Zening Fan; Suo Zhao
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# ON SECTIONAL NEWTONIAN GRAPHS

Zening Fan, Chengdu, Suo Zhao, Shanghai

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*Abstract.* In this paper, we introduce the so-called sectional Newtonian graphs for univariate complex polynomials, and study some properties of those graphs. In particular, we list all possible sectional Newtonian graphs when the degrees of the polynomials are less than five, and also show that every stable gradient graph can be realized as a polynomial sectional Newtonian graph.

*Keywords*: sectional Newtonian graph; level set; partition

*MSC 2020*: 05C75, 53C43

## 1. Introduction

In 1985, Smale in [8] introduced the Newtonian graph $N_f$ for any complex univariate polynomial $f \in \mathbb{C}[z]$ as follows. Vertices of $N_f$ are $f^{-1}(0) \cup f_z^{-1}(0)$, which consist of zeros of $f$ and its holomorphic derivative $f_z := \mathrm{d}f/\mathrm{d}z$. Edges of $N_f$ are the degenerate curves of flow of the associated Newtonian vector field

$$V_f(z) := -\frac{f(z)}{f_z(z)}, \quad z \in \mathbb{C}.$$

The word "Newtonian" comes from the fact that those vector fields are related to Newton's method, which was discussed in Newton's own book *Method of Fluxions*.

A Newtonian graph $N_f$ is completely determined by the polynomial $f$. However, the answer to the converse problem is not obvious, i.e.:

*Given a graph $G$, is it the Newtonian graph $N_f$ of some $f \in \mathbb{C}[z]$?*

Based on the contribution of Smale (see [8]), various successors went deeper into Newtonian graphs and provided their partial answers. In 1988, Shub, Tischler and

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William in [7] proved that given an acyclic dynamic graph $G \subset \mathbb{R}^2$, there always exists a polynomial $f$ such that the Newtonian graph $N_f$ is isotopic to $G$, where the *dynamic graph* is a finite directed graph with two types of vertices, which we call saddles and sinks subject to some conditions (see [7], page 251). In [5], Kahn detailed two families of Newtonian graphs of complex polynomials. Besides, Jongen, Jonker and Twilt in [4] proved some necessary and sufficient conditions to judge whether a plane graph is equivalent to a Newtonian graph. In 1995, Stefánsson generalized the definition of Newtonian graphs in his Ph.D. dissertation (see [9] or [6]) in two directions: on the one hand, he defined those graphs for rational functions on $\mathbb{C}$; on the other hand, he also defined those graphs on general Riemann surfaces which are not necessarily closed.

Although the Newtonian graph $N_f$ reflects many important properties of $f$, deficiencies still exist. For example, as mentioned in [7], multiple zeros of $f$ have no geometric effect on the corresponding sinks, instead, only the velocity of the flow increases.

The main purpose of this paper is to introduce sectional Newtonian graphs, SNG for short, on the Riemann sphere. Roughly speaking, it is a combinatorical description of a complex polynomial (or meromorphic function) via some specified holomorphic pullback CW-structures.

This paper is organized as follows. We devote Section 2 to the preliminaries of graph theory and complex analysis used in the paper. In Section 3, we define the first and second SNG of polynomials and generalize those definitions for meromorphic functions. In Section 4, we define the equivalence of SNG using gradient and vortex graphs on the Riemann sphere, and give a complete classification of polynomial SNG $G_f$, where $\deg f < 5$. Finally, in Section 5 we define stable graphs and prove that each stable gradient graph admits a polynomial realization.

## 2. Preliminaries

In this paper, the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ refers to the one-point compactification of the complex plane $\mathbb{C}$ as a 2-dimensional topological space. Let $st\colon \widehat{\mathbb{C}} - \{\infty\} \to \mathbb{C}$ be the stereographic projection, where $\infty$ is the north pole of $\mathbb{S}^2 \cong \widehat{\mathbb{C}}$. In geometry, we have

$$\widehat{\mathbb{C}} = st^{-1}(\mathbb{C}) \sqcup \{\infty\},$$

where $\sqcup$ is the disjoint union.

**2.1. Graph.** An *indirected graph* is an ordered pair $G = (V, E)$ consisting of a set $V$ of *vertices* and a set $E$, disjoint from $V$, of *edges*, together with an *incidence* *function* $\psi$ that associates with each edge of $G$ an unordered pair of (not necessarily distinct) vertices of $G$. If the incidence function $\psi$ maps an edge to an ordered pair of vertices, we get a *directed graph*. For notational simplicity, we write $uv$ for the unordered pair $\{u, v\}$ or the ordered pair $(u, v)$ if there is no confusion. Moreover, a *tail* or a *ray* is an edge with only one end point.

We say that $G' = (V', E')$ is a *subgraph* of $G$ if $V' \subset V$ and $E' \subset E$. For $V' \subsetneq V$, the *induced subgraph* $G[V'] = (V', E|_{V'})$ is a subgraph such that $E|_{V'} = \{e = uv \in E\colon\ u,\, v \in V'\}$.

In a directed graph, if $\psi(e) = uv$, then $u := \operatorname{init}(e)$ is the *initial point* and $v := \operatorname{term}(e)$ is the *terminal point* of $e$. Conversely, $e$ is an *outgoing edge* of $u$ and an *incoming edge* of $v$.

In an indirected graph $G$, a *cycle* $C = (V_C, E_C)$ is a subgraph of the form

$$E_C = \{e_1 = u_0u_1, e_2 = u_1u_2, \ldots, e_n = u_nu_0\},$$

where the $u_i$ are not necessarily distinct. Especially, all cycles mentioned in this paper are in the sense of indirected graphs. A graph without cycles is called a *forest*, and a connected forest is called a *tree*.

**2.2. Meromorphic function.** Let $f \in \mathbb{C}[z]$ be a complex polynomial. The *degree* of $f$, denoted by $\deg f$, is the highest of the degrees of its monomials with nonzero coefficient, and the degree of the zero polynomial is set to be $-\infty$. Given $n \in \mathbb{N}$, $L_n := \{f \in \mathbb{C}[z]\colon\ \deg f \leqslant n\} \cong \mathbb{C}^{n+1}$ is a vector subspace of $\mathbb{C}[z]$. Since $\{0\} \subset L_0 \subset L_1 \subset \ldots$, we may use

$$L_n^* := L_n - L_{n-1} = \{f \in \mathbb{C}[z]\colon\ \deg f = n\} \cong \mathbb{C}^* \times \mathbb{C}^n$$

to represent the collection of complex polynomials of degree $n$.

Let $\mathbb{M}(\widehat{\mathbb{C}})$ be the set of meromorphic functions on $\widehat{\mathbb{C}}$. Note that a polynomial $f$ is also a meromorphic function from the Riemann sphere $\widehat{\mathbb{C}}$ to itself by setting $f(\infty) = \infty$.

**Proposition 2.1** ([2], page 168). *Every nonconstant meromorphic function $r$ on $\widehat{\mathbb{C}}$ is a rational function, i.e.,*

$$r(z) = \frac{p(z)}{q(z)},$$

*where $p, q \in \mathbb{C}[z]$ are complex polynomials without common factors.*

The degree of a meromorphic function $r \in \mathbb{M}(\widehat{\mathbb{C}})$ is determined by the degrees of $p$ and $q$ as:

$$\deg r := \max\{\deg p, \deg q\}.$$

A well-known fact is that there exist exactly $\deg r$ solutions, counted with multiplicities, for the equation

$$r(z) = w \quad \text{for all } w \in \widehat{\mathbb{C}}.$$

**2.3. Multiset and partition.**

**Definition 2.2.** A *multiset* is a pair $\mathbb{A} = (A, m)$, where $A$ is the *underlying set of elements* and $m\colon A \to \mathbb{Z}_+ := \{1, 2, 3, \ldots\}$ is the *multiplicity function*. Moreover, we say $\mathbb{A}$ is a finite multiset if the underlying set $A$ is finite.

In this paper, a multiset $\mathbb{A}$ takes the form of $\{a_{m(a)}\colon\, a \in A\}$, where $m(a)$ is the *multiplicity* of $a$. For example, we write the multiset $\{1, 1, 3, 4, 4\}$ as $\{1_2, 3, 4_2\}$ and omit the subscript 1 in $3_1$ if there is no confusion. The *multiplicity decomposition* of $\mathbb{A}$ is

$$\mathbb{A} = \mathbb{A}_1 \sqcup \mathbb{A}_2 \sqcup \ldots,$$

where $\mathbb{A}_j = m^{-1}(j) = \{a_{m(a)}\colon\, a \in A,\, m(a) = j\}$ is the *sub-multiset* of multiplicity $j$.

An *(integer) partition* $\sigma$ of $n$ is a solution of the equation

$$1 \cdot d_1 + 2 \cdot d_2 + \ldots + n \cdot d_n = n,$$

where $d_j \in \mathbb{N}$. Even a partition $\sigma$ of $n$ forms a multiset $\{j_{d_j}\colon\, 1 \leqslant j \leqslant n\}$, for some historical reason, we may write $\sigma$ as $(1^{d_1}, 2^{d_2}, \ldots)$ and drop $j^{d_j}$-terms if $d_j = 0$. The *order* $|\sigma|$ of a partition $\sigma$ is the number of blocks $\sum\limits_j d_j$. For example, we write $(1^0, 2^2, 3^0, 4^0)$ as $(2^2)$. Moreover, the collection of partitions of $n$ is denoted by $P(n)$.

