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# P-INJECTIVE GROUP RINGS

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*Abstract.* A ring $R$ is called right P-injective if every homomorphism from a principal right ideal of $R$ to $R_R$ can be extended to a homomorphism from $R_R$ to $R_R$. Let $R$ be a ring and $G$ a group. Based on a result of Nicholson and Yousif, we prove that the group ring RG is right P-injective if and only if (a) $R$ is right P-injective; (b) $G$ is locally finite; and (c) for any finite subgroup $H$ of $G$ and any principal right ideal $I$ of RH, if $f \in \mathrm{Hom}_R(I_R, R_R)$, then there exists $g \in \mathrm{Hom}_R(\mathrm{RH}_R, R_R)$ such that $g|_I = f$. Similarly, we also obtain equivalent characterizations of $n$-injective group rings and F-injective group rings.

*Keywords*: group ring; P-injective ring; $n$-injective ring; F-injective ring

*MSC 2020*: 16S34, 16D50

## 1. Introduction

Throughout this paper rings are associative with identity and modules are unitary modules. Let $R$ be a ring, we use $\mathrm{Hom}_R(M_R, N_R)$ to denote the set of all $R$-homomorphisms between two right $R$-modules $M_R$ and $N_R$. If $G$ is a group, we use RG to denote the group ring of $G$ over $R$. For $\alpha = \sum\limits_{g\in G} a_g g \in \mathrm{RG}$, define $\mathrm{Supp}(\alpha) = \{g \in G \colon a_g \neq 0\}$ to be the *support* of $\alpha$. If $h \in G$, the projection $\pi_h \colon \mathrm{RG} \to R$ given by $\pi_h\Big(\sum\limits_{g\in G} a_g g\Big) = a_h$ is right and left $R$-linear. We write $\pi = \pi_{1_G}$. And $\pi(\alpha)$ is also called the *trace* of $\alpha$. Note that, if $\alpha \in \mathrm{RG}$, then $\pi_h(\alpha) = \pi(\alpha h^{-1}) = \pi(h^{-1}\alpha)$, and hence

$$\alpha = \sum_{g\in G} \pi_g(\alpha) g = \sum_{g\in G} \pi(\alpha g^{-1}) g = \sum_{g\in G} \pi(g^{-1}\alpha) g.$$

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Let $H$ be a subgroup of $G$. The map $\pi_H\colon \mathrm{RG} \to \mathrm{RH}$ given by $\pi_H\Big(\sum\limits_{g\in G} a_g g\Big) = \sum\limits_{h\in H} a_h h$ is called the *projection* of RG onto RH. As RH is a subring of RG, RG is naturally a two-sided RH-module.

Recall that a ring $R$ is called *right self-injective* if every homomorphism from a right ideal of $R$ to $R_R$ can be extended to an endomorphism of $R_R$. And $R$ is called *right P-injective* if every homomorphism from a principal right ideal of $R$ to $R_R$ can be extended to an endomorphism of $R_R$. Right P-injective rings were first introduced by Ikeda, see [3] in 1951. In 1963, Connell in [1] proved that for a finite group $G$, RG is right self-injective if and only if $R$ is right self-injective. In 1971, Renault in [7] showed that $G$ must be finite if RG is right self-injective. Thus RG is right self-injective if and only if $R$ is right self-injective and $G$ is finite. In 1975, Farkas in [2] showed that, if $F$ is a field, $FG$ is right P-injective if and only if $G$ is locally finite. In 1995, Nicholson and Yousif in [5], Theorem 4.1 proved the following result on P-injective group rings:

Let $R$ be a ring and $G$ a group.

(i) If RG is right P-injective, then $R$ is right P-injective and $G$ is locally finite.

(ii) If $R$ is right self-injective and $G$ is locally finite, then RG is right P-injective.

