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FAULT ESTIMATION FOR TIME-VARYING SYSTEMS WITH ROUND–ROBIN PROTOCOL

Haijing Fu, Hongli Dong, Jinbo Song, Nan Hou and Gongfa Li

This paper is concerned with the design problem of finite-horizon H_∞ fault estimator for a class of nonlinear time-varying systems with Round–Robin protocol scheduling. The faults are assumed to occur in a random way governed by a Bernoulli distributed white sequence. The communication between the sensor nodes and fault estimators is implemented via a shared network. In order to prevent the data from collisions, a Round-Robin protocol is utilized to orchestrate the transmission of sensor nodes. By means of the stochastic analysis technique and the completing squares method, a necessary and sufficient condition is established for the existence of fault estimator ensuring that the estimation error dynamics satisfies the prescribed H_∞ constraint. The time-varying parameters of fault estimator are obtained by recursively solving a set of coupled backward Riccati difference equations. A simulation example is given to demonstrate the effectiveness of the proposed design scheme of the fault estimator.

Keywords: fault estimation, Round–Robin protocol, randomly occurring faults, Riccati difference equations, nonlinear time-varying system

Classification: 93C10, 93E10

1. INTRODUCTION

In the past decades, fault diagnosis and fault-tolerant control problems [18, 30] have been a more and more popular research topic due primarily to the growing demand for higher performance, higher security and stricter reliability in an automated control system. Fault estimation, as a crucial stage for the implementation of the desired fault detection, is capable of providing the accurate size and shape of the fault and then helping reconstruct the fault signals. As such, fault estimation is further needed for the purpose of active fault-tolerant control. So far, considerable research attention has been devoted to the theoretical research on the fault estimation problem, and a number of fault estimation schemes have been proposed in the existing literature, see e. g. [6, 7, 8, 9, 14, 22, 33, 37] and the references therein. Besides, in reality, in response to the increasing complexity of industrial systems, the time-varying nature [26, 27] has gradually been an indispensable way of reflecting the fast changes in dynamic systems. For example, in [8], the finite-horizon fault estimation problem has been presented for a class of nonlinear time-varying systems with randomly occurring faults (ROFs).

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With the rapid development of network technologies, severe yet complex network-induced problems have taken place such as data collisions/congestion, which may cause a negative impact on the system performance. It has been recognized that the protocol scheduling can deal with these issues and prevent a waste of the communication resource, then a great deal of research attention has been paid to the study of protocols. Among various communication protocols, the basic principle is to permit one sensor node/sensor to access the communication network to transmit data at each time step, and their respective scheduling rules are different (e. g. the specified ordered rule for the Round–Robin protocol [1, 10, 29, 35], the weight computation and comparison rule for the Weighted-Try-Once-Discard protocol [25, 34, 36], the random selection rule for the Random Access protocol [37]). Up to now, the analysis and synthesis problems of networked systems under various communication protocols have begun to attract some research interest, and some research results have been obtained about the communication protocols which are widely employed in practice. Particularly, the Round–Robin protocol, also known as the periodic protocol, has been widely used in communication and signal processing communities. At each transmission instant, depending on a predetermined circular order, the RR protocol determines whether or not a node is accessible to the network.

In fact, in networked control systems, due to bandwidth limitation of the shared links and unpredictable variation of the network conditions, some networked-induced intermittent phenomena may occur, including packet loss, missing measurements, communication delays, signal quantization, channel fading, failure of the sensor/actuators or data collision [15, 16, 28, 31, 32]. Also, the ROFs [8, 9] are regarded as a class of networked-induced phenomena, which may stem from the abrupt changes of the communication networks, and influence the reliability of the system dynamics. Then, how to execute the processes of fault detection, fault isolation, fault estimation or fault-tolerant control has become a concern in order to reduce the impact from different kinds of faults. It is known that such faults typically occur in a probabilistic way, then it would make practical sense to consider the random occurrence of faults where the occurrence probability can be estimated via statistical tests. It is worth mentioning that, up to now, the finite-horizon H_∞ fault estimation problem with randomly occurring faults and Round–Robin protocol has not been taken into account adequately yet, not to refer to the case that necessary and sufficient conditions are established to ensure the H_∞ fault estimator can exist, which becomes the main purpose of this paper.

Based on the above discussions, we aim to investigate the fault estimation problem for a class of nonlinear time-varying systems with a Round–Robin protocol. The main contributions of this paper are highlighted as follows:

1) the plant model addressed is quite comprehensive which covers time-varying parameters, stochastic nonlinearity with known statistical characteristics, ROFs as well as RR protocol, hence reflecting the reality more closely;

2) the finite-horizon estimation problem is, for the first time, settled for nonlinear time-varying systems with ROFs and RR protocol, where the faults are assumed to be incipient faults and abrupt faults;

3) a necessary and sufficient condition is obtained for the existence of the desired fault estimator by utilizing completing squares method and stochastic analysis technology.

Notation: Throughout this paper, $\mathbb{R}^n$ means the n dimensional Euclidean space. For a matrix B, B^T, $\mathrm{tr}(B)$ and $\|B\|_F$ refer to the transpose, the trace and the Frobenius norm, respectively. The notation $A > 0$, where A is a real symmetric matrix, means that A is positive definite. $N^\dagger \in \mathbb{R}^{n\times m}$ denotes the Moore-Penrose pseudo inverse of matrix $N \in \mathbb{R}^{m\times n}$, and $Y^{-1} \in \mathbb{R}^{m\times m}$ means the inverse of $Y \in \mathbb{R}^{m\times m}$. $\mathbb{E}\{y\}$ denotes the mathematical expectation of the random variable y about the known probability measure Prob. $l_2([0, T_1], \mathbb{R}^m)$ is the space of nonanticipatory square-summable m-dimensional vector-valued functions. In the symmetric matrix, $*$ represents the symmetric term. $\mathbf{1}_n = [1, 1, \ldots, 1]^T \in \mathbb{R}^n$. $I_{(m)}$ and $0_{m\times n}$ are the $m \times m$ identity matrix and the $m \times n$ zero matrix, respectively.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following class of discrete time-varying nonlinear stochastic systems defined on the finite horizon $k \in [0, N]$:

$$\begin{cases} x(k+1) = g(k, x(k)) + A(k)x(k) + \alpha(k)B(k)f(k) + E(k)w(k) \\ \quad\; y(k) = C(k)x(k) + D(k)v(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ represents the state vector; $y(k) \in \mathbb{R}^{n_y}$ is the process output; $w(k) \in \mathbb{R}^{n_w}$, $v(k) \in \mathbb{R}^{n_v}$ and $f(k) \in \mathbb{R}^{n_f}$ are, respectively, the disturbance input, the measurement noises and the fault signal, all of which belong to $l_2[0, N]$. $A(k)$, $B(k)$, $E(k)$, $C(k)$ and $D(k)$ are known time-varying matrices with proper dimensions.

We assume that the stochastic nonlinear function $g(k, x(k))$ with $g(k, 0) = 0$ is described by the following statistical characteristics:

$$\begin{aligned} &\mathbb{E}\left\{g(k, x(k))\Big|x(k)\right\} = 0 \\ &\mathbb{E}\left\{g(j, x(j))g^T(k, x(k))\big|x(k)\right\} = 0,\ k \neq j \\ &\mathbb{E}\left\{g(k, x(k))g^T(k, x(k))\big|x(k)\right\} \triangleq \sum_{l=1}^{q} \Theta_l(k)\mathbb{E}\left\{x^T(k)\bar{\Gamma}_l(k)x(k)\right\} \end{aligned} \tag{2}$$

where $\Theta_l(k)$ and $\bar{\Gamma}_l(k)$ are known matrices with appropriate dimensions, and q is the number of independent state components.

Remark 2.1. It is now widely recognized that, nonlinearities may have their existence in practical time-varying control systems, which lead to extra challenges when coping with the analysis/design issues. So far, several different types of nonlinearities appearing in the network nodes have been extensively studied in the literature which include, but are not limited to, Lipschitz-type nonlinearity [9], general sector-like nonlinearity [13, 19], nonlinearity subject to inequality constraints [24] and stochastic nonlinearity with known statistical characteristics [8]. In this paper, we adopt stochastic nonlinearity with known statistical characteristics in stochastic systems (1).

The variable $\alpha(k)$ reflects the phenomenon of randomly occurring faults, which is assumed to be a Bernoulli distributed white sequence taking values on 0 or 1 with

$$\mathrm{Prob}\{\alpha(k) = 0\} = 1 - \bar{\alpha},\ \mathrm{Prob}\{\alpha(k) = 1\} = \bar{\alpha} \tag{3}$$

where $\bar{\alpha} \in [0,1]$ is a known constant.

