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A BLOW-UP CRITERION FOR THE STRONG SOLUTIONS TO THE NONHOMOGENEOUS NAVIER-STOKES-KORTEWEG EQUATIONS IN DIMENSION THREE

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Abstract. This paper proves a Serrin's type blow-up criterion for the 3D density-dependent Navier-Stokes-Korteweg equations with vacuum. It is shown that if the density ϱ and velocity field u satisfy $\|\nabla\varrho\|_{L^\infty(0,T;W^{1,q})} + \|u\|_{L^s(0,T;L^r_\omega)} < \infty$ for some $q > 3$ and any (r,s) satisfying $2/s + 3/r \leqslant 1$, $3 < r \leqslant \infty$, then the strong solutions to the density-dependent Navier-Stokes-Korteweg equations can exist globally over $[0,T]$. Here L^r_ω denotes the weak L^r space.

Keywords: Navier-Stokes-Korteweg equations; capillary fluid; blow-up criterion; vacuum; strong solutions

MSC 2020: 35Q35, 76D45, 35D35

1. Introduction and main result

It is well-known that some available mathematical results on the classical incompressible Navier-Stokes equations between dimension three and two are very different. For example, the global well-posedness of the two-dimensional incompressible Navier-Stokes equations with large initial data has been proved long time ago. However, the three-dimensional global well-posedness for large initial data is still a famous open problem in the partial differential equations. And we believe that the similar dimensional differences also appear in the analysis of the nonhomogeneous fluid dynamics. This is a continuation of paper [9], in which the author established a blow-up criterion for the strong solutions to the initial and boundary value problem of the nonhomogeneous incompressible Navier-Stokes-Korteweg equations in dimension two. The purpose of this paper is to establish a blow-up criterion for the strong

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solutions to the initial and boundary value problem of the nonhomogeneous incompressible Navier-Stokes-Korteweg equations in dimension three, which will involve not only the density but also the velocity field. And our result also indicates the famous Serrin's criterion for the classical (homogeneous) incompressible Navier-Stokes equations.

The time evolution of the density $\varrho = \varrho(x,t)$, velocity field $u = (u_1, u_2, u_3)(x,t)$ and pressure $P = P(x,t)$ of a general viscous capillary fluid is governed by the so-called nonhomogeneous incompressible Navier-Stokes-Korteweg equations which are written as

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho u) = 0, \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) - \operatorname{div}(2\mu(\varrho)d(u)) + \nabla P + \operatorname{div}(\kappa(\varrho)\nabla\varrho \otimes \nabla\varrho) = 0, \\ \operatorname{div} u = 0, \end{cases} \tag{1.1}$$

where $x \in \Omega$ is the spatial coordinate, and $t \geqslant 0$ is the time. In this paper, Ω is a bounded domain with smooth boundary in $\mathbb{R}^3$,

$$d(u) = \frac{1}{2}[\nabla u + (\nabla u)^\top],$$

denotes the deformation tensor of the matrix form with the ij component

$$d_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big).$$

Moreover, $\kappa = \kappa(\varrho)$, which is a C^1 nonnegative function of density ϱ, standing for the capillary coefficient. And $\mu = \mu(\varrho)$ is the viscosity coefficient of the fluids, which is assumed to be a function of density ϱ satisfying

$$\mu \in C^1[0,\infty) \quad \text{and} \quad \mu \geqslant \underline{\mu} > 0 \text{ on } [0,\infty) \tag{1.2}$$

for a some positive constant $\underline{\mu}$.

We focus on system (1.1)–(1.2) with the initial and boundary conditions:

$$u = 0 \quad \text{on } \partial\Omega \times [0,T), \tag{1.3}$$

$$(\varrho, u)|_{t=0} = (\varrho_0, u_0) \quad \text{in } \Omega. \tag{1.4}$$

When $\kappa \equiv 0$, that is, the capillary effect is neglected, system (1.1)–(1.4) are the famous nonhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity. For nonhomogeneous incompressible Navier-Stokes equations with initial vacuum, Cho and Kim [2] proved the local existence of unique strong solution for all initial data satisfying a compatibility condition. And later Huang and Wang [6] proved the strong solution exists globally in time when the initial gradient of the velocity is suitably small. For the related progress, see [4]–[6] and the references therein.