**Definition 2.3.** Let $\mathbb{A} = \bigsqcup_j \mathbb{A}_j$ be the multiplicity decomposition of a finite multiset $\mathbb{A}$. Then the *partition* of $\mathbb{A}$ is

$$\sigma(\mathbb{A}) = (1^{a_1}, 2^{a_2}, \ldots),$$

where $a_j = |\mathbb{A}_j|$ is the cardinal of $\mathbb{A}_j$.

For example, the multiplicity decomposition of $\mathbb{A} = \{1_2, 2, 3_2, 4, 5_3\}$ is $\mathbb{A} = \mathbb{A}_1 \sqcup \mathbb{A}_2 \sqcup \mathbb{A}_3$, where

$$\mathbb{A}_1 = \{2, 4\},\ \mathbb{A}_2 = \{1_2, 3_2\},\ \mathbb{A}_3 = \{5_3\}.$$

Since we have $|\mathbb{A}_1| = |\mathbb{A}_2| = 2$ and $|\mathbb{A}_3| = 1$, the partition of $\mathbb{A}$ is $(1^2, 2^2, 3)$.

Note that either the zeros or poles of a meromorphic function $r \in \mathbb{M}(\widehat{\mathbb{C}})$ form a multiset as

$$Z_r := \{z_{\mathrm{ord}_r(z)}\colon\, r(z) = 0\} \quad \text{or} \quad P_r := \{z_{\mathrm{ord}_r(z)}\colon\, r(z) = \infty\},$$

where $\mathrm{ord}_r(z)$ is the order of the zero or pole of $r$. Finally, we have:

**Definition 2.4.** Let $r \in \mathbb{M}(\widehat{\mathbb{C}})$ be a rational function of degree $n$. Let $Z_r$ and $P_r$ be the multisets of zeros and poles of $r$, respectively. Then
(1) the zero symbol $\sigma_\bullet(r) := \sigma(Z_r)$ is the partition of $Z_r$;
(2) the pole symbol $\sigma_\times(r) = \sigma(P_r)$ is the partition of $P_r$.

**2.4. Harmonic level set.** Let $f(z) = u(x,y) + \mathrm{i}v(x,y)$ be a holomorphic function defined on $\Omega \subset \mathbb{C}$, which is not necessarily a polynomial. Then $u = \Re(f)$ and $v = \Im(f)$ are two *harmonic functions*, i.e., they satisfy Laplace's equation $\Delta h := h_{xx} + h_{yy} = 0$. Conversely, a pair of real smooth functions $(u,v)$ is called a *conjugate pair* if they solve the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x.$$

Note that if $(u,v)$ is a conjugate pair given by some $f$, then $uv = \Im(\frac{1}{2}f^2)$ is also a harmonic function since $\frac{1}{2}f^2$ is a holomorphic function.

Let $g\colon X \to Y$ be a continuous function and $\alpha \in Y$. Then $g^{-1}(\alpha)$ is called the $\alpha$ *level set* of $g$. Let $\mathbf{Z}_g := g^{-1}(0)$ be the zero level set of $g$.

**Definition 2.5.** Let $f(z) = u(x,y) + \mathrm{i}v(x,y)$ be a holomorphic function from $\mathbb{C}$ to itself. We call $\mathbf{RE}_f := \mathbf{Z}_v$, $\mathbf{IM}_f := \mathbf{Z}_u$ and $\mathbf{NIL}_f := \mathbf{RE}_f \cup \mathbf{IM}_f = \mathbf{Z}_{uv}$ the *real-pullback*, *imaginary-pullback* and *nil-pullback* of $f$, respectively.

**Remark 2.6.** In this paper, we use $Z_r$ to represent the MULTISET of zeros and $\mathbf{Z}_r$ to represent the SET of zeros. Moreover, all of $\mathbf{Z}_f, \mathbf{RE}_f, \mathbf{IM}_f$ and $\mathbf{NIL}_f$ are realized as real algebraic varieties by some harmonic functions.

Let $\mathrm{Crit}_z(f) := \{z\colon f_z(z) = 0\}$ be the critical points of $f$ with respect to $z$ and $\mathrm{Crit}_{x,y}(g) := \{(x,y)\colon g_x = g_y = 0\}$ the critical points of $g$ with respect to $(x,y)$. Then we have the following lemma.

**Lemma 2.7.** *For any holomorphic function* $f(z) = u(x,y) + \mathrm{i}v(z,y)$ *in* $\Omega \subset \mathbb{C}$, *we have*

$$\mathrm{Crit}_{x,y}(u) = \mathrm{Crit}_{x,y}(v) = \mathrm{Crit}_z(f).$$

Proof. Note that $\mathrm{Crit}_z(f) = \{z\colon f_z = 0\}$, $\mathrm{Crit}_{x,y}u = \{(x,y)\colon u_x = 0, u_y = 0\}$ and $\mathrm{Crit}_{x,y}v = \{(x,y)\colon v_x = 0, v_y = 0\}$. By $\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$ and the Cauchy-Riemann equations, we have

$$\begin{aligned} f_z &= \tfrac{1}{2}(\partial_x - \mathrm{i}\partial_y)(u + \mathrm{i}v) = \tfrac{1}{2}(u_x + \mathrm{i}v_x - \mathrm{i}u_y - \mathrm{i}^2 v_y) \\ &= \frac{u_x + v_y}{2} + \mathrm{i}\frac{v_x - u_y}{2} = u_x - \mathrm{i}u_y = v_y + \mathrm{i}v_x, \end{aligned}$$

so $z \in \mathrm{Crit}_z(f)$ if and only if $(x,y) \in \mathrm{Crit}_{x,y}(u) = \mathrm{Crit}_{x,y}(v)$. $\square$

Owing to the fact that zeros of holomorphic functions are isolated (see [3], page 3) and Lemma 2.7, we have:

**Corollary 2.8.** *The set of critical points of a nonconstant harmonic function $h$ is isolated.*

Furthermore, the local picture of a level set of a harmonic function is simple, since we have:

**Theorem 2.9** ([1], page 19). *The level set of a nonconstant harmonic function $h$ through a critical point $z_0$ consists locally of two or more analytic arcs intersecting with equal angles at $z_0$.*

## 3. Sectional Newtonian graphs

In this section, we always view the polynomial $f \in \mathbb{C}[z]$ as a meromorphic function $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by setting $f(\infty) = \infty$.

**3.1. Holomorphic pullback of CW complex structure.** Historically, as Stefánsson in [9] pointed out, it was Gauss who first paid attention to the zeros of flows for a Newtonian vector field. In fact, the curves of a Newtonian vector field are also the curves of fixed argument under $f \in \mathbb{C}[z]$ (see [9]), i.e., those curves are given by

$$A_f^\theta := \{z \in \mathbb{C}\colon \arg(f(z)) = \theta,\ \theta \in [0, 2\pi)\}.$$

Let $\mathbf{P}_f := f^{-1}(\infty)$ be the unique pole of $f$. Then the real-, imaginary- and nil-pullback of $f$ on $\widehat{\mathbb{C}}$ are

$$\mathbf{RE}_f = \mathbf{Z}_f \sqcup A_f^0 \sqcup A_f^\pi \sqcup \mathbf{P}_f, \quad \mathbf{IM}_f = \mathbf{Z}_f \sqcup A_f^{\pi/2} \sqcup A_f^{3\pi/2} \sqcup \mathbf{P}_f,$$

and

$$\mathbf{NIL}_f = \mathbf{Z}_f \sqcup A_f^0 \sqcup A_f^{\pi/2} \sqcup A_f^\pi \sqcup A_f^{3\pi/2} \sqcup \mathbf{P}_f,$$

respectively.

Let $z\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the identity map of $\widehat{\mathbb{C}}$ and $\Omega$ be the first quadrant of $\widehat{\mathbb{C}}$ (recall that $\Omega = st^{-1}(\{(x, y)\colon x > 0,\ y > 0\})$ where $st\colon \widehat{\mathbb{C}} \to \mathbb{C}$ is the stereographic projection). The nil-pullback $\mathbf{NIL}_z$ implies a natural CW complex structure $C_* = \{C_0, C_1, C_2\}$ on $\widehat{\mathbb{C}}$ (and hence on $\mathbb{C}$), where $C_0 = \{0, \infty\}$, $C_1 = \{A_z^{(k-1)\pi/2}\colon k = 1, 2, 3, 4\}$ and $C_2 = \{\mathrm{i}^{k-1} \cdot \Omega,\ k = 1, 2, 3, 4\}$. For notational simplicity, we use $(k)$ rather than $\mathrm{i}^{k-1} \cdot \Omega$ (see Figure 1).

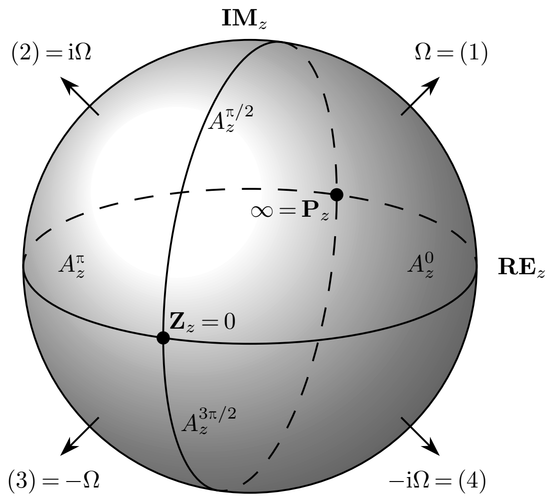


Figure 1. $\mathbf{NIL}_z$ on $\widehat{\mathbb{C}}$.