The faults concerned are assumed to satisfy the condition $\Delta(\Delta(f(k))) = 0$, (namely, the second-order difference of the fault signal should be zero), the characteristics of the fault $f(k)$ could be depicted in the following form:

$$f(k+1) = \Delta(f(k)) + f(k). \tag{4}$$

Letting $\bar{x}(k) \triangleq \begin{bmatrix} x^T(k) & f^T(k) & \Delta^T(f(k)) \end{bmatrix}^T$ and $\bar{w}(k) \triangleq \begin{bmatrix} w^T(k) & v^T(k) \end{bmatrix}^T$, we have

$$\begin{cases} \bar{x}(k+1) = I_g g(k, I_g^T \bar{x}(k)) + \bar{A}(k)\bar{x}(k) + \tilde{\alpha}(k)\bar{B}(k)\bar{x}(k) + \bar{E}(k)\bar{w}(k) \\ \quad y(k) = \bar{C}(k)\bar{x}(k) + \bar{D}(k)\bar{w}(k) \\ \quad f(k) = \bar{L}\bar{x}(k) \end{cases} \tag{5}$$

where

$$\begin{aligned} I_g &\triangleq \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T, \ \bar{A}(k) \triangleq \begin{bmatrix} A(k) & \bar{\alpha}B(k) & 0 \\ 0 & I & I \\ 0 & 0 & I \end{bmatrix}, \\ \bar{B}(k) &\triangleq \begin{bmatrix} 0 & B(k) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \bar{E}(k) \triangleq \begin{bmatrix} E(k) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ \bar{C}(k) &\triangleq \begin{bmatrix} C(k) & 0 & 0 \end{bmatrix}, \ \bar{D}(k) \triangleq \begin{bmatrix} 0 & D(k) \end{bmatrix}, \ \bar{L} \triangleq \begin{bmatrix} 0 & I & 0 \end{bmatrix}. \end{aligned}$$

Under the scheduling of the RR protocol, $\zeta(k) \in \{1, 2, \ldots, n_y\}$ represents the sensor node which can get access to the communication network to transmit data at time step k. The value of $\zeta(k)$ satisfies $\zeta(k + n_y) = \zeta(k)$ for all $k \in \mathbb{N}^+$ where $\zeta(k) = k$ for $k \in \{1, 2, \ldots, n_y\}$. In other words, $\zeta(k)$ can be calculated as:

$$\zeta(k) = \mathrm{mod}(k-1, n_y) + 1.$$

By analysis, the measurement output before being transmitted can be rewritten as:

$$y(k) \triangleq \begin{bmatrix} y_1^T(k) & y_2^T(k) & \cdots & y_{n_y}^T(k) \end{bmatrix}^T$$

where $y_i(k)(i \in \{1, 2, \ldots, n_y\})$ is the measurement of the ith sensor node before transmission.

Next, let us denote the measurement output after transmission through the network by

$$\bar{y}(k) \triangleq \begin{bmatrix} \bar{y}_1^T(k) & \bar{y}_2^T(k) & \cdots & \bar{y}_{n_y}^T(k) \end{bmatrix}^T.$$

The updating rule of $\bar{y}_i(k)(k \in \mathbb{N}^+, i \in \{1, 2, \ldots, n_y\})$ is set to be

$$\bar{y}_i(k) = \begin{cases} y_i(k) & \text{if } i = \zeta(k) \\ 0 & \text{else.} \end{cases} \tag{6}$$

According to the updating rule (6), it can be defined that

$$\bar{y}(k) = \Phi_{\zeta(k)} y(k)$$

$$= \Phi_{\zeta(k)}\bar{C}(k)\bar{x}(k) + \Phi_{\zeta(k)}\bar{D}(k)\bar{w}(k) \tag{7}$$

where $\Phi_{\zeta(k)} = \text{diag}\{\delta(\zeta(k) - 1), \ \ldots, \ \delta(\zeta(k) - n_y)\}$ $(1 \leq \zeta(k) \leq n_y)$ and $\delta(\cdot)$ is the Kronecker delta function.

Remark 2.2. Among various communication protocols, the RR protocol [5] is widely used by communication and signal processing. The RR protocol can be employed to determine which node obtains the access to the network at a certain transmission instant. It is well known that the RR protocol is one of the static protocols utilized by network schedulers, where the transmission order of nodes is predetermined according to a circular order. Therefore, the burden of communication transmission is mitigated significantly and the pressure on the communication transmission is relieved at each time instant.

In this paper, we adopt the following time-varying fault estimator for system (1):

$$\begin{cases} \hat{x}(k+1) = G(k)\hat{x}(k) + K(k)\bar{y}(k) \\ \quad\ \hat{f}(k) = \bar{L}\hat{x}(k) \end{cases} \tag{8}$$

where $\hat{x}(k)$ is the estimate of the state $\bar{x}(k)$ and $\hat{f}(k)$ is the estimate of the fault $f(k)$. The matrices $G(k)$ and $K(k)$ are the estimator gains to be designed.

Letting $e(k) \triangleq \bar{x}(k) - \hat{x}(k)$ and $\tilde{f}(k) \triangleq f(k) - \hat{f}(k)$, the estimation error can be obtained from (1), (5) and (8) as follows:

$$\begin{cases} e(k+1) = I_g g(k, I_g^T\bar{x}(k)) + (\bar{A}(k) - G(k) - K(k)\Phi_{\zeta(k)}\bar{C}(k))\bar{x}(k) \\ \qquad\qquad + \tilde{\alpha}(k)\bar{B}(k)\bar{x}(k) + G(k)e(k) + (\bar{E}(k) - K(k)\Phi_{\zeta(k)}\bar{D}(k))\bar{w}(k) \\ \quad\ \tilde{f}(k) = \bar{L}e(k). \end{cases} \tag{9}$$

Furthermore, denoting $\eta(k) \triangleq \begin{bmatrix}\bar{x}^T(k) & e^T(k)\end{bmatrix}^T$, we have the following dynamic system to be investigated:

$$\begin{cases} \eta(k+1) = \hat{I}_g g(k, I_e\eta(k)) + \hat{A}_{\zeta(k)}(k)\eta(k) + \tilde{\alpha}(k)\hat{B}(k)\eta(k) + \hat{E}_{\zeta(k)}(k)\bar{w}(k) \\ \quad\ \tilde{f}(k) = \hat{L}\eta(k) \end{cases} \tag{10}$$

where

$$\begin{aligned} \hat{I}_g &\triangleq \begin{bmatrix} I_g \\ I_g \end{bmatrix}, \ \hat{A}_{\zeta(k)}(k) \triangleq \begin{bmatrix} \bar{A}(k) & 0 \\ \bar{A}(k) - G(k) - K(k)\Phi_{\zeta(k)}\bar{C}(k) & G(k) \end{bmatrix}, \\ I_e &\triangleq \begin{bmatrix} I_g^T & 0 \end{bmatrix}, \ \hat{E}_{\zeta(k)}(k) \triangleq \begin{bmatrix} \bar{E}(k) \\ \bar{E}(k) - K(k)\Phi_{\zeta(k)}\bar{D}(k) \end{bmatrix}, \\ \hat{B}(k) &\triangleq \begin{bmatrix} \bar{B}(k) & 0 \\ \bar{B}(k) & 0 \end{bmatrix}, \ \hat{L} \triangleq \begin{bmatrix} 0 & \bar{L} \end{bmatrix}. \end{aligned}$$

Our objective of this paper is to design fault estimator (8) such that for the given positive scalar γ, the dynamic system (10) satisfies the following fault estimation performance requirement:

$$J_N \triangleq \mathbb{E}\left\{\sum_{k=0}^{N-1}\left(\|\tilde{f}(k)\|^2-\gamma^2\|\bar{w}(k)\|^2\right)\right\}-\gamma^2\eta^T(0)S\eta(0)<0,\quad \forall(\{\bar{w}(k)\},\eta(0))\neq 0 \tag{11}$$

where S is a known positive definite weighting matrix.

3. MAIN RESULTS

To begin with, three lemmas are introduced in order to facilitate the later analysis.

Lemma 3.1. (Penrose and Todd [23]) Let matrices A, B and C be nonzero matrices with proper dimensions. The solution S to $\min_S \|ASC-B\|_F$ is $A^\dagger BC^\dagger$.