Let us come back to the fluids with capillary effect, that is, $\kappa(\varrho)$ depends on the density ϱ. As far as I know, the first local existence of a unique strong solution was obtained by Tan and Wang [11] when the capillary coefficient κ is a nonnegative constant. And very recently, Wang [12] extended their result to the case when $\kappa(\varrho)$ is a C^1 function of the density.

First we give the definition of strong solutions to the initial and boundary problem (1.1)–(1.4) as follows.

Definition 1.1 (Strong solution). A pair of functions ($\varrho \geqslant 0,\ u, P$) is called a strong solution to problem (1.1)–(1.4) in $\Omega \times (0,T)$ if for some $q_0 \in (3,6]$,

$$\begin{aligned}(1.5)\quad &\varrho \in C([0,T];W^{2,q_0}),\quad u \in C([0,T];H_0^1 \cap H^2),\quad \nabla^2 u \in L^2(0,T;L^{q_0}),\\ &\varrho_t \in C([0,T];W^{1,q_0}),\quad \nabla P \in C([0,T];L^2) \cap L^2(0,T;L^{q_0}),\quad u_t \in L^2(0,T;H_0^1),\end{aligned}$$

and (ϱ, u, P) satisfies (1.1) a.e. in $\Omega \times (0,T)$.

In the case when the initial data may vanish in an open subset of Ω, that is, the initial vacuum is allowed, the following local well-posedness of strong solution to (1.1)–(1.4) was obtained by Wang [12].

Theorem 1.2. *Assume that the initial data (ϱ_0, u_0) satisfies the regularity condition*

$$0 \leqslant \varrho_0 \in W^{2,q},\quad 3 < q \leqslant 6,\quad u_0 \in H_{0,\sigma}^1 \cap H^2, \tag{1.6}$$

and the compatibility condition

$$-\mathrm{div}(\mu(\varrho_0)(\nabla u_0 + (\nabla u_0)^\top)) + \nabla P_0 + \mathrm{div}(\kappa(\varrho_0)\nabla\varrho_0 \otimes \nabla\varrho_0) = \varrho_0^{1/2} g, \tag{1.7}$$

for some $(P_0, g) \in H^1 \times L^2$. Then there exist a small time T and a unique strong solution (ϱ, u, P) to the initial boundary value problem (1.1)–(1.4).

Motivated by the work of Kim [7], in which a Serrin's type blow-up criterion for the 3D nonhomogeneous incompressible Navier-Stokes flow was established, we derive a similar blow-up criterion for the nonhomogeneous Navier-Stokes-Korteweg equations with density-dependent viscosity and capillary coefficients in dimension three. More precisely, our main result can be stated as follows.

Theorem 1.3. *Assume that the initial data (ϱ_0, u_0) satisfies the regularity condition* (1.6) *and the compatibility condition* (1.7). *Let (ϱ, u, P) be a strong solution of problem* (1.1)–(1.4) *satisfying* (1.5). *If $0 < T^* < \infty$ is the maximal time of existence, then*

$$\lim_{T\to T^*}(\|\nabla\varrho\|_{L^\infty(0,T;W^{1,q})} + \|u\|_{L^s(0,T;L^r_\omega)}) = \infty \tag{1.8}$$

for any r and s satisfying

$$\frac{2}{s} + \frac{3}{r} \leqslant 1, \quad 3 < r \leqslant \infty, \tag{1.9}$$

where L^r_ω denotes the weak L^r space.

Remark 1.4. Compared to the two-dimensional blow-up criterion established in [9] by the author, the blow-up criterion obtained in this paper involves not only the density but aslo the velocity field, see (1.8). And when $\varrho_0 \equiv 1$, the nonhomogeneous incompressible Navier-Stokes-Korteweg equations (1.1) reduce to the classical incompressible Navier-Stokes equations. Therefore, our blow-up criterion indicates the generalization of Serrin's criterion using weak Lesbegue spaces for incompressible Navier-Stokes equations, see the work of Sohr [10], Bosia et al. [1].

