F(\cdot,t)$ converges to a point as $t \to T_{\max}$, i.e., the curvature of the limit curve becomes unbounded as $t \to T_{\max}$;*
- ▷ *the curvature k of the evolving curve is discontinuous as $t \to T_{\max}$, so the solution $F(\cdot,t)$ converges to a piecewise smooth curve.*

Remark 5.10. In Case (II) of Theorem 5.9 above, the condition $\delta^{-1}T_{\max} + \max_{u\in\mathbb{S}^1} f(u) < 0$ is not easy to check, since for a general strictly convex closed plane curve evolving under the hyperbolic flow (5.1), it is difficult to get the accurate value of the maximal time $T_{\max}$. However, as shown in the proof below, by Example 3.1 and Proposition 5.6 (Containment principle) we have

$$
T_{\max} \leqslant T^* = \frac{1}{2}\ln\left(\frac{-1 + \delta\max_{u\in\mathbb{S}^1} f(u)}{1 + \delta\max_{u\in\mathbb{S}^1} f(u)}\right).
$$

So, for the purpose of easier checking, one can use a weaker condition $\delta^{-1}T^* + \max_{u\in\mathbb{S}^1} f(u) < 0$ to replace the assumption $\delta^{-1}T_{\max} + \max_{u\in\mathbb{S}^1} f(u) < 0$. However, here we prefer to use the latter one, since it is sharper than the former.

P r o o f. Let $[0, T_{\max})$ be the maximal time interval of the IVP (5.1) with F_0 and f as the initial curve and initial velocity of the initial curve, respectively.

By Proposition 5.7, we know that the solution $F(\cdot, t)$ remains strictly convex on $[0, T_{\max})$ and the curvature of $F(\cdot, t)$ has a uniformly positive lower bound $\delta > 0$ on $\mathbb{S}^1 \times [0, T_{\max})$.

Case I: When $\zeta^{-1} + \min\limits_{u \in \mathbb{S}^1} f(u) > 0$.

Since $\zeta = \max\limits_{\mathbb{S}^1} k_0(\theta) \geqslant \delta > 0$, the initial curve F_0 can enclose a circle $\mathcal{C}_0$ with radius ζ^{-1}. Let the normal initial velocity of $\mathcal{C}_0$ be equal to $\min\limits_{u \in \mathbb{S}^1} f(u)$. Evolve $\mathcal{C}_0$ by the hyperbolic flow (5.1) to get a solution $\mathcal{C}(\cdot, t)$. By Example 3.1, we know that if $\zeta^{-1} + \min\limits_{u \in \mathbb{S}^1} f(u) > 0$, the evolving circle $\mathcal{C}(\cdot, t)$ exists for all the time, and its radius tends to infinity as $t \to \infty$. By Proposition 5.6, we can get that $\mathcal{C}(\cdot, t)$ always lies in the domain $\mathcal{D}$ enclosed by the closed curve $F(\cdot, t)$ for all $t \geqslant 0$, and moreover, $\mathcal{D}$ tends to be the whole plane as $t \to \infty$. So, in this case, the IVP (5.1) has the long-time existence, i.e., $T_{\max} = \infty$.

Case II: When $\delta^{-1} + \max\limits_{u \in \mathbb{S}^1} f(u) < 0$.

Since $\delta = \min\limits_{\mathbb{S}^1} k_0(\theta) > 0$, the initial curve F_0 can be enclosed by a circle $\mathcal{C}_1$ with radius δ^{-1}. Let the normal initial velocity of $\mathcal{C}_1$ be equal to $\max\limits_{u \in \mathbb{S}^1} f(u)$. Evolve $\mathcal{C}_1$ by the hyperbolic flow (5.1) to get a solution $\widetilde{\mathcal{C}}(\cdot, t)$. By Example 3.1, we know that if $\delta^{-1} + \max\limits_{u \in \mathbb{S}^1} f(u) < 0$, the solution exists at a finite time interval $[0, T^*)$ and the evolving circle $\widetilde{\mathcal{C}}(\cdot, t)$ converges to a single point as $t \to T^*$. By Proposition 5.6, we know that the evolving curve $F(\cdot, t)$ always lies in the domain $\widetilde{\mathcal{D}}$ (i.e., a disk) enclosed by $\widetilde{\mathcal{C}}(\cdot, t)$ for all $t \in [0, T^*)$. Hence, we can get that $F(\cdot, t)$ must become singular at some time $T_{\max} \leqslant T^* < \infty$.

Now, we need the following conclusion from convex geometry (see, e.g., [20]).