In this short paper, based on the above result of Nicholson and Yousif, an equivalent characterization of right P-injective group rings is given in Theorem 2.7. By a similar discussion, we also obtain an equivalent characterization of right $n$-injective group rings (see Theorem 2.9) and right F-injective group rings (see Corollary 2.10), respectively. Let $n$ be a positive integer. Recall that a ring $R$ is called *right $n$-injective* (*right F-injective*) if every homomorphism from an $n$-generated (finitely generated) right ideal of $R$ to $R_R$ can be extended to an endomorphism of $R_R$.

## 2. Results

**Lemma 2.1** ([5], Theorem 4.1 (1))**.** *Let $R$ be a ring and $G$ a group. If* RG *is right* P*-injective, then $R$ is right* P*-injective and $G$ is locally finite.*

Let $H$ be a subgroup of a group $G$. A complete set of representatives of left (right) cosets of $H$ in $G$ is called a *left* (*right*) *transversal* of $H$ in $G$.

**Proposition 2.2.** *Let $R$ be a ring, $H$ a subgroup of a group $G$ and $\{g_i\}_{i\in A}$ a right transversal of $H$ in $G$. Assume that $I$ is a right ideal of the group ring* RH *and $\{\alpha_j \in \mathrm{RH}\}_{j\in B}$ is a set of generators for the right* RH*-module $I$. Set $J = \sum\limits_{i\in A} I g_i$. Then $J$ is a right ideal of* RG *and $\{\alpha_j \in \mathrm{RH}\}_{j\in B}$ is also a set of generators for the right* RG*-module $J$.*

Proof. As $I = \sum\limits_{j \in B} \alpha_j(\mathrm{RH})$, we have

$$\begin{aligned} J &= \sum_{i \in A} \Big( \sum_{j \in B} \alpha_j(\mathrm{RH}) \Big) g_i = \sum_{i \in A} \sum_{j \in B} (\alpha_j(\mathrm{RH}) g_i) \\ &= \sum_{j \in B} \alpha_j \Big( \sum_{i \in A} (\mathrm{RH}) g_i \Big) = \sum_{j \in B} \alpha_j(\mathrm{RG}). \end{aligned}$$

So $J$ is a right ideal of RG generated by $\{\alpha_j \in \mathrm{RH}\}_{j \in B}$. □

**Lemma 2.3.** *Let $H$ be a subgroup of a group $G$ and $n$ a positive integer. If* RG *is right $n$-injective, then* RH *is also right $n$-injective.*

Proof. Let $\{g_i\}_{i \in A}$ be a right transversal of $H$ in $G$. It suffices to show that for any $n$-generated right ideal $I$ of RH, the following diagram of RH-homomorphisms can be completed:

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{\;i\;} & \mathrm{RH} \\ & & {\scriptstyle f}\big\downarrow & & \\ & & \mathrm{RH} & & \end{array}$$

Set $J = \sum\limits_i I g_i$. By Proposition 2.2, $J$ is an $n$-generated right ideal of RG. Define $\tilde{f}\colon J \to \mathrm{RG}$ by

$$\tilde{f}\Big(\sum \alpha_i g_i\Big) = \sum f(\alpha_i) g_i, \quad \alpha_i \in I.$$

If $u \in G$, then $g_i u = h_{ij} g_j$ for some $h_{ij} \in H$, and $j \in A$. So

$$\begin{aligned} \tilde{f}\Big(\Big(\sum \alpha_i g_i\Big) u\Big) &= \tilde{f}\Big(\sum \alpha_i h_{ij} g_j\Big) = \sum f(\alpha_i h_{ij}) g_j = \sum f(\alpha_i) h_{ij} g_j \\ &= \sum f(\alpha_i) g_i u = \tilde{f}\Big(\sum \alpha_i g_i\Big) u. \end{aligned}$$

Thus $\tilde{f}$ is a well-defined right RG-linear map. Since RG is right $n$-injective, there exists a right RG-homomorphism $\widetilde{\varphi}$ from RG to RG such that $\widetilde{\varphi}|_J = \tilde{f}$. Now set $\varphi = \pi_H \widetilde{\varphi}|_{\mathrm{RH}}$. Then $\varphi$ is a right RH-linear map and

$$\varphi|_I = \pi_H \widetilde{\varphi}|_I = \pi_H \tilde{f}|_I = \pi_H f = f.$$

So RH is also right $n$-injective. □

Taking $H = \{1_G\}$ in the above lemma and using Lemma 2.1, we have the following corollary.