The proof of Theorem 1.3 is based on the contradiction argument. In view of the local existence result, to prove Theorem 1.3, it suffices to verify that (ϱ, u) satisfy (1.6) and (1.7) at the time T^* under the assumption that the left-hand side of (1.8) is finite. Unlike the Navier-Stokes equations treated in Kim [7], the use of weak Lesbegue space makes it more difficult to obtain some estimates because of the apperance of capillary effect. To overcome the difficulty, we make good use of the finiteness of $\|\nabla\varrho\|_{W^{1,q}}$ and other interpolation techniques in Lorentz space.

The remainder of this paper is arranged as follows. In Section 2, we give some auxiliary lemmas which are useful in our later analysis. The proof of Theorem 1.3 will be done by combining the contradiction argument with the estimates derived in Section 3.

2. Preliminaries

2.1. Notations and general inequalities. Assume that Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. For notations simplicity below, we omit the integration domain Ω. And for $1 \leqslant r \leqslant \infty$ and $k \in \mathbb{N}$, the Lesbegue and Sobolev spaces are defined in a standard way,

$$L^r = L^r(\Omega), \quad W^{k,r} = \{f \in L^r \colon \nabla^k f \in L^r\}, \quad H^k = W^{k,2}.$$

The following Gagliardo-Nirenberg inequality will be used frequently in the later analysis.

Lemma 2.1 (Gagliardo-Nirenberg inequality)**.** *Let Ω be a domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. For $p \in [2,6]$, $q \in (1,\infty)$ and $r \in (3,\infty)$, there exist some generic constants $C > 0$ that may depend on q and r such that for $f \in H^1$ satisfying $f|_{\partial\Omega} = 0$ and $g \in L^q \cap D^{1,r}$, we have*

$$\|f\|_{L^p}^p \leqslant C\|f\|_{L^2}^{(6-p)/2}\|\nabla f\|_{L^2}^{(3p-6)/2}, \tag{2.1}$$

$$\|g\|_{L^\infty} \leqslant C\|g\|_{L^q}^{q(r-3)/(3r+q(r-3))}\|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \tag{2.2}$$

See the proof of this lemma in Ladyzhenskaya et al. [8], page 62. Denote the Lorentz space and its norm by $L^{p,q}$ and $\|\cdot\|_{L^{p,q}}$, respectively, where $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. We recall the weak-$L^p$ space L^p_ω which is defined as follows:

$$L^p_\omega := \{f \in L^1_{\text{loc}}\colon\ \|f\|_{L^p_\omega} = \sup_{\lambda>0} \lambda|\{|f(x)| > \lambda\}|^{1/p} < \infty\}.$$

And it should be noted that

$$L^p \subsetneqq L^p_\omega, \quad L^\infty_\omega = L^\infty, \quad L^p_\omega = L^{p,\infty}.$$

For the details of Lorentz space, we refer to the first chapter in Grafakos [3]. The following lemma involving the weak Lesbegue spaces has been proved in Kim [7], Xu and Zhang [13], which will play an important role in the subsequent analysis.

Lemma 2.2. *Assume $g \in H^1$ and $f \in L^r_\omega$ with $r \in (3,\infty]$. Then $f \cdot g \in L^2$. Furthermore, for any $\varepsilon > 0$ we have*

$$\|f \cdot g\|_{L^2}^2 \leqslant \varepsilon\|g\|_{H^1}^2 + C(\varepsilon)(\|f\|_{L^r_\omega}^s + 1)\|g\|_{L^2}^2, \tag{2.3}$$

where C is a positive constant depending only on ε, r and the domain Ω.

2.2. Higher order estimates on u. High-order a priori estimates of velocity field u rely on the following regularity results for density-dependent Stokes equations.