Blaschke Selection Theorem. *Let K_j be a sequence of convex sets which are contained in a bounded set. Then there exists a subsequence K_{jk} and a convex set K such that K_{jk} converges to K in the Hausdorff metric.*

In Case II, since $\widetilde{\mathcal{C}}(\cdot, t)$ shrinks as t increases and the evolving curve $F(\cdot, t)$ is contained in the circle $\widetilde{\mathcal{C}}(\cdot, t)$ for each $t \in [0, T_{\max})$, this strictly convex closed plane curve $F(\cdot, t)$ is contained in the circle $\mathcal{C}_1$ for all $t \in [0, T_{\max})$. By Blaschke Selection Theorem, we know that in the sense of the *Hausdorff metric*, $F(\cdot, t)$ converges to a weakly convex curve $F(\cdot, T_{\max})$ which may be degenerate and non-smooth.

We claim that $F(\cdot, t)$ *converges to either a single point or a limit curve which has a discontinuous curvature under the further assumption* $\delta^{-1} T_{\max} + \max\limits_{u \in \mathbb{S}^1} f(u) < 0$.

By Proposition 5.7 and Lemma 5.8, we have

$$\frac{\mathrm{d}^2\mathcal{L}(t)}{\mathrm{d}t^2} = \int_0^{2\pi} \left[k\left(\frac{\partial\widetilde{\sigma}}{\partial\theta}\right)^2 + k^{-1}\right] \mathrm{d}\theta > 0 \quad \text{for all } t \in [0, T_{\max}).$$

Besides, by Proposition 5.7, we have

$$\begin{aligned}\widetilde{\sigma}(\theta, t) = \sigma(u, t) = f(u) + \int_0^t k^{-1}(u, \xi)\,\mathrm{d}\xi \leqslant \delta^{-1}t + \max_{u\in\mathbb{S}^1} f(u) \\ \leqslant \delta^{-1}T_{\max} + \max_{u\in\mathbb{S}^1} f(u) < 0\end{aligned}$$

for all $t \in [0, T_{\max})$, which implies

$$\frac{\mathrm{d}\mathcal{L}(t)}{\mathrm{d}t} = \int_0^{2\pi} \widetilde{\sigma}(\theta, t)\,\mathrm{d}\theta < 0 \quad \text{for all } t \in [0, T_{\max}).$$

So, for all $t \in [0, T_{\max})$, we have

$$\frac{\mathrm{d}\mathcal{L}(t)}{\mathrm{d}t} < 0, \qquad \frac{\mathrm{d}^2\mathcal{L}(t)}{\mathrm{d}t^2} > 0,$$

which implies that there exists a finite time T_0 such that $\mathcal{L}(T_0) = 0$. The following two situations may occur:

- ▷ $T_0 \leqslant T_{\max}$. On the one hand, by Theorem 5.4 there exists a unique classical solution $F(\cdot, t)$ to the IVP (5.1) on $[0, T_0)$. On the other hand, since $\mathcal{L}(t)$ is decreasing on $[0, T_0)$ and $\mathcal{L}(T_0) = 0$, we have $\mathcal{L}(T_0) \to 0$ as $t \to T_0$. This implies the curvature k tends to infinity as $t \to T_0$, and the solution will blow up at T_0. Therefore, by the definition of $T_{\max}$, we have $T_0 = T_{\max}$. So, $F(\cdot, t)$ converges to a point as $t \to T_{\max}$.
- ▷ $T_0 > T_{\max}$. In this situation, $\mathcal{L}(T_{\max}) > 0$, which implies that $F(\cdot, T_{\max})$ must be non-smooth. Then there are three possibilities:
  (1) $\|F(u, T_{\max})\| = \sup |F(u, T_{\max})| = \infty$. However, $F(\cdot, t)$ is always contained in the circle $\mathcal{C}_1$, which implies that $\|F(u, T_{\max})\|$ must be bounded. This is a contradiction. So, case (1) is impossible.
  (2) $\|F_u(u, T_{\max})\| = \infty$. However, the length of the limit curve $\mathcal{L}(T_{\max})$ satisfies

$$\begin{aligned}\mathcal{L}(T_{\max}) &= \lim_{t\to T_{\max}} \int_{F(u,t)} \mathrm{d}s = \lim_{t\to T_{\max}} \int_{F(u,t)} |F_u(u, t)|\,\mathrm{d}u \\ &= \int_{F(u,t)} \lim_{t\to T_{\max}} |F_u(u, t)|\,\mathrm{d}u = \infty\end{aligned}$$

  which contradicts $\mathcal{L}(T_{\max}) < \mathcal{L}_0$ with $\mathcal{L}_0$ being the length of the initial curve F_0. So, case (2) is also impossible.

(3) The curvature function k is discontinuous. We cannot exclude this possibility. This phenomenon will occur if the above shocks are not possible.

Our claim is true. The proof of Theorem 5.9 is finished. □

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