**Corollary 2.4.** *Let $R$ be a ring and $G$ a group. If the group ring* RG *is right $n$-injective (F-injective), then $R$ is right $n$-injective (F-injective) and $G$ is locally finite.*

**Lemma 2.5** ([1], Proposition 7). *Let $R$ be a ring and $G$ a group. Assume that $M$ is a right* RG*-module. Then there is a group monomorphism*

$$t\colon \operatorname{Hom}_{\mathrm{RG}}(M_{\mathrm{RG}}, \mathrm{RG}_{\mathrm{RG}}) \to \operatorname{Hom}_R(M_R, R_R)$$

*such that $t(\varphi) = \pi\varphi$ for all $\varphi \in \operatorname{Hom}_{\mathrm{RG}}({}_{\mathrm{RG}}M, {}_{\mathrm{RG}}\mathrm{RG})$. In addition, if $G$ is a finite group, then $t$ is an isomorphism.*

Let RG be the group ring of a group $G$ over a ring $R$ and let $M$ be a right $R$-module. According to [4], elements of the *group module* MG are defined as follows:

$$\sum_{g\in G} m_g g, \quad \text{where } m_g \in M \text{ and } m_g = 0 \text{ for almost every } g.$$

The sum in MG is defined componentwise:

$$\sum_{g\in G} m_g g + \sum_{g\in G} n_g g = \sum_{g\in G} (m_g + n_g) g.$$

And the scalar product of $\sum\limits_{g\in G} m_g g$ by $\sum\limits_{g\in G} a_g g \in \mathrm{RG}$ is defined by

$$\Bigl(\sum_{g\in G} m_g g\Bigr)\Bigl(\sum_{g\in G} a_g g\Bigr) = \sum_{g\in G} k_g g, \quad \text{where } \ k_g = \sum_{hh'=g} m_h a_{h'}.$$

With the above two operations, MG becomes a right RG-module. It is also clear that MG is a right $R$-module with the canonical scalar product

$$\Bigl(\sum_{g\in G} m_g g\Bigr) r = \sum_{g\in G} (m_g r) g, \quad r \in R.$$

The following result was given in [4] without proof. To be self-contained, we write down the proof.

**Lemma 2.6** ([4], Lemma 5.1). *Let $M_R$ be a module and let $H$ be a subgroup of a group $G$. Then*

$$(\mathrm{MG})_{\mathrm{RG}} \cong (\mathrm{MH} \otimes_{\mathrm{RH}} \mathrm{RG})_{\mathrm{RG}}.$$

P r o o f. Let $K$ be a right transversal of $H$ in $G$. Then $\mathrm{RG} = \bigoplus\limits_{k\in K} (\mathrm{RH})k$ is a free left RH-module with basis $K$. It is easy to see that every element of $\mathrm{MH} \otimes_{\mathrm{RH}} \mathrm{RG}$ has the form $\sum\limits_{k\in K} \alpha_k \otimes k$, $\alpha_k \in \mathrm{MH}$. Now define a map

$$\Phi\colon \mathrm{MH} \otimes_{\mathrm{RH}} \mathrm{RG} \to (\mathrm{MG})_{\mathrm{RG}}$$

such that

$$\Phi\Big(\sum_{k\in K}\alpha_k\otimes k\Big)=\sum_{k\in K}\alpha_k k,\quad \alpha_k\in \mathrm{MH}.$$

It is clear that $\Phi$ is a right RG-homomorphism. Since MG is a direct sum of $(\mathrm{MH})k$, $k\in K$, $\Phi$ is an isomorphism. □

Now we prove the main result of this paper.