Lemma 2.3. *Assume that $\varrho \in W^{2,q}$, $3 < q < \infty$, and $0 \leqslant \varrho \leqslant \bar{\varrho}$. Let $(u,P) \in H^1_{0,\sigma} \times L^2$ be the unique weak solution to the boundary value problem*

$$-\operatorname{div}(\mu(\varrho)(\nabla u + (\nabla u)^\top) + \nabla P = F, \quad \operatorname{div} u = 0 \quad \textit{in } \Omega \quad \textit{and} \quad \int P\,\mathrm{d}x = 0, \tag{2.4}$$

where

$$\mu \in C^1[0,\infty), \quad \underline{\mu} \leqslant \mu(\varrho) \leqslant \bar{\mu} \quad \text{on } [0,\bar{\varrho}].$$

Then we have the following regularity results:

(1) *If* $F \in L^2$, *then* $(u,P) \in H^2 \times H^1$ *and*

$$\|u\|_{H^2} + \|P\|_{H^1} \leqslant C(1+\|\nabla\varrho\|_{L^\infty})\|F\|_{L^2}. \tag{2.5}$$

(2) *If* $F \in L^r$ *for some* $r \in (2,\infty)$, *then* $(u,P) \in W^{2,r} \times W^{1,r}$ *and*

$$\|u\|_{W^{2,r}} + \|P\|_{W^{1,r}} \leqslant C(1+\|\nabla\varrho\|_{L^\infty})\|F\|_{L^r}. \tag{2.6}$$

The proof of Lemma 2.3 has been given by Wang [12]. Refer also to Lemma 2.1 in his paper.

3. Proof of Theorem 1.3

Let (ϱ,u,P) be a strong solution to the initial and boundary value problem (1.1)–(1.4) as derived in Theorem 1.2. Then it follows from the standard energy estimate that:

Lemma 3.1. *For any* $T>0$ *it holds that for any* $p \in [1,\infty]$,

$$\sup_{0\leqslant t\leqslant T}(\|\varrho\|_{L^p} + \|\sqrt{\varrho}u\|_{L^2}^2 + \|\sqrt{\kappa(\varrho)}\nabla\varrho\|_{L^2}^2) + \int_0^T\int |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant C. \tag{3.1}$$

As is mentioned in Section 1, the main theorem will be proved by using a contradiction argument. Denote $0 < T^* < \infty$ the maximal existence time for the strong solution to the initial and boundary value problem (1.1)–(1.4). Suppose that (1.8) were false, that is

$$M_0 := \lim_{T\to T^*}(\|\nabla\varrho\|_{L^\infty(0,T;W^{1,q})} + \|u\|_{L^s(0,T;L^r_\omega)}) < \infty. \tag{3.2}$$

Under condition (3.2), one will extend the existence time of the strong solutions to (1.1)–(1.4) beyond T^*, which contradicts the definition of maximum of T^*.

The first key step is to derive the L^2-norm of the first order spatial derivatives of u under the assumption of initial data and (3.2). Here we define the material derivative $\dot{u} := u_t + u\cdot\nabla u$.

Lemma 3.2. *Under condition* (3.2), *it holds that for any* $0<T<T^*$,

$$\sup_{0\leqslant t\leqslant T}\|\nabla u\|_{L^2}^2 + \int_0^T \|\sqrt{\varrho}\dot{u}\|_{L^2}^2\,\mathrm{d}t \leqslant C. \tag{3.3}$$

P r o o f. Multiplying the momentum equations $(1.1)_2$ by u_t and integrating the resulting equations over Ω, we have

$$\begin{aligned}
(3.4)\qquad & \int \varrho|\dot{u}|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int \mu(\varrho)|d|^2\,\mathrm{d}x \\
&= \int \varrho\dot{u}\cdot(u\cdot\nabla u)\,\mathrm{d}x - \int \mu'(\varrho)u\cdot\nabla\varrho|d|^2\,\mathrm{d}x + \int \kappa(\varrho)\nabla\varrho\otimes\nabla\varrho : \nabla u_t\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int \kappa(\varrho)\nabla\varrho\otimes\nabla\varrho : \nabla u\,\mathrm{d}x + \int \kappa'(\varrho)(u\cdot\nabla\varrho)\nabla\varrho\otimes\nabla\varrho : \nabla u\,\mathrm{d}x \\
&\quad + \int \kappa(\varrho)\nabla(u\cdot\nabla\varrho)\otimes\nabla\varrho : \nabla u\,\mathrm{d}x + \int \varrho\dot{u}\cdot(u\cdot\nabla u)\,\mathrm{d}x \\
&\quad - \int \mu'(\varrho)u\cdot\nabla\varrho|d|^2\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int \kappa(\varrho)\nabla\varrho\otimes\nabla\varrho : \nabla u\,\mathrm{d}x + \sum_{k=1}^{4} I_k.
\end{aligned}$$