Factually, $\mathbf{NIL}_f$ induces a CW complex structure on $\widehat{\mathbb{C}}$ as follows. According to Lemma 2.7, there are at most finitely many critical points on $\mathbf{NIL}_f$. If $\mathbf{Z}_{f_z} \cap \mathbf{NIL}_f = \emptyset$, then each connected component of $A_f^\theta$, $\theta \in \{\frac{1}{2}k\pi\colon\ k = 1, 2, 3, 4\}$ is homeomorphic to $\mathbb{R}$. Otherwise, the local picture of $\mathbf{NIL}_f$ is described by Theorem 2.9: $\mathbf{NIL}_f$ is the 1-dimensional skeleton of the pullback CW complex structure $f^{-1}(C_*)$ on $\widehat{\mathbb{C}}$, since $f$ is a ramified covering mapping.

**Definition 3.1.** Let $C_f := f^{-1}(C_*)$ be the pullback CW complex structure and $C'_f$ be the (Poincaré) dual cell-complex of $C_f$. Then, the *first sectional Newtonian graph* $G_f$ is the 1-dimensional skeleton of $C_f$ and the *second sectional Newtonian graph* $G'_f$ is the 1-dimensional skeleton of $C'_f$.

**Example 3.2** ($f \in L_1^*$)**.** Consider $f = z$. Then the first SNG $G_z$ consists of 2 vertices and 4 edges, and the second SNG $G'_z$ is a cycle of length 4, see Figure 2.

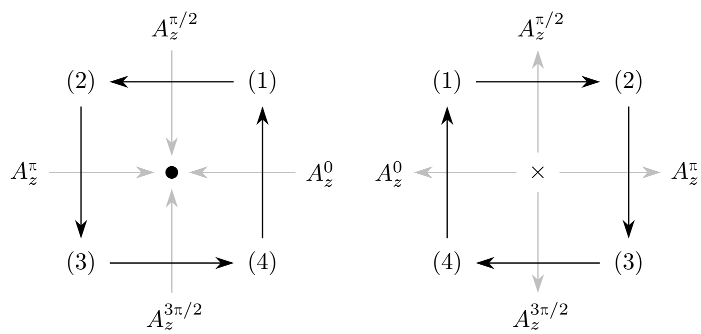


Figure 2. $G_z$ and $G'_z$.

In $G_z$, $\bullet$ is the zero and $\times$ is the pole. For a general $f = a_1 z + a_0 \in L_1^*$, $G_f$ is similar to $G_z$ with a proper rotation of angle $\arg a_1$. The directions of edges will be explained in the next subsection.

**3.2. Vertices and edges in SNG.** Let $G_f = (V_f, E_f)$ be the first SNG of $f \in \mathbb{C}[z]$. The edge set $E_f$ is relatively simple: if $V_f$ is given, then $E_f$ consists of all connected components in $\mathbf{NIL}_f - V_f$. Similar to the names of vertices in [7], the vertex set $V_f$ consists of all zeros and poles of $\hat{f}$, and some zeros of $f_z$ as follows:

$$V_f = \mathbf{Z}_f \sqcup \mathbf{P}_f \sqcup \mathbf{S}_f,$$

where $\mathbf{S}_f := \{w\colon\ w \in \mathrm{Crit}_z(f) \cap \mathbf{NIL}_f\}$. Elements of $\mathbf{Z}_f$, $\mathbf{P}_f$ and $\mathbf{S}_f$ are called *sinks*, *sources* and *saddles*, respectively. Those names come from the fact that there is a natural poset structure $\prec$ on $V_f$ as follows. Assume that $e \in E_f$ connects $p, q \in V_f$, then by definition of SNG we know that $e \subset A_z^\theta$ for some $\theta \in \{(k-1)\pi/2\colon$ $k = 1, 2, 3, 4\}$ and either $|f(p)| > |f(q)|$ or $|f(p)| < |f(q)|$ holds. Then we define $p \prec q$ if $|f(p)| < |f(q)|$ and $p \succ q$, otherwise. Since $f$ is a ramified covering, there are 3 possibilities of the directions of edges around a vertex.

**Proposition 3.3.** *Let $G_f = (V_f, E_f)$ be the first SNG of $f \in \mathbb{C}[z]$ and $v \in E_f$ be a vertex. Then we have:*

(1) *$v$ is a sink if and only if all edges meeting at $v$ are incoming edges;*
(2) *$v$ is a saddle if and only if incoming edges and outgoing edges appear alternatively around it;*
(3) *$v$ is a source if and only if all edges meeting at $v$ are outgoing edges.*

P r o o f. It is a direct corollary of the fact that $f$ is a ramified covering. $\square$

**Remark 3.4.** For a meromorphic function $f \in \mathbb{M}(\widehat{\mathbb{C}})$, if we use

$$\mathrm{Crit}_{\mathbb{M}}(f) = \{z \in \widehat{\mathbb{C}}\colon \text{ for all neighborhoods } U \text{ of } z, f\big|_U \text{ is not injective}\}$$

to represent the set of critical points, then we have a simple description of vertices in $G_f$ as $V_f = \{z \in \mathrm{Crit}_{\mathbb{M}}(f^2)\colon\ f^2(z) \in \widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}\}$.

**Definition 3.5.** An open set $K \subset \mathbb{C}$ is called a *basic domain* of $f$ if it is a connected component in $\mathbb{C} - \mathbf{NIL}_f$. Moreover, the inverse image $\widehat{K} = st^{-1}(K)$ of stereographic projection $st$ is called a *spherical basic domain* on $\widehat{\mathbb{C}}$.

Let $G'_f = (V'_f, E'_f)$ be the second SNG of $f \in \mathbb{C}[z]$. By the Poincaré duality, each vertex in $V'_f$ names a spherical basic domain in the pullback CW structure $C_f$. Moreover, there is a one-to-one correspondence between $E_f$ and $E'_f$.

**Remark 3.6.** From now on, we always assume that both $G_f$ and $G'_f$ are directed graphs. For $G_f$, the directions follow the partial order $\prec$ on $V_f$. For $G'_f$, the directions follow the pullback of cyclic order of quadrants on $\mathbb{C}$ as:

$$\ldots \leftarrow (1) \leftarrow (4) \leftarrow (3) \leftarrow (2) \leftarrow (1) \leftarrow (4) \leftarrow \ldots.$$

The so-called *vertex-cycle correspondence* is as follows. For each vertex $v \in V_f$, there is a cycle $P \subset G'_f$ around $v$. Let $\mathrm{ord}_f(v)$ be the order of zeros or poles of $f$ at $v$. According to the type of vertex, we call such cycle

(1) a *sink-cycle* of length $4 \cdot \mathrm{ord}_f(v)$:

$$\overbrace{\ldots \leftarrow (1) \leftarrow (4) \leftarrow (3) \leftarrow (2) \leftarrow (1) \leftarrow (4) \leftarrow \ldots}^{\text{counterclockwise}};$$

(2) a *saddle-cycle* of length $2 + 2 \cdot \mathrm{ord}_{f_z}(v)$:

$$\ldots \leftarrow (i) \rightarrow (j) \leftarrow (i) \rightarrow \ldots,$$

where $ij \in \{12, 23, 34, 41\}$;

(3) a *source-cycle* of length $4 \cdot \mathrm{ord}_f(v)$:

$$\overbrace{\ldots \rightarrow (4) \rightarrow (1) \rightarrow (2) \rightarrow (3) \rightarrow (4) \rightarrow (1) \rightarrow \ldots}^{\text{clockwise}}.$$

Then we have:

**Definition 3.7.** Let $G'_f$ be the second SNG of $f \in L_n^*$. A *basic cycle* $B$ in $G'_f$ is a cycle (may have self-intersections at some vertices) of length $4d$ of the form

$$\ldots \rightarrow (4) \rightarrow (1) \rightarrow (2) \rightarrow (3) \rightarrow (4) \rightarrow (1) \rightarrow \ldots,$$

where $1 \leqslant d \leqslant n$.

**3.3. Examples.** Let $f \in L_n^*$ be a polynomial of degree $n$. In $G_f$, we mark sinks, sources and saddles with $\bullet$, $\times$ and $\circ$, respectively. In $G'_f$, we use $\{(i)\colon\ i = 1, 2, 3, 4\}$ to represent the $i$th quadrant on $\mathbb{C}$. For convenience, we drop the unique pole $\infty$ from $G_f$ and draw $G_f$ as a planar graph.

**Example 3.8** ($f \in L_2^*$). Let $f = a_2z^2+a_1z+a_0$, where $a_k = x_k+\mathrm{i}y_k$, $k = 0, 1, 2$. In general, we have

$$u(x, y) = x_2(x^2 - y^2) - 2y_2xy + x_1x - y_1y + x_0,$$
$$v(x, y) = 2x_2xy + y_2(x^2 - y^2) + x_1y + y_1x + y_0$$

and $G_f$ is the union of two hyperbolic curves (if the $a_k$ are in general positions) with two intersections:

$$z_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2}.$$

For example, if $f = z^2 \pm 1$, then $G_f$ consists of grey lines and $G_f'$ consists of black lines, see Figure 3.

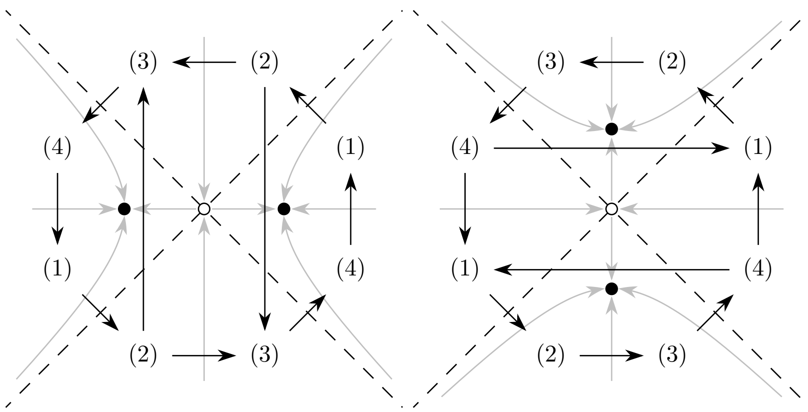


Figure 3. SNG: $z^2 - 1$ and $z^2 + 1$.