Lemma 3.2. For the noise signal $\bar{w}(k)$ and the initial value $\eta(0)$, denote by $\eta(k)$ the proper solution of the augmented system (10) over $[0,N]$. Then, one has

$$\begin{aligned}
\mathcal{J}_1(\eta(0),\bar{w}(k)) &\triangleq \sum_{k=0}^{N-1}\mathbb{E}\left\{\|\tilde{f}(k)\|^2-\gamma^2\|\bar{w}(k)\|^2\right\}\\
&=\sum_{k=0}^{N-1}\mathbb{E}\left\{\begin{bmatrix}\eta(k)\\ \bar{w}(k)\end{bmatrix}^T\begin{bmatrix}\Phi_{11}(k+1)-R(k) & \Phi_{12}(k+1)\\ * & -\Phi_{22}(k+1)\end{bmatrix}\begin{bmatrix}\eta(k)\\ \bar{w}(k)\end{bmatrix}\right\}\\
&\quad+\mathbb{E}\{\eta^T(0)R(0)\eta(0)-\eta^T(N)R(N)\eta(N)\}.
\end{aligned} \tag{12}$$

If $|\Phi_{22}(k+1)|\neq 0$ for all $k\in[0,N-1]$, and $\bar{w}(k)\triangleq\Phi_{22}^{-1}(k+1)\Phi_{12}(k+1)^T\eta(k)$, one further has

$$\begin{aligned}
\mathcal{J}_2(\tilde{\Omega}(k),\bar{w}(k)) &\triangleq \sum_{k=0}^{N-1}\mathbb{E}\left\{\|\tilde{f}(k)\|^2+\|\tilde{\Omega}(k)\|^2\right\}\\
&=\sum_{k=0}^{N-1}\mathbb{E}\left\{\begin{bmatrix}\eta(k)\\ \tilde{\Omega}(k)\end{bmatrix}^T\begin{bmatrix}\Psi(k+1)+\hat{L}^T\hat{L}-Z(k) & Y_1(k+1)\\ * & Y_2(k+1)\end{bmatrix}\begin{bmatrix}\eta(k)\\ \tilde{\Omega}(k)\end{bmatrix}\right\}\\
&\quad+\mathbb{E}\{\eta^T(0)Z(0)\eta(0)-\eta^T(N)Z(N)\eta(N)\}
\end{aligned} \tag{13}$$

where $\{R(k)\}_{0\leq k\leq N}>0$ and $\{Z(k)\}_{0\leq k\leq N}>0$ are two groups of matrices and

$$\begin{aligned}
\tilde{\Omega}(k) &\triangleq \begin{bmatrix}0\\ X(k)\mathcal{C}(k)\eta(k)\end{bmatrix},\ \mathcal{C}(k)\triangleq\begin{bmatrix}-I & I\\ -\Phi_{\zeta(k)}\bar{C}(k) & 0\end{bmatrix},\ \mathcal{A}(k)\triangleq\begin{bmatrix}\bar{A}(k) & 0\\ \bar{A}(k) & 0\end{bmatrix},\\
X(k) &\triangleq \begin{bmatrix}G(k) & K(k)\end{bmatrix},\ Y_2(k+1)\triangleq Z(k+1)+I,\\
\Phi_{11}(k+1) &\triangleq \hat{L}^T\hat{L}+\mathrm{tr}\left[\hat{I}_g^T R(k+1)\hat{I}_g\bar{\Gamma}_l(k)\right]I_e^T\sum_{l=1}^{q}\Theta_l(k)I_e\\
&\quad+\hat{A}_{\zeta(k)}^T(k)R(k+1)\hat{A}_{\zeta(k)}(k)+\alpha^*\hat{B}^T(k)R(k+1)\hat{B}(k),\\
\Phi_{12}(k+1) &\triangleq \hat{A}_{\zeta(k)}^T(k)R(k+1)\hat{E}_{\zeta(k)}(k),
\end{aligned}$$

$$\begin{aligned}
\Phi_{22}(k+1) &\triangleq \gamma^2 I - \hat{E}^T_{\zeta(k)}(k)R(k+1)\hat{E}_{\zeta(k)}(k),\\
\Xi(k+1) &\triangleq \hat{E}_{\zeta(k)}(k)\Phi^{-1}_{22}(k+1)\Phi^T_{12}(k+1),\\
\Psi(k+1) &\triangleq \operatorname{tr}\left[\hat{I}^T_g Z(k+1)\hat{I}_g\bar{\Gamma}_l(k)\right] I^T_e \sum_{l=1}^{q}\Theta_l(k)I_e\\
&\quad +(\mathcal{A}(k)+\Xi(k+1))^T Z(k+1)(\mathcal{A}(k)+\Xi(k+1))\\
&\quad +\alpha^*\hat{B}^T(k)R(k+1)\hat{B}(k),\ \alpha^*=\bar{\alpha}(1-\bar{\alpha}),\\
Y_1(k+1) &\triangleq (\mathcal{A}(k)+\Xi(k+1))^T Z(k+1).
\end{aligned}$$

Proof. Letting $Q(k) \triangleq \eta^T(k+1)R(k+1)\eta(k+1) - \eta^T(k)R(k)\eta(k)$, we can obtain

$$\begin{aligned}
&\mathbb{E}\{Q(k)\}\\
=&\mathbb{E}\Big\{g^T(k,I_e\eta(k))\hat{I}^T_g R(k+1)\hat{I}_g g(k,I_e\eta(k)) + \eta^T(k)\hat{A}^T_{\zeta(k)}(k)R(k+1)\hat{A}_{\zeta(k)}(k)\eta(k)\\
&+ 2\eta^T(k)\hat{A}^T_{\zeta(k)}(k)R(k+1)\hat{E}_{\zeta(k)}(k)\bar{w}(k) + \alpha^*\eta^T(k)\hat{B}^T(k)R(k+1)\hat{B}(k)\eta(k)\\
&+ \bar{w}^T(k)\hat{E}^T_{\zeta(k)}(k)R(k+1)\hat{E}_{\zeta(k)}(k)\bar{w}(k) - \eta^T(k)R(k)\eta(k)\Big\}.
\end{aligned} \tag{14}$$

Taking the nonlinearity (2) and the property of the trace operation of matrices into account, one has

$$\begin{aligned}
&\mathbb{E}\{g^T(k,I_e\eta(k))\hat{I}^T_g R(k+1)\hat{I}_g g(k,I_e\eta(k))\}\\
=&\mathbb{E}\left\{\operatorname{tr}\left[g^T(k,I_e\eta(k))\hat{I}^T_g R(k+1)\hat{I}_g g(k,I_e\eta(k))\right]\right\}\\
=&\mathbb{E}\left\{\operatorname{tr}\left[\hat{I}^T_g R(k+1)\hat{I}_g g(k,I_e\eta(k))g^T(k,I_e\eta(k))\right]\right\}\\
=&\mathbb{E}\left\{\operatorname{tr}\left[\hat{I}^T_g R(k+1)\hat{I}_g\sum_{l=1}^{q}\Theta_l(k)\eta^T(k)I^T_e\bar{\Gamma}_l(k)I_e\eta(k)\right]\right\}\\
=&\mathbb{E}\left\{\eta^T(k)I^T_e\sum_{l=1}^{q}\operatorname{tr}\left[\hat{I}^T_g R(k+1)\hat{I}_g\bar{\Gamma}_l(k)\right]\Theta_l(k)I_e\eta(k)\right\}.
\end{aligned} \tag{15}$$

According to (14) and (15), add zero term

$$\sum_{k=0}^{N-1}Q(k) - \sum_{k=0}^{N-1}Q(k) - \sum_{k=0}^{N}\gamma^2\{\|\bar{w}(k)\|^2\} + \sum_{k=0}^{N}\gamma^2\{\|\bar{w}(k)\|^2\}$$

to the following equation:

$$\begin{aligned}
&\sum_{k=0}^{N-1}\mathbb{E}\{\|\tilde{f}(k)\|^2\}\\
=&\mathbb{E}\{\eta^T(0)R(0)\eta(0) - \eta^T(N)R(N)\eta(N)\} + \sum_{k=0}^{N}\mathbb{E}\{\gamma^2\|\bar{w}(k)\|^2\}
\end{aligned}$$

$$+\sum_{k=0}^{N}\mathbb{E}\left\{\begin{bmatrix}\eta(k)\\ \bar{w}(k)\end{bmatrix}^T\begin{bmatrix}\Phi_{11}(k+1)-R(k) & \Phi_{12}(k+1)\\ * & -\Phi_{22}(k+1)\end{bmatrix}\begin{bmatrix}\eta(k)\\ \bar{w}(k)\end{bmatrix}\right\}. \tag{16}$$

Likewise, denoting $\tilde{\Omega}_{\zeta(k)}(k)=\begin{bmatrix}0 & (X(k)\mathcal{C}(k)_{\zeta(k)}\eta(k))^T\end{bmatrix}^T$, we have

$$\hat{A}_{\zeta(k)}(k)\eta(k)=\mathcal{A}(k)\eta(k)+\tilde{\Omega}_{\zeta(k)}(k). \tag{17}$$