**Theorem 2.7.** *Let $R$ be a ring and $G$ a group. The following are equivalent:*

(i) RG *is right* P*-injective;*

(ii) (a) *$R$ is right* P*-injective;*

(b) *$G$ is a locally finite group;*

(c) *for each finite subgroup $H$ of $G$ and any principal right ideal $I$ of* RH*, if $f\in \mathrm{Hom}_R(I_R,R_R)$, there exists $g\in\mathrm{Hom}_R(\mathrm{RH}_R,R_R)$, such that $g|_I=f$.*

Proof. (i)⇒(ii). By Lemma 2.1, (a) and (b) are satisfied. For any finite subgroup $H$ of $G$, by Lemma 2.3, RH is right P-injective. For (c), let $H$ be a finite subgroup of $G$ and $I$ a principal right ideal of RH with $f\in\mathrm{Hom}_R(I_R,R_R)$. Since $H$ is finite, by Lemma 2.5, there exists $\varphi\in\mathrm{Hom}_{\mathrm{RH}}(I_{\mathrm{RH}},\mathrm{RH}_{\mathrm{RH}})$ such that $f=t(\varphi)=\pi\varphi$. As RH is right P-injective, there exists $\psi\in\mathrm{Hom}_{\mathrm{RH}}(\mathrm{RH}_{\mathrm{RH}},\mathrm{RH}_{\mathrm{RH}})$ such that $\psi|_I=\varphi$. Take $g=\pi\psi$. Then $g$ is a right $R$-linear map from $\mathrm{RH}_R$ to $R_R$ and $g|_I=f$.

(ii)⇒(i). First, we show that for any finite subgroup $H$ of $G$, RH is right P-injective. Assume $I=\alpha\mathrm{RH}$ is a principal right ideal of RH and $\varphi$ is a right RH-homomorphism from $I_{\mathrm{RH}}$ to $\mathrm{RH}_{\mathrm{RH}}$. We want to find an endomorphism $\psi$ of $\mathrm{RH}_{\mathrm{RH}}$ such that $\psi|_I=\varphi$. Let $f=\pi\varphi$. Then $f\in\mathrm{Hom}_R(I_R,R_R)$. By the assumption, there exists $g\in\mathrm{Hom}_R(\mathrm{RH}_R,R_R)$ such that $g|_I=f$. Since $H$ is finite, by Lemma 2.5 there exists $\psi\in\mathrm{Hom}_{\mathrm{RG}}(\mathrm{RH}_{\mathrm{RH}},\mathrm{RH}_{\mathrm{RH}})$ such that $\pi\psi=g$. Thus, $\pi\psi|_I=g_I=f=\pi\varphi$. For each $x\in I$, we have

$$\begin{aligned}\varphi(x)&=\sum_{h\in H}\pi(\varphi(x)h^{-1})h=\sum_{h\in H}(\pi\varphi(xh^{-1}))h=\sum_{h\in H}(\pi\psi(xh^{-1}))h\\&=\sum_{h\in H}\pi(\psi(x)h^{-1})h=\psi(x).\end{aligned}$$

Thus, $\psi|_I=\varphi$.

Next we show that RG is right P-injective; this needs to show that, for any principal right ideal $K=\alpha\mathrm{RG}$ of RG, every right RG-homomorphism $\varphi\colon K\to\mathrm{RG}$ can be extended to an endomorphism of $\mathrm{RG}_{\mathrm{RG}}$. Since $G$ is locally finite, there exists a finite subgroup $H$ of $G$ such that $\varphi(\alpha)\in\mathrm{RH}$ and $K'=\alpha(\mathrm{RH})\subseteq\mathrm{RH}\subseteq\mathrm{RG}$. Let $\iota\colon K\to\mathrm{RG}$ and $\iota'\colon K'\to\mathrm{RH}$ be the natural inclusions. If $\{g_i\}_{i\in A}$ is a right transversal of $H$, then $\mathrm{RG}=\oplus_{i\in A}(\mathrm{RH})g_i$ is a free left RH-module. So RG is a free left RH-module. Hence we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 & \longrightarrow & K' \otimes_{\mathrm{RH}} \mathrm{RG} & \xrightarrow{\iota'\otimes 1} & \mathrm{RH} \otimes_{\mathrm{RH}} \mathrm{RG} \\ & & \Big\downarrow{\scriptstyle \Phi_1} & & \Big\downarrow{\scriptstyle \Phi_2} \\ 0 & \longrightarrow & K & \xrightarrow{\quad\iota\quad} & \mathrm{RG}, \end{array}$$