**Example 3.9** ($f = z^n - a_0$). Let $f = z^n - a_0$ for some $n \geqslant 2$, where $a_0 = x_0+\mathrm{i}y_0$. Since 0 is the unique zero of $f_z = nz^{n-1}$, $\mathbf{RE}_f$ and $\mathbf{IM}_f$ are both smooth varieties unless $x_0y_0 = 0$. According to the sign of $x_0$ and $y_0$, there are 3 types of SNG, denoted by A, B and C:

| | $x_0 < 0$ | $x_0 = 0$ | $x_0 > 0$ |
|---|---|---|---|
| $y_0 > 0$ | C | B | C |
| $y_0 = 0$ | B | A | B |
| $y_0 < 0$ | C | B | C |

Let $(ijkl) \in \{(1234), (2341), (3412), (4123)\}$. Since $G_f$ and $G_f'$ determine each other, we only draw $G_f'$ for $n = 4$, see Figure 4:

*Type A*: $G_f'$ is a $4n$-polygon ($|E_f'| = 4n$).

*Type B*: $G_f'$ is a $2n$-polygon attached with $n$ 4-polygons ($|E_f'| = 2n+4n-n = 5n$).

*Type C*: $G_f'$ is an $n$-flower ($|E_f'| = 4n$).

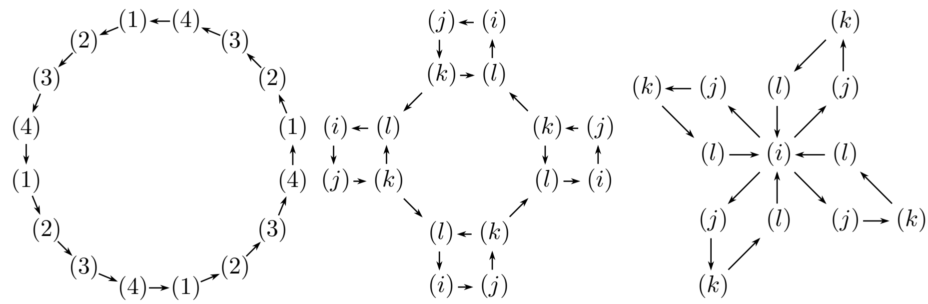

Figure 4. $G'_f$ of $f = z^n - a_0$: type A, B and C.

**Example 3.10** ($f = r_0(z - r_1)\dots(z - r_n)$, $r_i \in \mathbb{R}$)**.** First, we assume that $f$ is a real polynomial with different real roots, see Figure 5. By Rolle's theorem, any zero of $f_z$ is located between two different zeros of $f$. So $G_f - \{\infty\}$ is a tree containing $2n - 1$ points ($n$ zeros and $n - 1$ saddles), $2(n - 1)$ edges and $4n$ tails, while $G'_f$ contains $2n - 1$ squares.

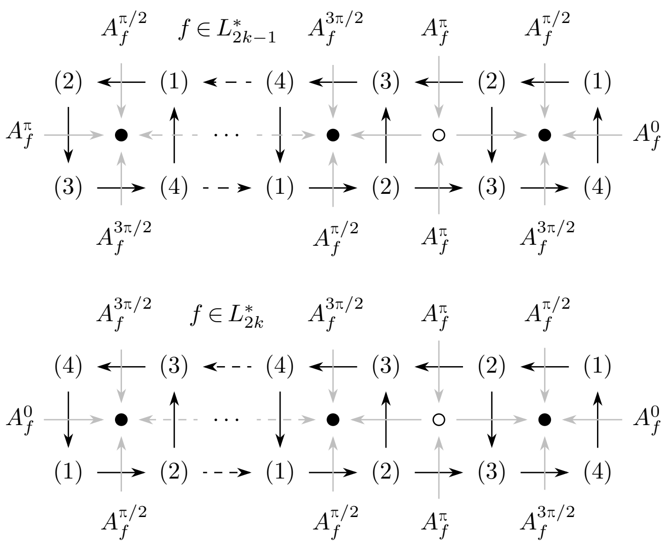


Figure 5. SNG: $f \in \mathbb{R}[z]$ with different real zeros.

If $f$ contains multiple roots, then $G_f$ looks as in Figure 6 (we omit all edges connecting to $\infty$ and the up-scripts are the degrees of vertices).

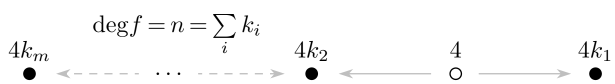


Figure 6. $G_f$: $f$ with multiple zeros.

If the zero symbol of $f$ is $\sigma_\bullet = (1^{d_1}, 2^{d_2}, \ldots)$, then $G'_f$ consists of $d_i$ $4i$-polygons (around the zeros of $f$) and $|\sigma_\bullet| - 1$ squares (around the zeros of $f_z$) since each saddle point in $G_f$ is of order 1.

**3.4. Meromorphic SNG.** The definition of SNG works for entire functions on $\mathbb{C}$ or algebraic functions on Riemann surfaces. However, those graphs may be WILD: some consist of infinitely many vertices and edges, such as $f = \cos z$; while some others do not contain any vertex, such as $f = \exp z$. For simplicity, we only discuss SNG of meromorphic functions on $\widehat{\mathbb{C}}$.

Let $G_r = (V_r, E_r)$ and $G'_r = (V'_r, E'_r)$ be the first and second SNG induced by the pullback CW structure $C_r := r^{-1}(C_*)$ and its dual $C'_r$, respectively. Then we have:

**Proposition 3.11.** *Let $r \in \mathbb{M}(\widehat{\mathbb{C}})$ be a rational function, then $G_r$ and $G'_r$ are finite planar graphs.*

Similarly, the classification of vertices and Proposition 3.3 for polynomial SNG also work for meromorphic SNG. Moreover, we have:

**Corollary 3.12.** *Let $G_r = (V_r, E_r)$ be the first SNG of $r \in \mathbb{M}(\widehat{\mathbb{C}})$. If $e = pq \in E_r$ and $p$, $q$ share the same type of vertices, then both of them are saddles.*

**Example 3.13** ($r = (z-1)/(z+1)$). The SNG of $r = (z-1)/(z+1)$, see Figure 7, is isotopic to the SNG of $r = z$ (see Example 3.2). For meromorphic SNG, there is no canonical projection from $\widehat{\mathbb{C}}$ to $\mathbb{C}$ since the north pole is no longer a special point. Note that in $G'_r$, the dashed cycle in Figure 7 is just the solid one.

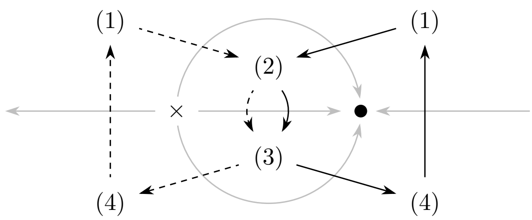


Figure 7. SNG of $(z-1)/(z+1)$.

**Theorem 3.14.** *For any $s, t \in \mathbb{M}(\widehat{\mathbb{C}})$, if there exists a Möbius transformation $T\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $z \mapsto (az+b)/(cz+d)$, where $ad - bc \in \mathbb{C}^*$ such that $s = t \circ T$, then $T$ induces an isomorphism between the pullback CW complex structures $C_s = T^{-1}(C_t)$.*

Proof. Trivial, since $\mathbf{NIL}_s = \mathbf{NIL}_{t\circ T} = T^{-1}(\mathbf{NIL}_t)$. □

**Corollary 3.15.** *Let $f \in L_2^*$. Then $G'_f$ must be of one of the types A, B and C in Example* 3.9.

Proof. Note that

$$f = a_2 z^2 + a_1 z + a_0 = \Big(\sqrt{a_2} z + \frac{a_1}{2\sqrt{a_2}}\Big)^2 + a_0 - \frac{a_1}{4a_2}$$
$$= \Big(\frac{2a_2 \cdot z + a_1}{0 \cdot z + 2\sqrt{a_2}}\Big)^2 + a_0 - \frac{a_1}{4a_2} =: w^2 + w_0.$$

By Theorem 3.14,

$$T(z) = \frac{2a_2 \cdot z + a_1}{2\sqrt{a_2}},$$

which completes the proof. □

**Example 3.16.** Consider $\prod_{k=1}^{4} (z - \mathrm{i}^k)^{p(k)}$, where $p(k) = \pm 1$. Because of the obvious symmetry, we always assume that $z = \mathrm{i}$ is a zero of $r$. Then we have the $G_r$ of

$$\frac{z-\mathrm{i}}{(z+\mathrm{i})(z+1)(z-1)}, \quad \frac{(z-\mathrm{i})(z-1)}{(z+\mathrm{i})(z+1)}, \quad \frac{(z-\mathrm{i})(z+\mathrm{i})}{(z-1)(z+1)},$$
$$\frac{(z-\mathrm{i})(z+1)}{(z+\mathrm{i})(z-1)}, \quad \frac{(z-\mathrm{i})(z-1)(z+1)}{z+\mathrm{i}}$$

as shown in Figure 8.

Note that the dotted circle is the $\infty \in \widehat{\mathbb{C}}$, since $\mathbb{S}^2 = D^2/\partial D^2$ as a quotient space of the closed disk on $\mathbb{R}^2$.