Moreover, under $|\Phi_{22}(k+1)|\neq 0$ for all $k\in[0,N]$, by choosing $\bar{w}(k)=\Phi_{22}^{-1}(k+1)\Phi_{12}^T(k+1)\eta(k)$, it is acquired that

$$\begin{aligned}
&\sum_{k=0}^{N-1}\mathbb{E}\{\|\tilde{f}(k)\|^2\}\\
=&\sum_{k=0}^{N-1}\mathbb{E}\{\|\tilde{f}(k)\|^2-\|\tilde{\Omega}(k)\|^2+\|\tilde{\Omega}(k)\|^2\}+\mathbb{E}\left\{\eta^T(0)\ Z(0)\eta(0)-\eta^T(N)Z(N)\eta(N)\right\}\\
=&\sum_{k=0}^{N-1}\mathbb{E}\Big\{g^T(k,I_e\eta(k))\hat{I}_g^TZ(k+1)\hat{I}_gg(k,I_e\eta(k))+\eta^T(k)\Big((\mathcal{A}(k)+\Xi(k+1))^T\\
&\times Z(k+1)(\mathcal{A}(k)+\Xi(k+1))+\alpha^*\hat{B}^T(k)Z(k+1)\hat{B}(k)-Z(k)\Big)\eta(k)\\
&+2\eta^T(k)(\mathcal{A}(k)+\Xi(k+1))^TZ(k+1)\tilde{\Omega}(k)+\tilde{\Omega}^T(k)Z(k+1)\tilde{\Omega}(k)\Big\}\\
=&\mathbb{E}\left\{\eta^T(0)\ Z(0)\eta(0)-\eta^T(N)Z(N)\eta(N)\right\}-\sum_{k=0}^{N-1}\|\tilde{\Omega}(k)\|^2\\
&+\sum_{k=0}^{N-1}\mathbb{E}\Bigg\{\begin{bmatrix}\eta(k)\\ \tilde{\Omega}(k)\end{bmatrix}^T\begin{bmatrix}\Psi(k+1)+\hat{L}^T\hat{L}-Z(k) & (\mathcal{A}(k)+\Xi(k+1))^TZ(k+1)\\ * & Z(k+1)+I\end{bmatrix}\\
&\times\begin{bmatrix}\eta(k)\\ \tilde{\Omega}(k)\end{bmatrix}\Bigg\}.
\end{aligned} \tag{18}$$

The proof is complete. □

Lemma 3.3. Consider the nonlinear time-varying systems (1) − (4). Let the disturbance attenuation level $\gamma>0$ and the matrix $S>0$ be given. The performance constraint (11) is satisfied if and only if there exist a group of real-valued matrices $\{X(k)\}_{0\leq k\leq N-1}$ and a group of matrices $\{R(k)>0\}_{0\leq k\leq N-1}$ (with the final condition $R(N)=0$) such that the following backward recursive Riccati difference equation (RDE):

$$\Phi_{11}(k+1)+\Phi_{12}(k+1)\Phi_{22}^{-1}(k+1)\Phi_{12}^T(k+1)=R(k) \tag{19}$$

holds with

$$\Phi_{22}(k+1)>0 \quad \text{and} \quad R(0)<\gamma^2S. \tag{20}$$

P r o o f. *Sufficiency:* For matrices $\{R(k) > 0\}_{0\le k\le N}$ in (19), taking (16) into consideration, we have

$$\begin{aligned}
&\sum_{k=0}^{N-1}\mathbb{E}\{\|\tilde{f}(k)\|^2\} - \sum_{k=0}^{N-1}\mathbb{E}\{\gamma^2\|\bar{w}(k)\|^2\}\\
=&\mathbb{E}\{\eta^T(0)R(0)\eta(0) - \eta^T(N)R(N)\eta(N)\} + \sum_{k=0}^{N-1}\mathbb{E}\Bigg\{\eta^T(k)(\Phi_{11}(k+1) - R(k))\eta(k)\\
&+ 2\eta^T(k)\Phi_{12}(k+1)\bar{w}(k) - \bar{w}^T(k)\Phi_{22}(k+1)\bar{w}(k)\Bigg\}\\
=&\mathbb{E}\{\eta^T(0)R(0)\eta(0) - \eta^T(N)R(N)\eta(N)\} + \sum_{k=0}^{N-1}\mathbb{E}\Bigg\{\eta^T(k)\Big(\Phi_{11}(k+1) - R(k)\\
&+ \Phi_{12}(k+1)\Phi_{22}^{-1}(k+1)\Phi_{12}^T(k+1)\Big)\eta(k) - (\bar{w}(k) - \bar{w}^\star(k))^T\\
&\times \Phi_{22}(k+1)(\bar{w}(k) - \bar{w}^\star(k))\Bigg\}\\
=&\mathbb{E}\{\eta^T(0)R(0)\eta(0) - \eta^T(N)R(N)\eta(N)\} + \sum_{k=0}^{N-1}\mathbb{E}\{-(\bar{w}(k) - \bar{w}^\star(k))^T\\
&\times \Phi_{22}(k+1)(\bar{w}(k) - \bar{w}^\star(k))\}
\end{aligned}\tag{21}$$

where $\bar{w}^\star(k) \triangleq \Phi_{22}^{-1}(k+1)\Phi_{12}^T(k+1)\eta(k)$.

Since $\Phi_{22}(k+1) > 0$ and $R(0) < \gamma^2 S$, for all nonzero $\bar{w}(k)$, we obtain from $R(N) = 0$ that

$$\begin{aligned}
&\sum_{k=0}^{N-1}\mathbb{E}\{\|\tilde{f}(k)\|^2\} - \gamma^2\mathbb{E}\Bigg\{\|\eta(0)\|_S^2 + \sum_{k=0}^{N-1}\|\bar{w}(k)\|^2\Bigg\}\\
<&\sum_{k=0}^{N-1}\mathbb{E}\{\|\tilde{f}(k)\|^2\} - \mathbb{E}\Bigg\{\|\eta(0)\|_{R(0)}^2 + \sum_{k=0}^{N-1}\gamma^2\|\bar{w}(k)\|^2\Bigg\}\\
=&-\sum_{k=0}^{N-1}\mathbb{E}\Bigg\{(\bar{w}(k) - \bar{w}^\star(k))^T\Phi_{22}(k+1)(\bar{w}(k) - \bar{w}^\star(k))\Bigg\}\\
<&0,
\end{aligned}\tag{22}$$

which is the same as (11).

Necessity: Now, we will prove that, if (11) is satisfied, then there exists a feasible solution $R(k)$ $(0 \le k \le N)$ to (19), which satisfies (20) for all nonzero $(\{\bar{w}(k)\}, \eta(0))$. In fact, according to the final condition $R(N) = 0$, the recursion (19) can always be solved backward if $\Phi_{22}(k+1) > 0$ and $R(0) < \gamma^2 S$ for $k \in [0,\ N-1]$, which means that (19) cannot carry on the recursion for some $k = k_0 \in [0,\ N-1]$ if $\Phi_{22}(k_0+1)$ or $\gamma^2 - R(0)$ has at least one non-positive eigenvalue.

The following part of the proof is done by contradiction. Assuming that $\Phi_{22}(k+1)$ or $\gamma^2 - R(0)$ has one or more non-positive eigenvalues at some instant $k = k_0 \in [0,\ N-1]$, now we begin to prove that $J_N < 0$ cannot be satisfied.

Case 1: Let us prove

$$\lambda_\ell(\Phi_{22}(k+1)) \leq 0, \forall k \in [0,\ N-1], \ell = 1, 2, \ldots, n_w + n_v \Rightarrow \quad J_N \geq 0 \tag{23}$$

where $\lambda_\ell(\Phi_{22}(k+1))$ represents the ℓth eigenvalue of $\Phi_{22}(k+1)$.

For convenience of notation, we represent the non-positive eigenvalue of $\Phi_{22}(k+1)$ at instant k_o as $\lambda(k_o)$ ($\lambda(k_o) \leq 0$). Then, we will choose $\lambda(k_o) \leq 0$ to indicate that there is certain $(\{\bar{w}(k)\}, \eta(0)) \neq 0$ so as to $J_N \geq 0$. Firstly, choose $\eta(0) = 0$ and

$$\bar{w}(k) = \begin{cases} \psi(k_o), & k = k_o \\ \bar{w}^\star(k), & k_o < k < N \\ 0, & 0 \leq k < k_o \end{cases} \tag{24}$$

where $\psi(k_o)$ is the eigenvector of $\Phi_{22}(k_o+1)$ with $\lambda(k_o)$.

For $0 \leq k < k_o$, according to (9) with $\eta(0) = 0$, $g(k, 0) = 0$ and $\bar{w}(k) = 0$, we obtain $\eta(k) = 0$ $(0 \leq k \leq k_o)$, then $\bar{w}^\star(k) = \Phi_{22}^{-1}(k+1)\Phi_{12}^T(k+1)\eta(k) = 0$ $(0 \leq k \leq k_o)$.