where $\Phi_2$ is defined accordingly in Lemma 2.6. Since $\Phi_2$ is a right RG-isomorphism by Lemma 2.6, it is clear that $\Phi_1$ is also a right RG-isomorphism. As $\sigma = \pi_H \varphi|_{K'}\colon K' \to \mathrm{RH}$ is a right RH-homomorphism and RH is right P-injective, there exists a right RH-homomorphism $\overline{\sigma}\colon \mathrm{RH} \to \mathrm{RH}$ such that $\sigma = \overline{\sigma}\iota'$. Thus, we have the following diagrams:

$$\begin{array}{ccccc} & & \mathrm{RH} & & \\ & & \Big\uparrow{\scriptstyle \sigma} & \nwarrow{\scriptstyle \overline{\sigma}} & \\ 0 & \longrightarrow & K' & \xhookrightarrow{\ \iota'\ } & \mathrm{RH} \end{array}$$

and

$$\begin{array}{ccc} K & \xrightarrow{\qquad\varphi\qquad} & \mathrm{RG} \\ \Big\downarrow{\scriptstyle \Phi_1^{-1}} & & \Big\uparrow{\scriptstyle \Phi_2} \\ K' \otimes_{\mathrm{RH}} \mathrm{RG} & \xrightarrow[\sigma\otimes 1]{} & \mathrm{RH} \otimes_{\mathrm{RH}} \mathrm{RG}. \end{array}$$

Thus, $\varphi = \Phi_2(\sigma \otimes 1)\Phi_1^{-1} = \Phi_2(\overline{\sigma} \otimes 1)(\iota' \otimes 1)\Phi_1^{-1} = \Phi_2(\overline{\sigma} \otimes 1)\Phi_2^{-1}\iota$. So the right RG-homomorphism $\Phi_2(\overline{\sigma} \otimes 1)\Phi_2^{-1}\colon \mathrm{RG} \to \mathrm{RG}$ extends $\varphi$. □

**Remark 2.8.** By [6], Example 5.70, if $R$ is right P-injective and $G$ is locally finite (even finite), RG need not be right P-injective.

By Corollary 2.4, using discussions similar to those in Theorem 2.7, we have

**Theorem 2.9.** *Let $R$ be a ring, $G$ a group and $n \geqslant 1$ an integer. The following are equivalent:*

(i) RG *is right $n$-injective;*

(ii) (a) *$R$ is right $n$-injective;*

(b) *$G$ is a locally finite group;*

(c) *for each finite subgroup $H$ of $G$ and any $n$-generated right ideal $I$ of* RH, *if $f \in \mathrm{Hom}_R(I_R, R_R)$, then there exists $g \in \mathrm{Hom}_R(\mathrm{RH}_R, R_R)$ such that $g|_I = f$.*

**Corollary 2.10.** *Let $R$ be a ring and $G$ a group. The following are equivalent:*

(i) RG *is right F-injective;*

(ii) (a) *R is right F-injective;*

(b) *G is a locally finite group;*

(c) *for each finite subgroup H of G and any finitely right ideal I of* RH*, if* $f \in \mathrm{Hom}_R(I_R, R_R)$*, then there exists* $g \in \mathrm{Hom}_R(\mathrm{RH}_R, R_R)$ *such that* $g|_I = f$.

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