**Example 3.17** ($r(k) = z^k(z^2 - z^{-2})$, $k \in \mathbb{N}$)**.** Note that $r_z = z^{k-3}((k+2)z^4 - (k-2))$. Set $Z = \{\mathrm{i}^m \colon m = 1,2,3,4\}$ and $S = \{\mathrm{i}^m \sqrt[4]{(k-2)/(k+2)} \colon m = 1,2,3,4\}$. Then we have

$$V_r = \begin{cases} \mathbf{Z}_r = \{0_{k-2}\} \cup Z, & \mathbf{P}_r = \{\infty_{k+2}\}, & \mathbf{S}_r = S, & k > 2, \\ \mathbf{Z}_r = Z, & \mathbf{P}_r = \{\infty_4\}, & \mathbf{S}_r = \{0_3\} = S, & k = 2, \\ \mathbf{Z}_r = Z, & \mathbf{P}_r = \{0_1, \infty_3\}, & \mathbf{S}_r = \emptyset, & k = 1, \\ \mathbf{Z}_r = Z, & \mathbf{P}_r = \{0_2, \infty_2\}, & \mathbf{S}_r = S, & k = 0, \\ \mathbf{Z}_r = Z, & \mathbf{P}_r = \{0_3, \infty_1\}, & \mathbf{S}_r = \emptyset, & k = -1, \\ \mathbf{Z}_r = Z, & \mathbf{P}_r = \{0_4\}, & \mathbf{S}_r = \{\infty_3\} = S, & k = -2, \\ \mathbf{Z}_r = \{\infty_{-k-2}\} \cup Z, & \mathbf{P}_r = \{0_{-k+2}\}, & \mathbf{S}_r = S, & k < -2, \end{cases}$$

where the subscript of 0 or $\infty$ is its order. Note that there is a natural duality between $r(k)$ and $r(-k)$, since $w = z^{-1}$ maps the local picture of $G_{r(k)}$ near 0 to the local picture of $G_{r(-k)}$ near $\infty$.

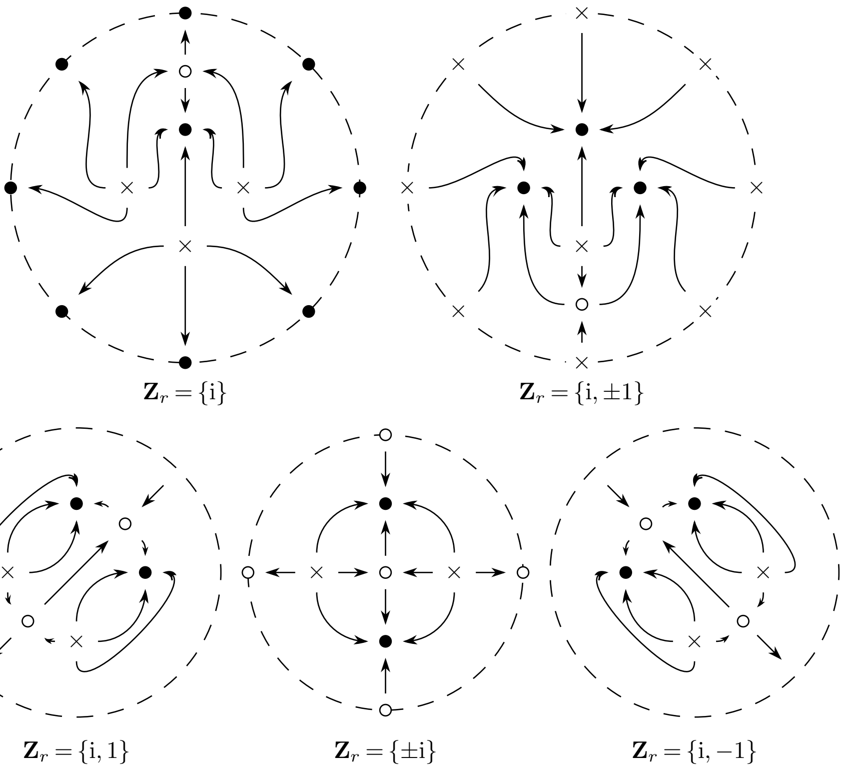


Figure 8. SNG of $\prod_{k=1}^{4} (z - \mathrm{i}^k)^{p(k)}$, $p(k) = \pm 1$.

## 4. Properties of SNG

**4.1. Equivalence of SNG.** To define the equivalence between SNG, we need to introduce gradient graphs as follows.

**Definition 4.1.** Let $D = (G, \phi)$ be an ordered pair where $G = (V, E)$ is a finite directed graph embedded in $\mathbb{S}^2$ with an edge-coloring map

$$\varphi\colon\ E \to \mathbb{Z}_4 := \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}.$$

We say that $D$ is a *gradient graph* if:

(1) $V = Z \sqcup P \sqcup S$: the set of vertices consists of zeros, poles and saddles and the type of a vertex $v \in V$ is determined by Proposition 3.3;

(2) the edge-coloring map $\varphi$ is compatible with $V$, i.e., the edge-coloring looks like a sink-cycle at each zero, like a source-cycle at each pole and like a saddle-cycle at each saddle.

Similar to SNG, $D$ induces a CW complex structure $C$ on $\mathbb{S}^2$. Then the dual graph $D' = (G', \phi) = ((V', E'), \phi)$ is the one-dimensional skeleton of the Poincaré duality of $C$. Note that $\phi$ is also an edge-coloring of $E'$.

**Definition 4.2.** Let $D = (G, \phi)$ be a gradient graph and $D' = (G', \phi)$ be its dual graph induced by the Poincaré duality. Then $D'$ is called the *vortex graph* of $D$.

The names *gradient* and *vortex* come from the fact that the directed graph structure on $V$ follows the gradient field $\nabla f$ and the tangent field $\mathrm{e}^{\mathrm{i}\pi/2} \cdot \nabla f$ on $\mathbb{C}$, where $f(r, \theta) = r$ is the radius function in the polar coordinate of $\mathbb{C}$. Since one can read $D'$ from $D$ via duality, we only define the equivalence of gradient graphs as follows.

**Definition 4.3.** Let $D_i = (G_i, \phi_i) = ((V_i, E_i), \phi_i) = (Z_i \sqcup P_i \sqcup S_i, E_i)$ be two gradient graphs and $C_i$ be the CW complex structures induced by $G_i$ where $i = 1, 2$. We say that they are *equivalent*, denoted by $D_1 \sim D_2$, if there is an orientation-preserving map $\psi\colon \mathbb{S}^2 \to \mathbb{S}^2$ such that:

(1) $\psi$ is a graph isomorphism compatible with the types of vertices, which sends zeros/poles/saddles of $D_1$ to zeros/poles/saddles of $D_2$ as $\psi(Z_1) = Z_2, \psi(P_1) = P_2$ and $\psi(S_1) = S_2$.

(2) $\psi$ is compatible with the edge-coloring maps, i.e,

$$k_\psi = \phi_1(e) - \phi_2(\psi(e))$$

is independent of the choice of $e \in E_1$.

**Remark 4.4.** We list some remarks here:

(1) The equivalence of gradient graphs is much stronger than the usual equivalence in graph theory.

(2) Condition (1) in Definition 4.3 does NOT imply condition (2) alone. For example, consider the complex conjugation $\psi\colon\ f \to \overline{f}$.

(3) $\phi$ is not necessarily a Möbius transformation. Note that all SNG of the form

$$f_k(z) = (z+1)z(z-1)(z-k), \quad k > 1,$$

are equivalent. But the restricted mapping $\psi_{ba}\colon \mathbf{NIL}_{f_a} \to \mathbf{NIL}_{f_b}$ where $a > b > 1$ cannot be a Möbius transformation.

**4.2. Basic cycle.** Note that the Definition 3.7 of basic cycle also works for meromorphic functions on $\widehat{\mathbb{C}}$.

Let $\mathbb{S}^1 \hookrightarrow \mathbb{S}^2 \cong \widehat{\mathbb{C}}$ be an embedded circle without self-intersections which is compatible with the orientation on $\mathbb{S}^2$. Then we have a decomposition of the Riemann sphere along $\mathbb{S}^1$ as

$$\mathbb{S}^2 = \Omega^\bullet \sqcup \mathbb{S}^1 \sqcup \Omega^\times,$$

where $\partial\Omega^\bullet$ shares the direction of $\mathbb{S}^1$ and $\partial\Omega^\times$ has the opposite one. Roughly speaking, a basic cycle $P$ of $f$ is a piecewise linear $\mathbb{S}^1$ on $\widehat{\mathbb{C}}$. Even though there may exist some self-intersections on $P$, we can get rid of them via some homotopic pushing. Finally, a basic cycle $P$ of a polynomial $f$ gives two domains in $\mathbb{C}$: the unbounded one is called the outer domain $\Omega_P^\times$, and the bounded one is called the inner domain $\Omega_P^\bullet$ which may be a disjoint union of open disks in $\mathbb{C}$.