On the basis of (21), we acquire

$$\begin{aligned} &\sum_{k=0}^{k_o-1} \left(\mathbb{E}\left\{ \|\tilde{f}(k)\|^2 \right\} - \gamma^2 \|\bar{w}(k)\|^2 \right) \\ =&\mathbb{E}\{\eta^T(0)R(0)\eta(0) - \eta^T(k_o)R(N)\eta(k_o)\} \\ &+ \sum_{k=0}^{k_o-1} \mathbb{E}\big\{ - (\bar{w}(k) - \bar{w}^\star(k))^T \Phi_{22}(k+1)(\bar{w}(k) - \bar{w}^\star(k))\big\} \\ =&0, \end{aligned} \tag{25}$$

$$\begin{aligned} &\mathbb{E}\left\{ \|\tilde{f}(k_o)\|^2 \right\} - \gamma^2 \|\bar{w}(k_o)\|^2 + \mathbb{E}\left\{ Q(k_o) - Q(k_o) \right\} \\ =&\mathbb{E}\left\{ -\bar{w}^T(k_o)\Phi_{22}(k_o+1)\bar{w}(k_o) - \eta^T(k_o+1)\ R(k_o+1)\eta(k_o+1) \right\} \end{aligned} \tag{26}$$

and

$$\begin{aligned} &\sum_{k=k_o+1}^{N-1} \left(\mathbb{E}\left\{ \|\tilde{f}(k)\|^2 \right\} - \gamma^2 \|\bar{w}(k)\|^2 \right) \\ =&\mathbb{E}\big\{\eta^T(k_o+1)R(k_o+1)\eta(k_o+1) - \eta^T(N)R(N)\eta(N)\big\} \\ &- \sum_{k=k_o+1}^{N-1} \mathbb{E}\big\{(\bar{w}(k) - \bar{w}^\star(k))^T \Phi_{22}(k+1)(\bar{w}(k)\ - \bar{w}^\star(k))\big\} \\ =&\mathbb{E}\{\eta^T(k_o+1)R(k_o+1)\eta(k_o+1)\}. \end{aligned} \tag{27}$$

Then, we derive from (25) − (27) that

$$\begin{aligned}
J_N =& \sum_{k=0}^{N-1} \left(\mathbb{E}\left\{ \|\tilde{f}(k)\|^2 \right\} - \gamma^2 \|\bar{w}(k)\|^2 \right) - \gamma^2 \|\eta(0)\|_S^2 \\
=& - \bar{w}^T(k_o)\Phi_{22}(k_o+1)\bar{w}(k_o) \\
=& - \psi^T(k_o)\Phi_{22}(k_o+1)\psi(k_o) \\
=& - \lambda(k_o)\|\psi(k_o)\|^2 \geq 0,
\end{aligned} \tag{28}$$

which does not conform to the condition $J_N < 0$. Thus, it is concluded that $\Phi_{22}(k+1) > 0$.

Case 2: We are to prove

$$\Phi_{22}(k+1) > 0 \text{ and } R(0) \geq \gamma^2 S,\ \forall k \in [0, N-1] \quad \Rightarrow \quad J_N \geq 0. \tag{29}$$

Selecting $\bar{w}(k) = \bar{w}^\star(k)$, we obtain from (21) that

$$\begin{aligned}
J_N =& \sum_{k=0}^{N-1} \left(\mathbb{E}\left\{ \|\tilde{f}(k)\|^2 \right\} - \gamma^2 \|\bar{w}(k)\|^2 \right) - \gamma^2 \|\eta(0)\|_S^2 \\
=& \mathbb{E}\{\eta^T(0)R(0)\eta(0) - \eta^T(N)R(N)\eta(N)\} - \gamma^2\|\eta(0)\|_S^2 \\
&+ \sum_{k=0}^{N-1} \mathbb{E}\big\{ - (\bar{w}(k) - \bar{w}^\star(k))^T \Phi_{22}(k+1)(\bar{w}(k) - \bar{w}^\star(k))\big\} \\
=& \mathbb{E}\{\eta^T(0)(R(0) - \gamma^2\eta(0)\}.
\end{aligned} \tag{30}$$

For $\eta(0) \neq 0$, it is obvious that $J_N \geq 0$, which is against the condition $J_N < 0$. The proof is now complete. □

In what follows, we aim to determine the gain matrices $G(k)$ and $K(k)$ of the desired fault estimator under the situation of the worst case disturbance $\bar{w}^\star(k)$.

Theorem 3.4. Consider the time-varying nonlinear stochastic system described by (1) − (4). Let the disturbance attenuation level $\gamma > 0$ and the positive definite matrices $S > 0$ and $R(k) > 0$ be given. The time-varying fault estimator (8) satisfies the performance criterion (11) if (19) and the following recursive RDE:

$$\Psi(k+1) + \hat{L}^T\hat{L} - Y_1(k+1)Y_2^{-1}(k+1)Y_1^T(k+1) = Z(k) \tag{31}$$

have a group of solutions $\{R(k), Z(k), G(k), K(k)\}_{0\leq k \leq\ N-1}$ satisfying

$$R(N) = Z(N) = 0 \tag{32}$$

$$Y_2(k+1) > 0, \Phi_{22}(k+1) > 0, R(0) < \gamma^2 S, \tag{33}$$

$$\begin{aligned}
X^\star(k) =& \begin{bmatrix} G^\star(k) & K^\star(k) \end{bmatrix} \\
&= \arg\min_{X(k)} \|\check{L}^T X(k)\mathcal{C}(k) + Y_2^{-1}(k+1)Y_1^T(k+1)\|_F
\end{aligned} \tag{34}$$

where $\check{L} = \begin{bmatrix} 0 & I \end{bmatrix}$ and other matrices are defined in Lemma 3.2.

Proof. If there is $\{R(k)\}_{0\le k\le N-1}$ satisfying (19) and (33), according to Lemma 3.3, we obtain that the dynamics (10) satisfies the performance constraint (11). Under this circumstance, the worst-case disturbance is expressed by $\bar{w}^\star(k) = \Phi_{22}^{-1}(k+1)\Phi_{12}^T(k+1)\eta(k)$. By employing the completing squares method, we obtain from Lemma 3.2 that:

$$
\begin{aligned}
&\mathcal{J}_2(\tilde{\Omega}(k),\bar{w}(k))\\
=&\sum_{k=0}^{N-1}\mathbb{E}\{\|\tilde{f}(k)\|^2\}+\sum_{k=0}^{N-1}\mathbb{E}\{\|\tilde{\Omega}(k)\|^2\}\\
=&\mathbb{E}\{\eta^T(0)Z(0)\eta(0)-\eta^T(N)Z(N)\eta(N)\}+\sum_{k=0}^{N-1}\mathbb{E}\Big\{\eta^T(k)(\Psi(k+1)\\
&+\hat{L}^T\hat{L}-Z(k)-Y_1(k+1)Y_2^{-1}(k+1)Y_1^T(k+1))\eta(k)\\
&+(\tilde{\Omega}(k)-\tilde{\Omega}^\star(k))^TY_2(k+1)(\tilde{\Omega}(k)-\tilde{\Omega}^\star(k))\Big\}\\
\le&\mathbb{E}\{\eta^T(0)Z(0)\eta(0)-\eta^T(N)Z(N)\eta(N)\}+\sum_{k=0}^{N-1}\mathbb{E}\Big\{\eta^T(k)(\Psi(k+1)\\
&+\hat{L}^T\hat{L}-Y_1(k+1)Y_2^{-1}(k+1)Y_1^T(k+1)-Z(k))\eta(k)\\
&+\|\check{L}^TX(k)\mathcal{C}(k)+Y_2^{-1}(k+1)Y_1^T(k+1)\|_F^2\|Y_2(k+1)\|_F\|\eta(k)\|^2\Big\}
\end{aligned}\tag{35}
$$

where $\tilde{\Omega}^\star(k)\triangleq -Y_2^{-1}(k+1)Y_1^T(k+1)\eta(k)$. Besides, the estimator gains $K(k)$ and $G(k)$ are proved to satisfy (31) and (34) simultaneously, which completes the proof. □

Theorem 3.5. Let the disturbance attenuation level $\gamma>0$, scalars $\sigma(k)>0$, $\delta(k)>0$, matrices $S>0$ and $Z(k)>0$ be given. System (10) achieves the H_∞ performance specification (11) if there is a group of solutions $\{R(k),Z(k),K(k),G(k)\}_{\,0\le k\le N-1}$ to the following recursive RDEs:

$$\Phi_{11}(k+1)+\bar{\Phi}_{12}(k+1)\bar{\Phi}_{22}^{-1}(k+1)\bar{\Phi}_{12}^T(k+1)=R(k)\tag{36}$$

$$\bar{\Psi}(k+1)+\hat{L}^T\hat{L}-\bar{Y}_1(k+1)Y_2^{-1}(k+1)\bar{Y}_1^T(k+1)=Z(k)\tag{37}$$

with

$$R(N)=Z(N)=0\tag{38}$$
$$\bar{\Phi}_{22}(k+1)>0, R(0)<\gamma^2S, Y_2(k+1)>0\tag{39}$$
$$X^\star(k)=\begin{bmatrix}G^\star(k) & K^\star(k)\end{bmatrix}=\Pi^\dagger(k+1)\Gamma(k+1)\mathcal{C}^\dagger(k)\tag{40}$$
$$M(k)\le\delta(k)I\tag{41}$$

where

$$\tilde{E}(k)\ \triangleq\ \begin{bmatrix}\bar{E}(k) & 0\\ \bar{E}(k) & -\sigma^{-1}(k)I\end{bmatrix},\ \hat{S}\triangleq\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix},$$

$$\begin{aligned}
\bar{\Phi}_{12}(k+1) &\triangleq \hat{A}^T_{\zeta(k)}(k)R(k+1)\tilde{E}(k),\\
\bar{\Phi}_{22}(k+1) &\triangleq \gamma^2 I - \tilde{E}^T(k)R(k+1)\tilde{E}(k) - \delta(k)\mathcal{U}^T\mathcal{U},\\
\bar{\Xi}(k+1) &\triangleq \tilde{E}(k)\bar{\Phi}^{-1}_{22}(k+1)\bar{\Phi}^T_{12}(k+1),\\
\bar{Y}_1(k+1) &\triangleq (\mathcal{A}(k)+\bar{\Xi}(k+1))^T Z(k+1),\\
\bar{\Psi}(k+1) &\triangleq \operatorname{tr}\left[\hat{I}^T_g Z(k+1)\hat{I}_g\bar{\Gamma}_l(k)\right] I^T_e \sum_{l=1}^{q}\Theta_l(k)I_e\\
&\quad +(\mathcal{A}(k)+\bar{\Xi}(k+1))^T Z(k+1)(\mathcal{A}(k)+\bar{\Xi}(k+1))\\
&\quad +\alpha^*\hat{B}^T(k)R(k+1)\hat{B}(k),\ \mathcal{U}\triangleq\begin{bmatrix} I & 0\end{bmatrix},\\
\Pi(k+1) &\triangleq (I+Y_2^{-1}(k+1)Z(k+1)\tilde{E}(k)\bar{\Phi}^{-1}_{22}(k+1)\tilde{E}^T(k)R(k+1))\check{L}^T,\\
\Gamma(k+1) &\triangleq -Y_2^{-1}(k+1)Z(k+1)(I+\tilde{E}(k)\bar{\Phi}^{-1}_{22}(k+1)\tilde{E}^T(k)R(k+1))\mathcal{A}(k),\\
M(k) &\triangleq \gamma^2\sigma^2(k)\bar{D}^T(k)\Phi^T_{\zeta(k)}K^T(k)K(k)\Phi_{\zeta(k)}\bar{D}(k).
\end{aligned}$$

Proof. Let $\tilde{w}(k)\triangleq\sigma(k)K(k)\Phi_{\zeta(k)}\bar{D}(k)\bar{w}(k)$, where $\sigma(k)>0$ is utilized to offer extra degree of freedom in the design of the fault estimator. Selecting $\hat{w}(k)\triangleq\begin{bmatrix}\bar{w}^T(k) & \tilde{w}^T(k)\end{bmatrix}^T$, we rewrite (10) in the following form:

$$\begin{cases}\eta(k+1)=\hat{I}_g g(k,I_e\eta(k))+\hat{A}_{\zeta(k)}(k)\eta(k)+\tilde{\alpha}(k)\hat{B}(k)\eta(k)+\tilde{E}_{\zeta(k)}(k)\hat{w}(k)\\ \quad\tilde{f}(k)=\hat{L}\eta(k).\end{cases}\tag{42}$$

Furthermore, based on Lemma 3.1, it is seen that (40) is a solution of the following optimization problem:

$$\min_{X(k)}\|\Pi(k+1)X(k)\mathcal{C}(k)-\Gamma(k+1)\|_F,$$

which can be further shown as

$$\min_{X(k)}\|\check{L}^T X(k)\mathcal{C}(k)+Y_2^{-1}(k+1)\bar{Y}_1^T(k+1)\|_F.\tag{43}$$

On the basis of (16) and Theorem 3.4, provided that there is a group of solutions to the recursive RDEs (36) and (37) with (38)–(41), we can have

$$\begin{aligned}
&\sum_{k=0}^{N-1}\mathbb{E}\left\{\|\tilde{f}(k)\|^2\right\}\\
&=\sum_{k=0}^{N-1}\mathbb{E}\Big\{\eta^T(k)(\Phi_{11}(k+1)-R(k)+\bar{\Phi}_{12}(k+1)\bar{\Phi}^{-1}_{22}(k+1)\bar{\Phi}^T_{12}(k+1))\\
&\quad\times\eta(k)-(\hat{w}(k)-\hat{w}^\star(k))^T\bar{\Phi}_{22}(k+1)(\hat{w}(k)-\hat{w}^\star(k))\Big\}\\
&\quad-\mathbb{E}\big\{\eta^T(N)R(N)\eta(N)-\eta^T(0)R(0)\eta(0)\big\}\\
&\quad-\sum_{k=0}^{N-1}\mathbb{E}\left\{\delta(k)(\mathcal{U}\hat{w}(k))^T(\mathcal{U}\hat{w}(k))\right\}+\gamma^2\sum_{k=0}^{N-1}\mathbb{E}\left\{\hat{w}^T(k)\hat{S}\hat{w}(k)\right\}
\end{aligned}$$

$$
\begin{aligned}
&<\gamma^2\mathbb{E}\{\eta^T(0)S\eta(0)\} - \sum_{k=0}^{N-1}\mathbb{E}\left\{\delta(k)(\mathcal{U}\hat{w}(k))^T(\mathcal{U}\hat{w}(k))\right\} \\
&\quad + \gamma^2\sum_{k=0}^{N-1}\mathbb{E}\left\{\bar{w}^T(k)\bar{w}(k) + \tilde{w}_2^T(k)\tilde{w}_2(k)\right\} \\
&=\gamma^2\mathbb{E}\{\|\eta(0)\|_S^2\} + \gamma^2\sum_{k=0}^{N-1}\mathbb{E}\left\{\|\bar{w}(k)\|^2\right\} + \sum_{k=0}^{N-1}\mathbb{E}\left\{\bar{w}^T(k)(M(k)-\delta(k)I)\bar{w}(k)\right\}
\end{aligned}
\tag{44}
$$

where $\hat{w}(k) = \hat{w}^\star(k) = \bar{\Phi}_{22}^{-1}(k+1)\bar{\Phi}_{12}^T(k+1)\eta(k)$.

By using (41), it is indicated from (44) that

$$
\sum_{k=0}^{N-1}\mathbb{E}\left\{\|\tilde{f}(k)\|^2\right\} < \gamma^2\|\eta(0)\|_S^2 + \gamma^2\sum_{k=0}^{N-1}\|\bar{w}(k)\|^2. \tag{45}
$$

Finally, it is concluded that the fault estimator (8) ensures that the system (10) satisfies the performance criterion (11). The proof is now complete. □

By means of Theorem 3.5, we can propose the Finite-Horizon Fault Estimator Design (*FHFED*) algorithm as follows:

Algorithm *FHFED*	
Step 1.	Set the H_∞ disturbance attenuation level γ, the matrix $S > 0$, and let $k = N-1$.
Step 2.	Compute $Y_2(k+1)$ and $\bar{\Phi}_{22}(k+1)$ with known $Z(k+1)$ and $R(k+1)$ through equations (13) and (36), respectively. Moreover, the gain matrices $G(k)$ and $K(k)$ of the fault estimator are obtained by the equation (40).
Step 3.	If $\bar{\Phi}_{22}(k+1) \neq 0$ and $Y_2(k+1) \neq 0$, then solve equations (36) and (37) to obtain $R(k)$ and $Z(k)$, respectively, and proceed to *Step 4*, else this algorithm is not feasible, stop.
Step 4.	If $k \neq 0$, $\bar{\Phi}_{22}(k+1) > 0$ and $Y_2(k+1) > 0$, let $k = k-1$ and return to *Step 2*, else proceed to *Step 5*.
Step 5.	If $R(0) \geq \gamma^2$ or $\bar{\Phi}_{22}(k+1) \leq 0$ or $Y_2(k+1) \leq 0$, then this algorithm is not feasible, stop.

Remark 3.6. In this paper, by solving a couple of Riccati difference equations, the finite-horizon fault estimation problem with a RR protocol is investigated. It can be observed from Algorithm *FHFED* that, all the important factors contributing to the system complexity are reflected in the estimators' design procedure, which include (1) the time-varying system parameters, (2) the occurrence probability of the random faults, (3) statistics information about the stochastic nonlinearities, and (4) the RR protocol. By solving the coupled backward recursive RDEs, the gains of the fault estimator are acquired. It is worth pointing out that a necessary and sufficient condition of the H_∞ fault estimator is set up in this paper.