**Proposition 4.5.** *Let $P$ be a basic cycle of $f \in L_n^*$ of length $4d$, then*

$$d = \sum_{z \in \mathbf{Z}_f \cap \Omega_P^\bullet} \operatorname{ord}(z)$$

*is the number of zeros in $\Omega_P^\bullet$ (counted with multiplicities).*

This proposition is a direct corollary of the following theorem which may be called the discrete argument principle:

**Theorem 4.6.** *Let $P$ be a basic cycle of $r \in \mathbb{M}(\widehat{\mathbb{C}})$ of length $4d$, then*

$$d = \sum_{z \in \mathbf{Z}_r \cap \Omega_P^\bullet} \operatorname{ord}(z) - \sum_{p \in \mathbf{P}_r \cap \Omega_P^\bullet} \operatorname{ord}(p) = \sum_{p \in \mathbf{P}_r \cap \Omega_P^\times} \operatorname{ord}(p) - \sum_{z \in \mathbf{Z}_r \cap \Omega_P^\times} \operatorname{ord}(z).$$

Using the homotopic pushing of curves on $\mathbb{S}^2$, this theorem is nothing but the classical argument principle:

**Theorem 4.7.** *Let $\Omega$ be a bounded domain in $\mathbb{C}$ with continuous boundary $\Gamma := \partial\Omega$ in the counterclockwise direction. Then we have*

$$\frac{1}{2\pi}\Delta_\Gamma \arg f(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f_z(w)}{f(w)}\,\mathrm{d}z = \sum_{z \in \mathbf{Z}_f \cap \Omega} \operatorname{ord}(z) - \sum_{p \in \mathbf{P}_f \cap \Omega} \operatorname{ord}(p),$$

*where $\Delta_\Gamma \arg f(z)$ is the change of argument of $f(z)$ along $\Gamma$.*

### 4.3. Basic domain.

**Proposition 4.8.** *Let $f = a_n z^n + \ldots + a_0 \in L_n^*$ be a polynomial and $K$ be a basic region of $f$ on $\mathbb{C}$. Then $K$ is unbounded on $\mathbb{C}$. Moreover, there are at most $4n$ vertices in the second SNG $G'_f$.*

P r o o f. We claim that if $K$ is bounded, then $f(K)$ is bounded, too. Otherwise, for any $z \in K$ we have $|z| < M$, $M \in \mathbb{R}$ and $|f(z)| \leqslant |a_n|M^n + \ldots + |a_0| < \infty$. However, $f(K)$ is one quadrant in $\mathbb{C}$ which is unbounded. Moreover, there are at most $4n$ unbounded domains (all around $\infty$). By the correspondence between vertices in $G'_f$ and basic domains in $\mathbb{C}$, the proof is completed. □

**Definition 4.9.** Let $G_f$ be the first SNG of a polynomial $f \in L_n^*$. The *planar SNG* $G_f^* := G_f[V_f - \{\infty\}]$ is the induced subgraph of $G_f$.

**Corollary 4.10.** *Let $f \in L_n^*$ be a polynomial. Then $G_f^*$ is a forest.*

P r o o f. If there exists a cycle in $G_f^*$, then by Jordan's closed curve theorem we have at least one bounded basic region, which contradicts Proposition 4.8. □

Let $f \in L_n^*$ be a polynomial of degree $n$. Recall that $V_f = \mathbf{Z}_f \sqcup \mathbf{P}_f \sqcup \mathbf{S}_f$ for the first SNG $G_f = (V_f, E_f)$. Similar to the multiset of zeros and poles, the saddles in $G_f$ also form a multiset

$$S_f = \{s_{\mathrm{ord}_{f_z}(s)} \colon\ s \in \mathbf{S}_f\}$$

and the multiplicity decomposition of $S_f$ is

$$S_f = S_f(1) \sqcup S_f(2) \sqcup \ldots,$$

where $S_f(i) := \{s_i \in \mathbf{S}_f \colon\ \mathrm{ord}_{f_z}(s) = i\}$. Since $S_f$ is a finite multiset, we can define the saddle symbol $\sigma_\circ$ as the partition $\sigma(S_f) \in P(k)$ of $S_f$:

$$\sigma(S_f) := (1^{d_1}, \ldots, (n-1)^{d_{n-1}}) \in P(k),$$

where $d_i = |S_f(i)|$ and $0 \leqslant k = d_1 + \ldots + d_{n-1} \leqslant n - 1$.

Let $\sigma_\bullet = \sigma(Z_f)$, $\sigma_\circ = \sigma(S_f)$ and $\sigma_\times = \sigma(P_f)$ be the zero, saddle and pole symbols of $f \in L_n^*$, respectively.

**Proposition 4.11.** *Let $f \in L_n^*$ and $\chi_f = \chi(G_f^*)$ be the number of trees in the planar SNG $G_f^* = (V_f^*, E_f^*)$, where $\chi$ is the Euler characteristics. Then we have*

$$\chi_f = |\sigma_\bullet| - \sum_{s \in \mathbf{S}_f} \mathrm{ord}_{f_z}(s) \in \{1, 2, \ldots, n\}.$$

Proof. By Proposition 3.3 and Corollary 3.12, the edge set $E_f^*$ consists of all outgoing edges starting from some saddle point. Note that

$$|V_f^*| = |\sigma_\bullet| + |\sigma_\circ|, \quad |E_f^*| = \sum_{s\in\mathbf{S}_f} (\mathrm{ord}_{f_z}(s) + 1) = \sum_{s\in\mathbf{S}_f} \mathrm{ord}_{f_z}(s) + |\sigma_\circ|.$$

Then we have

$$\chi_f = |V_f^*| - |E_f^*| = |\sigma_\bullet| - \sum_{s\in\mathbf{S}_f} \mathrm{ord}_{f_z}(s).$$

Moreover, since there is at least one tree and at most $n$ trees in $G_f^*$, we have $\chi_f \in \{1, 2, \ldots, n\}$. □

Finally, we have:

**Theorem 4.12.** *Let $f \in L_n^*$ be a polynomial, $G'_f = (V'_f, E'_f)$ be its second SNG and $\chi_f = \chi(G_f^*)$ be the number of trees in $G_f^*$. Then we have*

$$|V'_f| + \chi_f = 4n + 1.$$

Proof. Let $v$, $e$ and $d$ be the number of 0-, 1- and 2-cells in the pullback CW structure $C_f$ on $\mathbb{S}^2$. Then we have:

(1) For vertices, $v = |V_f^*| + |\sigma_\times| = |\sigma_\bullet| + |\sigma_\circ| + 1$.
(2) For edges, $e = 4n + |E_f^*| = 4n + \sum_{s\in\mathbf{S}_f} \mathrm{ord}_{f_z}(s) + |\sigma_\circ|$.
(3) For surfaces, $d = |V'_f|$.

On the other hand, we have $\chi(C_f) = \chi(\mathbb{S}^2) = v - e + d = 2$, which gives $|V'_f| + \chi_f = 4n + 1$ by a direct calculation. □

**4.4. Classification of polynomial SNG in degree 3 and 4.** As a direct corollary of Proposition 4.11 and Theorem 4.12, we know that there are at most $4n$ and at least $3n+1$ vertices in $G'_f$ if $f \in L_n^*$. In fact, we can construct all possible $G'_f$ from the basic cycle of length $4n$ via two kinds of operations:

(1) merging operation: merge some vertices of the same color (the double blue lines in $G'_f$ in Figure 9);
(2) connecting operation: connect some vertices of two adjacent colors $i$ and $j$ to form a saddle polygon (the grey arrows in $G'_f$ in Figure 9), where $(ij) \in \{(12), (23), (34), (41)\}$.

Let $\sigma_\bullet$ and $\sigma_\circ$ be the zero symbol and saddle symbol of $f$, respectively. If $\mathbf{S}_f = \emptyset$, we write $\sigma_\circ = (0)$ since the integer partition of 0 is empty. Through a straightforward but nontrivial calculation, we obtain Tables 1 and 2.

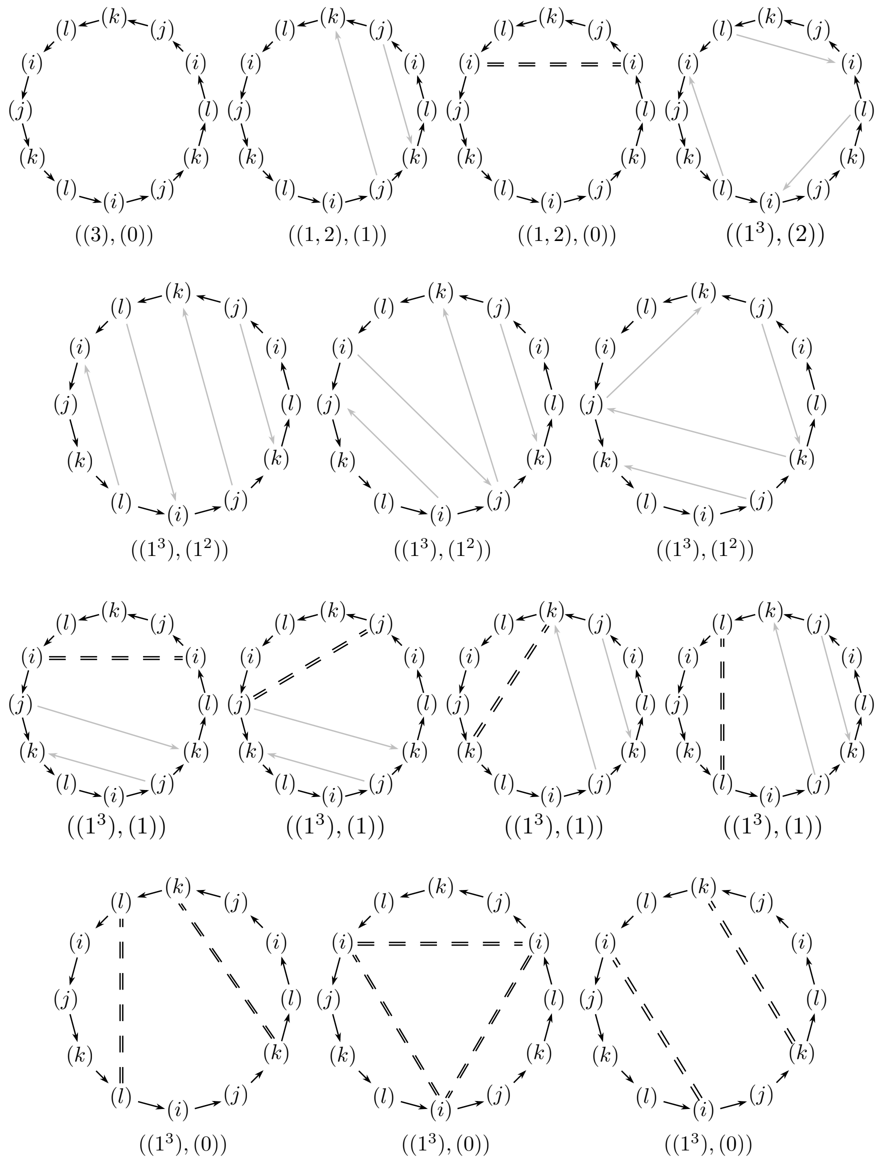


Figure 9. All possible $G'_f\colon\ f \in L_3^*$.