4. SIMULATION RESULTS

In this section, we present a simulation example to illustrate the effectiveness of the proposed fault estimation algorithm for a class of nonlinear time-varying systems. Consider an array of time-varying systems described by (1) − (4) with the following parameters over the finite horizon $[0, 100]$:

$$\begin{aligned} A(k) &= \begin{bmatrix} 1.01 & -0.7 \\ 0.2 + 0.3\sin(3k) & 0.53 \end{bmatrix},\ B(k) = \begin{bmatrix} 0.5 \\ 0.1 \end{bmatrix},\ E(k) = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}, \\ C(k) &= \begin{bmatrix} 0.2\sin(5k) & 0.5 \\ 0.1 & -0.2 \end{bmatrix},\ D(k) = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix},\ \bar{L} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \\ \Phi_1 &= \mathrm{diag}\{0, 1\},\ \Phi_2 = \mathrm{diag}\{1, 0\}. \end{aligned}$$

Select the following stochastic nonlinear function $g(k, x(k))$:

$$g(k, x(k)) = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix} \times (0.2x_1(k)\epsilon_1(k) + 0.3x_2(k)\epsilon_2(k))$$

where $x_i(k)$ $(i = 1, 2)$ represents the ith element of $x(k)$, and $\epsilon_r(k) \sim \mathcal{N}(0, 1)$ $(r = 1, 2)$ are uncorrelated Gaussian white noise sequences. It can be found that $g(k, x(k))$ satisfies

$$\begin{aligned} &\mathbb{E}\{g(k, x(k))|x(k)\} = 0, \\ &\mathbb{E}\left\{g(k, x(k))g^T(k, x(k))|x(k)\right\} \\ = \ &\begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}^T \mathbb{E}\left\{x^T(k) \begin{bmatrix} 0.04 & 0 \\ 0 & 0.09 \end{bmatrix} x(k)\right\}. \end{aligned}$$

Let $\bar{\alpha} = 0.6$. Choose the H_∞ performance indicator $\gamma = 1$ and the positive definite matrix $S = \mathrm{diag}\{1, 1, 1, 1, 2, 2, 2, 1\}$. Based on the obtained estimator design algorithm, we derive the estimator parameters. Besides, the initial values of the system states and the estimator states are chosen as $\bar{x}(0) = \begin{bmatrix} 0.2 & 1 & 0.5 & 0.5 \end{bmatrix}^T$ and $\hat{x}(0) = \begin{bmatrix} 1.5 & 0.5 & 1 & 0.1 \end{bmatrix}^T$, the exogenous disturbance input and the measurement noise are selected as $w(k) = 4\cos(0.2k)$ and $v(k) = 6\sin(0.4k)$. The fault signal is selected as $f(k) = 0.01\cos(0.01k)$. Simulation curves are demonstrated in Figures 1−2. Figure 1 plots the fault estimation error. Figure 2 shows the evolution of the fault estimation performance, from which we can see that

$$\varphi(k) = \sqrt{\frac{\sum_{\varrho=0}^{k} \mathbb{E}\left\{\|\tilde{f}(\varrho)\|^2\right\}}{\sum_{\varrho=0}^{k} \|\bar{w}(\varrho)\|^2 + \|\eta(0)\|_S^2}} < \gamma \quad (k = 0, 1, \ldots, N-1),$$

thus the proposed fault estimation method is capable of meeting the performance constraint (11).

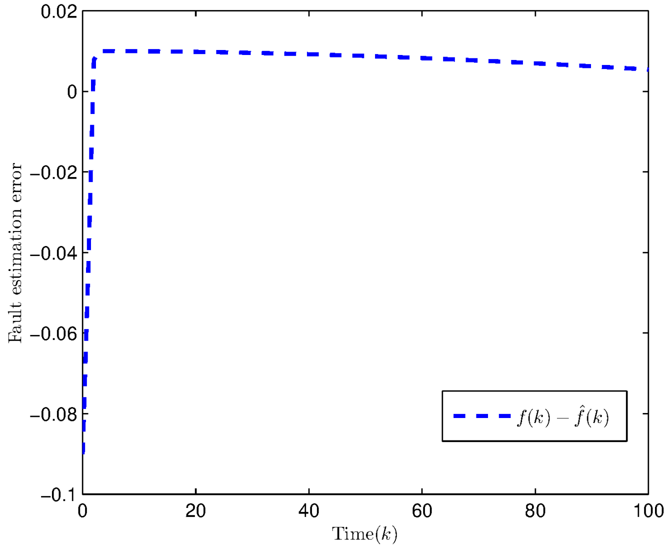

Fig. 1. Estimation errors $f(k) - \hat{f}(k)$.

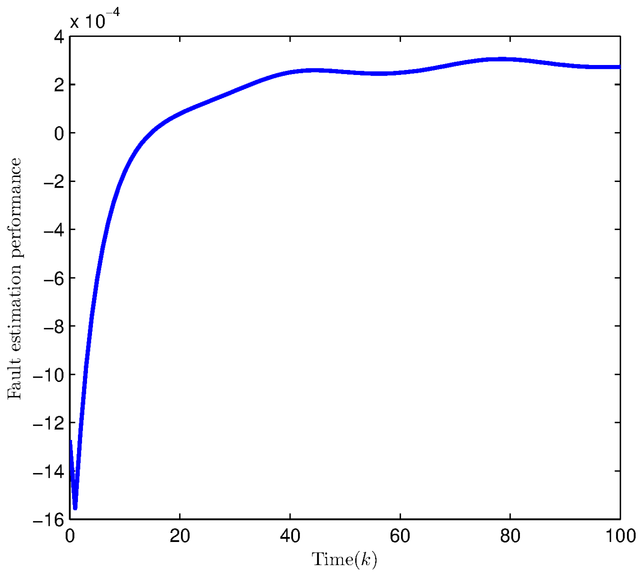

Fig. 2. Fault estimation performance $\varphi(k)$.

5. CONCLUSIONS

In this paper, the finite-horizon estimation problem has been addressed for a class of nonlinear time-varying systems with randomly occurring faults and Round–Robin protocol. The faults have been assumed to be either incipient faults or abrupt faults, which are the most common faults in practical systems. The RR protocol has been employed to determine which node obtains the access to the data transmission network. By utilizing the stochastic analysis technique and the completing squares method, a necessary and sufficient condition has been set up for the existence of the desired fault estimator. The time-varying parameters of fault estimator have been acquired via solving coupled backward Riccati difference equations. An illustrative example has been given to verify the usefulness of the proposed fault estimator design algorithm. It should be mentioned that the future research topics would be to investigate the fault estimation problems for discrete-time systems subject to dynamic event-triggered transmission scheme [11, 12, 17, 21], systems over sensor networks [3, 4, 20] and multi-agent systems under the communication protocols [2, 38].

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REFERENCES

[1] X. Bu, H. Dong, F. Han, N. Hou, and G. Li: Distributed filtering for time-varying systems over sensor networks with randomly switching topologies under the Round–Robin protocol. Neurocomputing *346* (2019), 58–64. DOI:10.1016/j.neucom.2018.07.087

[2] W. Chen, D. Ding, X. Ge, Q.-L. Han, and G. Wei: H-infinity containment control of multi-agent systems under event-triggered communication scheduling: The finite-horizon case. IEEE Trans. Cybernetics (2018), 1–11. DOI:10.1109/tcyb.2018.2885567

[3] D. Ding, Q.-L. Han, Z. Wang, and X. Ge: A survey on model-based distributed control and filtering for industrial cyber-physical systems. IEEE Trans. Industr. Inform. *15* (2019), 2483–2499. DOI:10.1109/tii.2019.2905295

[4] D. Ding, Z. Wang, and Q.-L. Han: A set-membership approach to event-triggered filtering for general nonlinear systems over sensor networks. IEEE Trans. Automat. Control (2019), 1. DOI:10.1109/tac.2019.2934389

[5] D. Ding, Z. Wang, Q.-L. Han, and G. Wei: Neural-network-based output-feedback control under Round–Robin scheduling protocols. IEEE Trans. Cybernetics *49* (2019), 2372–2384. DOI:10.1109/tcyb.2018.2827037

[6] H. Dong, N. Hou, and Z. Wang: Fault estimation for complex networks with randomly varying topologies and stochastic inner couplings. Automatica *112* (2020), 108734. DOI:10.1016/j.automatica.2019.108734

[7] H. Dong, N. Hou, Z. Wang, and H. Liu: Finite-horizon fault estimation under imperfect measurements and stochastic communication protocol: Dealing with finite-time boundedness. Int. J. Robust Nonlinear Control *29* (2019), 117–134. DOI:10.1002/rnc.4382

[8] H. Dong, Z. Wang, S. X. Ding, and H. Gao: Finite-horizon estimation of randomly occurring faults for a class of nonlinear time-varying systems. Automatica *50* (2014), 3182–3189. DOI:10.1016/j.automatica.2014.10.026

[9] H. Dong, Z. Wang, S. X. Ding, and H. Gao: On H_∞ estimation of randomly occurring faults for a class of nonlinear time-varying systems with fading channels. IEEE Trans. Automat. Control *61* (2015), 479–484. DOI:10.1109/tac.2015.2437526

[10] M. Gao, S. Yang, L. Sheng, and D. Zhou: Fault diagnosis for time-varying systems with multiplicative noises over sensor networks subject to Round–Robin protocol. Neurocomputing *346* (2019), 65–72. DOI:10.1016/j.neucom.2018.08.087

[11] X. Ge, Q.-L. Han, and Z. Wang: A threshold-parameter-dependent approach to designing distributed event-triggered H_∞ consensus filters over sensor networks. IEEE Trans. Cybernetics *49* (2019), 1148–1159. DOI:10.1109/tcyb.2017.2789296

[12] X. Ge, Q.-L. Han, and Z. Wang: A dynamic event-triggered transmission scheme for distributed set-membership estimation over wireless sensor networks. IEEE Trans. Cybernetics *49* (2019), 171–183. DOI:10.1109/tcyb.2017.2769722

[13] N. Hou, Z. Wang, D. W. C. Ho, and H. Dong: Robust Partial-Nodes-Based state estimation for complex networks under deception attacks. IEEE Trans. Cybernetics (2019), 1–10. DOI:10.1109/TCYB.2019.2918760

[14] J. Hu, Z. Wang, and H. Gao: Joint state and fault estimation for time-varying nonlinear systems with randomly occurring faults and sensor saturations. Automatica *97* (2018), 150–160. DOI:10.1016/j.automatica.2018.07.027

[15] J. Li, H. Dong, Z. Wang, and X. Bu: Partial-Neurons-Based passivity-guaranteed state estimation for neural networks with randomly occurring time-delays. IEEE Trans. Neural Networks Learn. Systems (2019), 1–7. DOI:10.1109/tnnls.2019.2944552

[16] J. Li, H. Dong, Z. Wang, N. Hou, and F. E. Alsaadi: On passivity and robust passivity for discrete-time stochastic neural networks with randomly occurring mixed time delays. Neural Computing Appl. *31* (2019), 65–78. DOI:10.1007/s00521-017-2980-1

[17] Q. Li, B. Shen, Z. Wang, T. Huan, and J. Luo: Synchronization control for a class of discrete time-delay complex dynamical networks: A dynamic event-triggered approach. IEEE Trans. Cybernetics *49* (2019), 1979–1986. DOI:10.1109/tcyb.2018.2818941

[18] Y. Li, H. R. Karimi, Q. Zhang, D. Zhao, and Y. Li: Fault detection for linear discrete time-varying systems subject to random sensor delay: a Riccati equation approach. IEEE Trans. Circuits and Systems I: Regular Papers *65* (2018), 1707–1716. DOI:10.1109/tcsi.2017.2763625

[19] Z. Li, H. Shu, and X. Kan: H_∞ fault detection with randomly occurring nonlinearities and channel fadings. Int. J. Systems Sci. *45* (2014), 1416–1426. DOI:10.1109/tcsi.2017.2763625

[20] Q. Liu, Z. Wang, X. He, and D. Zhou: Event-based distributed filtering over markovian switching topologies. IEEE Trans. Automat. Control *64* (2018), 99–115. DOI:10.1007/978-3-030-00157-5_6

[21] Q. Liu, Z. Wang, X. He, and D. Zhou: Event-triggered resilient filtering with measurement quantization and random sensor failures: Monotonicity and convergence. Automatica *94* (2018), 458–464. DOI:10.1016/j.automatica.2018.03.031

[22] Y. Liu, Z. Wang, and D. Zhou: UIO-based fault estimation for a class of time-varying systems with event-triggered transmissions. IFAC-PapersOnLine *51* (2018), 46–51. DOI:10.1016/j.ifacol.2018.09.527

[23] R. Penrose and J. A. Todd: On best approximate solutions of linear matrix equations. Math. Proc. Cambridge Philosoph. Soc. *52* (1956), 17–19. DOI:10.1017/s0305004100030929

[24] B. Shen, Z. Wang, D. Wang, and J. Luo: Finite-horizon filtering for a class of nonlinear time-delayed systems with an energy harvesting sensor. Automatica *100* (2019), 144–152. DOI:10.1016/j.automatica.2018.11.010

[25] Y. Shen, Z. Wang, B. Shen, F. E. Alsaadi, and F. E. Alsaadi: Fusion estimation for multi-rate linear repetitive processes under weighted Try-Once-Discard protocol. Inform. Fusion *55* (2020), 281–291. DOI:10.1016/j.inffus.2019.08.013

[26] L. Sheng, Y. Niu, L. Zou, Y. Liu, and F. E. Alsaadi: Finite-horizon state estimation for time-varying complex networks with random coupling strengths under Round–Robin protocol. J. Franklin Inst. *355* (2018), 7417–7442. DOI:10.1016/j.jfranklin.2018.07.026

[27] L. Sheng, Z. Wang, L. Zou, and F. E. Alsaadi: Event-based H_∞ state estimation for time-varying stochastic dynamical networks with state- and disturbance-dependent noises. IEEE Trans. Neural Networks Learn. Systems *28* (2017), 2382–2394. DOI:10.1109/tnnls.2016.2580601

[28] V. Vesely and M. Ernek: Design of robust gain scheduled controller using L_2 gain performance. Kybernetika *54* (2018), 593–609. DOI:10.14736/kyb-2018-3-0593

[29] X. Wan, Z. Wang, M. Wu, and X. Liu: State estimation for discrete time-delayed genetic regulatory networks with stochastic noises under the Round–Robin protocols. IEEE Trans. NanoBioscience *17* (2018), 145–154. DOI:10.1109/tnb.2018.2797124

[30] J. Wu, Z. Weng, Z. Tian, and S. Shi: Fault tolerant control for uncertain time-delay systems based on sliding mode control. Kybernetika *44* (2008), 617–632.

[31] X.-M. Zhang and Q.-L. Han: A decentralized event-triggered dissipative control scheme for systems with multiple sensors to sample the system outputs. IEEE Trans. Cybernet. *46* (2015), 2745–2757. DOI:10.1109/tcyb.2015.2487420

[32] X.-M. Zhang, Q.-L. Han, X. Ge, D. Ding, L. Ding, D. Yue, and C. Peng: Networked control systems: a survey of trends and techniques. IEEE/CAA J. Automat. Sinica (2019), 1–17. DOI:10.1109/jas.2019.1911651

[33] J.-W. Zhu, G.-H. Yang, H. Wang, and F. Wang: Fault estimation for a class of nonlinear systems based on intermediate estimator. IEEE Trans. Automat. Control *61* (2016), 2518–2524. DOI:10.1109/tac.2015.2491898

[34] L. Zou, Z. Wang, and H. Gao: Set-membership filtering for time-varying systems with mixed time-delays under Round–Robin and Weighted Try-Once-Discard protocols. Automatica *74* (2016), 341–348. DOI:10.1016/j.automatica.2016.07.025

[35] L. Zou, Z. Wang, H. Gao, and X. Liu: State estimation for discrete-time dynamical networks with time-varying delays and stochastic disturbances under the Round–Robin protocol. IEEE Trans. Neural Networks Learn. Systems *28* (2017), 1139–1151. DOI:10.1109/tnnls.2016.2524621

[36] L. Zou, Z. Wang, Q.-L. Han, and D. H. Zhou: Ultimate boundedness control for networked systems with Try-Once-Discard protocol and uniform quantization effects. IEEE Trans. Automat. Control *62* (2017), 6582–6588. DOI:10.1109/tac.2017.2713353

[37] L. Zou, Z. Wang and D. Zhou: Finite-horizon H_∞ fault estimation for time-varying systems with Random Access protocol. IFAC-PapersOnLine *51* (2018), 314–319. DOI:10.1016/j.ifacol.2018.09.595

[38] Z. Zuo, Q.-L. Han, B. Ning, X. Ge and X.-M. Zhang: An overview of recent advances in fixed-time cooperative control of multi-agent systems. IEEE Trans. Industr. Inform. *14*(2018), 2322–2334. DOI:10.1109/tii.2018.2817248

Haijing Fu, the Institute of Complex Systems and Advanced Control, Northeast Petroleum University, Daqing 163318. P. R. China. She is also with the Heilongjiang Provincial Key Laboratory of Networking and Intelligent Control, Northeast Petroleum University, Daqing 163318. P. R. China.
e-mail: haijingfu@126.com

Hongli Dong, the Institute of Complex Systems and Advanced Control, Northeast Petroleum University, Daqing 163318. P. R. China. She is also with the Heilongjiang Provincial Key Laboratory of Networking and Intelligent Control, Northeast Petroleum University, Daqing 163318. P. R. China.
e-mail: shiningdhl@vip.126.com

Jinbo Song, the Institute of Complex Systems and Advanced Control, Northeast Petroleum University, Daqing 163318. P. R. China. She is also with the Heilongjiang Provincial Key Laboratory of Networking and Intelligent Control, Northeast Petroleum University, Daqing 163318. P. R. China.
e-mail: sjb_nepu@126.com

Nan Hou, the Institute of Complex Systems and Advanced Control, Northeast Petroleum University, Daqing 163318. P. R. China. She is also with the Heilongjiang Provincial Key Laboratory of Networking and Intelligent Control, Northeast Petroleum University, Daqing 163318. P. R. China
e-mail: bayan2@163.com

Gongfa Li, the Key Laboratory of Metallurgical Equipment and Control Technology, Ministry of Education, Wuhan University of Science and Technology, Wuhan 430081. P. R. China.
e-mail: ligongfa@wust.edu.cn