| $\sigma_\bullet \backslash \sigma_\circ$ | (2) | $(1^2)$ | (1) | (0) |
|---|---|---|---|---|
| (3) | – | – | – | 1 |
| (1, 2) | – | – | 1 | 1 |
| $(1^3)$ | 1 | 3 | 4 | 3 |

Table 1. The number of possible SNG of deg $f = 3$.

| $\sigma_\bullet \backslash \sigma_\circ$ | (3) | (1,2) | $(1^3)$ | (2) | $(1^2)$ | (1) | (0) |
|---|---|---|---|---|---|---|---|
| (4) | – | – | – | – | – | – | 1 |
| (1,3) | – | – | – | – | – | 1 | 1 |
| $(2^2)$ | – | – | – | – | – | 1 | 1 |
| $(1^2,2)$ | – | – | – | 1 | 10 | 16 | 8 |
| $(1^4)$ | 1 | 5 | 20 | 4 | 40 | 32 | 11 |

Table 2. The number of possible SNG of $\deg f = 4$.

Since we do not know whether a possible $G'_f$ is realized by a polynomial, we have:

**Theorem 4.13.** *Let $K_n$ be the collection of equivalent SNG, where $f \in L_n^*$ and $\kappa_n = |K_n|$. Then we have:*

$$\kappa(1) = 1, \quad \kappa(2) = 3, \quad \kappa(3) \leqslant 14, \quad \kappa(4) \leqslant 153.$$

P r o o f. For $n = 1$, see Example 3.2; for $n = 2$, see Proposition 3.15; finally, by Tables 1 and 2 we have

$$\kappa(3) \leqslant 1 + 1 + 1 + 1 + 3 + 4 + 3 = 14,$$
$$\kappa(4) \leqslant 1 + 1 + 1 + 1 + 1 + 1 + 10 + 16 + 8 + 1 + 5 + 20 + 4 + 40 + 32 + 11 = 153.$$

□

## 5. Proofs of the main results

**Definition 5.1.** Let $G = (V, E)$ be a gradient graph. We say that $G$ is *stable* if $V$ contains no saddle.

If a stable $G = G_f$ is realized by some polynomial $f$, then $V_f = \mathbf{Z}_f \sqcup \mathbf{P}_f$ and $G'_f$ contains no saddle cycle, which means that we can construct $G'_f$ from the basic cycle of length $4 \deg f$ only with merging operations. In this case, for simplicity, we also say that $f$ is stable.

**Proposition 5.2.** *Let $f \in L_n^*$ be a polynomial of degree $n \geqslant 1$. There are only finitely many $\lambda \in \mathbb{S}^1 := \{z \in \mathbb{C}\colon\ |z| = 1\}$ such that $\mathbf{S}_{\lambda f} \neq \emptyset$.*

P r o o f. Note that there are at most $n - 1$ zeros in $\mathbf{Z}_{f_z}$. Let $v \in \mathbf{Z}_{f_z}$ of order $d$, then there are at most $4d$ different $\lambda \in \mathbb{S}^1$ such that $\lambda f(v) \in \mathbf{NIL}_{\lambda f}$. So there are at most

$$\sum_{v \in \mathbf{Z}_{f_z}} 4d = 4(n-1),$$

where $\lambda$ is such that $\mathbf{S}_{\lambda f} \neq \emptyset$. □

By this proposition, stable polynomials are dense in $\mathbb{C}[z]$.

Given a stable graph, for each sink $v \in \mathbf{Z}_f$ of order $d$, the induced subgraph $G_f[\{v,\infty\}]$ looks like $G_{z^d}$, and it cuts the Riemann sphere $\widehat{\mathbb{C}}$ into $4d$ pieces as

$$\widehat{\mathbb{C}} - G_f[\{v,\infty\}] = U_1^v \sqcup \ldots \sqcup U_{4d}^v,$$

where each $U_j^v$ is called a *conic domain* of $v$.

**Definition 5.3.** Let $v$ be a sink in a stable graph $G$. We say that $v$ is a *marginal sink* if all other sinks belong to the same conic domain of $v$.

**Example 5.4.** Consider the stable $G_f$ with sinks $u$, $v$ and $w$ as in Figure 10. There are 4 conic domains around each vertex. Note that there are two marginal sinks: $v$ and $w$, since the other sinks belong to $U_4^v$ or $U_4^w$.

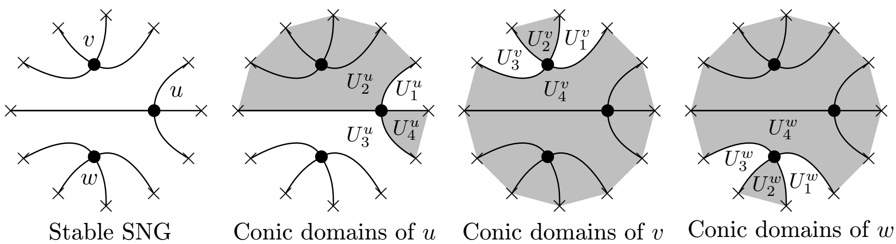


Figure 10. Conic domains of sinks.

**Lemma 5.5.** *Let $G$ be a stable gradient graph and $\alpha$, $\beta$ be two different sinks in $G$. Then there exists a conic domain $U^\alpha$ of $\alpha$ which contains all except one conic domains of $\beta$.*

P r o o f. Since $G$ is stable, $\beta$ belongs to a conic domain $U^\alpha$ of $\alpha$ and $\alpha$ belongs to a conic domain $U^\beta$. Finally $U^\alpha$ contains all conic domains of $\beta$ except $U^\beta$. □

**Theorem 5.6.** *Let $G$ be a stable gradient graph, then there exists a polynomial $f \in \mathbb{C}[z]$ such that $G \sim G_f$.*

P r o o f. Let $\sigma_\bullet = (1^{d_1}, 2^{d_2}, \ldots)$ be the zero symbol of $G = (V, E)$, $s = |\sigma_\bullet| = \sum d_j$ be its order and $d = \sum j d_j$. First we prove that for any stable $G$ there exists at least one marginal sink. We use induction on $s$.

If $s = 1$, then $f = z^d$ and 0 is the unique marginal sink since there is no other sink in $G$. Assume that the statement is true for $s < n$. Then for $s = n$, select an arbitrary sink $z_0$ and consider the induced graph $G_0 := G[V - \{z_0\}]$. By induction, $G_0$ contains at least one marginal sink, which is denoted by $z_1$, and there exists a conic domain $U(z_1)$ such that $V - \{z_0, \infty\} \subset U(z_1)$. Now we add back $z_0$. Since $G$

is stable, $z_0$ must belong to some conic domain $U'(z_1)$ of $z_1$: if $U'(z_1) = U(z_1)$, then $z_1$ is a marginal sink of $G$; otherwise, $z_0$ is a marginal sink of $G$, since there exists a conic domain $U(z_0)$ such that $z_1 \in U(z_0)$ and $U(z_1) \subset U(z_0)$ by Lemma 5.5.

Now we prove our main result by induction too. If $s = 1$, then we just take $f = z^d$; otherwise, $s \geqslant 2$. Note that there exists at least one marginal sink $m$ of order $k$ in $G$. Especially, for $s = 2$, by Proposition 5.2, $f_\lambda(z) = \lambda z^k (z-c)^{d-k}$, $c \neq 0$ realizes $G$ for some $\lambda \in \mathbb{S}^1$. By induction, we can find a polynomial $g$ such that $G_g$ is equivalent to the induced subgraph $G[V - \{m\}]$. By Theorem 3.14, $g$ and $h_s$ share the same SNG, where $h_s(z) = g(s(z-c))$, $s \in \mathbb{C}^*$ and $0 \ll c \in \mathbb{R}$. Let $z_i$ be a zero of $g(z)$, then $z_i' = z_i/s + c$ solves $h_s(z) = 0$. Without loss of generality, we may assume that all zeros of $h_s(z)$ are contained in a ball $B$ at $c$ with radius 1, and $0 \in \widehat{\mathbb{C}}$ is contained in the corresponding basic domain of $G[V - \{0\}]$. Then we claim that there exists a $\lambda \in \mathrm{e}^{\mathrm{i}\theta}$, $|\theta| \ll 1$ such that

$$f_\lambda(z) = \lambda z^k h_s(z)$$

realizes $G$. Note that the argument of $z \in B$ belongs to $(-\varepsilon, \varepsilon)$ where $\varepsilon \ll 1$. By Theorem 4.6, there are $4(d-k)$ incoming edges in the outside of $B$. Roughly speaking, if we can view $B$ as a sink and 0 as another sink, we complete the proof by the analysis of $s = 2$.

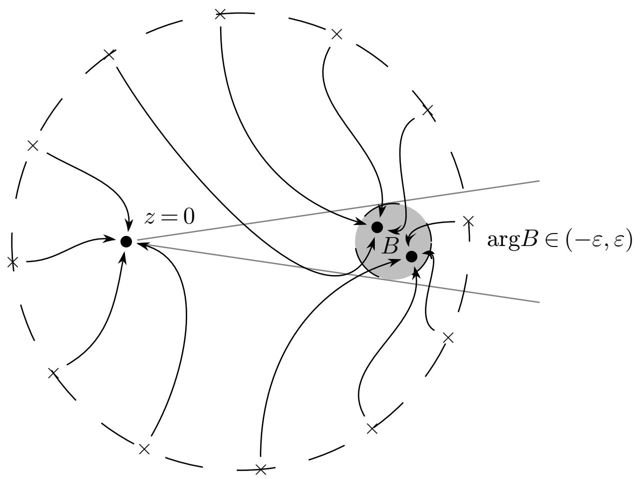


Figure 11. Reduction from $s > 2$ to $s = 2$.

Exactly, the local picture of $f_\lambda(z)$ in $B$ or at 0 is determined by $h_s(z)$ and $z^k$, together. Since $z^k$ is almost equal $c^k$ as a pure scaling, the local picture of $f_\lambda(z)$ is isotopic to $h_s(z)$; and $h_s(0)$ only causes a rotation on the incoming edges meeting at 0. By Proposition 5.2, we can find a sufficiently small rotation $\lambda = \mathrm{e}^{\mathrm{i}\theta}$ such that there is no saddle in $G_{f_\lambda}$ and which does not change the local picture in $B$ or at 0.

All in all, we can find a $\lambda \in \mathbb{S}^1$ such that 0 is a marginal sink of $G$, i.e., $G$ is realized by a polynomial $f_\lambda(z)$. $\square$

### A. Some examples of meromorphic SNG.

We draw the SNG of $r = z^k(z^2 - z^{-2})$ as follows in Example 3.17, see Figures 12 and 13.

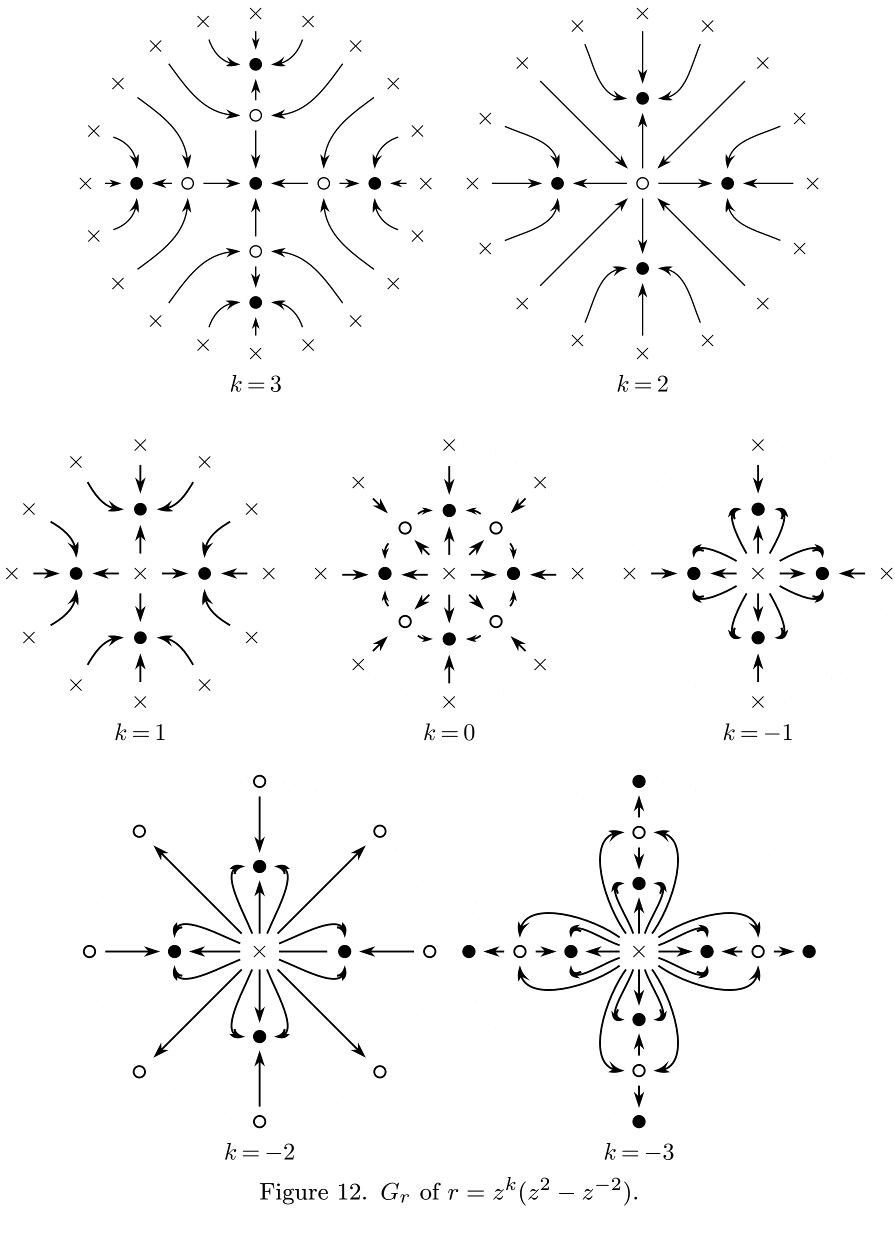


Figure 12. $G_r$ of $r = z^k(z^2 - z^{-2})$.

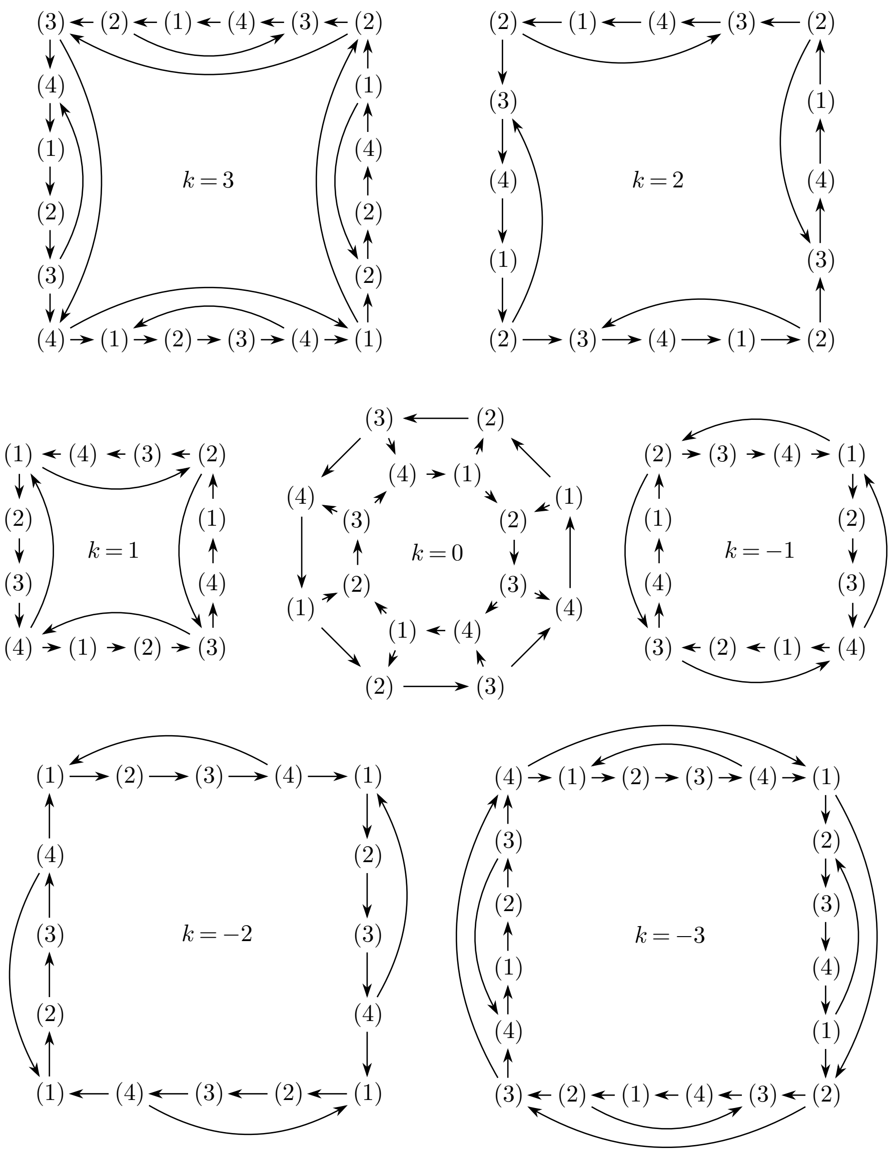


Figure 13. $G'_r$ of $r = z^k(z^2 - z^{-2})$

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*Authors' addresses*: Zening Fan, College of Mathematics, Sichuan University, No.24 South Section 1 Yihuan Road, Chengdu 610064, P. R. China, e-mail: zeningfan@126.com; Suo Zhao (corresponding author), Jiajia Research Center, Shanghai Yuanlu Jiajia Information & Technology Co., Ltd., No.1166 Shuangqiao Road, Shanghai 201208, P. R. China, e-mail: saaki@163.com.