To complete the proof, we should bound the terms I_1 to I_4. First, for I_1, we use assumption (3.2) and apply Hölder's inequality:

$$\begin{aligned}
(3.5)\qquad I_1 &= \int \kappa'(\varrho)(u\cdot\nabla\varrho)\nabla\varrho\otimes\nabla\varrho : \nabla u\,\mathrm{d}x \\
&\leqslant \|\kappa'(\varrho)\|_{L^\infty}\|\nabla\varrho\|_{L^6}^3\|u\cdot\nabla u\|_{L^2} \leqslant \|u\cdot\nabla u\|_{L^2}^2 + C.
\end{aligned}$$

For I_2 we devide it into two parts and simply use Hölder's inequality to get

$$\begin{aligned}
(3.6)\qquad I_2 &= \int \kappa(\varrho)\nabla(u\cdot\nabla\varrho)\otimes\nabla\varrho : \nabla u\,\mathrm{d}x \\
&\leqslant \|\kappa(\varrho)\|_{L^\infty}\|\nabla\varrho\|_{L^\infty}\|\nabla^2\varrho\|_{L^2}\|u\cdot\nabla u\|_{L^2} + \|\kappa(\varrho)\|_{L^\infty}\|\nabla\varrho\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 \\
&\leqslant C\|u\cdot\nabla u\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2).
\end{aligned}$$

For I_3, using Cauchy-Schwarz inequality with ε to get

$$(3.7)\qquad I_3 = \int \varrho\dot{u}\cdot(u\cdot\nabla u)\,\mathrm{d}x \leqslant \varepsilon\|\sqrt{\varrho}\dot{u}\|_{L^2}^2 + C(\varepsilon)\|u\cdot\nabla u\|_{L^2}^2,$$

and finally remark that $d = \frac{1}{2}(\nabla u + (\nabla u)^\top)$, one has

$$\begin{aligned}
(3.8)\qquad I_4 = \int \mu'(\varrho)u\cdot\nabla\varrho|d|^2\,\mathrm{d}x &\leqslant \|\mu'(\varrho)\|_{L^\infty}\|\nabla\varrho\|_{L^\infty}\|\nabla u\|_{L^2}\|u\cdot\nabla u\|_{L^2} \\
&\leqslant C\|\nabla u\|_{L^2}^2 + C\|u\cdot\nabla u\|_{L^2}^2.
\end{aligned}$$

To obtain the estimates of second order spatial derivatives of the velocity u, we make good use of the Stokes type estimates on the momentum equations $(1.1)_2$ by simply putting $F = -\varrho\dot{u} - \operatorname{div}(\kappa(\varrho)\nabla\varrho\otimes\nabla\varrho)$ in Lemma 2.3. Then (2.5) indicates that

$$
\begin{aligned}
\|\nabla u\|_{H^1} + \|P\|_{H^1} &\leqslant C(1+\|\nabla\varrho\|_{L^\infty})\|F\|_{L^2} \\
&\leqslant C(1+\|\nabla\varrho\|_{L^\infty})\|\varrho\dot{u} + \operatorname{div}(\kappa(\varrho)\nabla\varrho\otimes\nabla\varrho)\|_{L^2} \\
&\leqslant C_*\|\sqrt{\varrho}\dot{u}\|_{L^2} + C\|\nabla\varrho\|_{L^6}^3 + C\|\nabla\varrho\|_{L^\infty}\|\nabla^2\varrho\|_{L^2} \\
&\leqslant C_*\|\sqrt{\varrho}\dot{u}\|_{L^2} + C,
\end{aligned} \tag{3.9}
$$

where C_* is a positive number.

Now we substitute (3.5)–(3.8) into (3.4), deducing that

$$
\begin{aligned}
&\int \varrho|\dot{u}|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int \mu(\varrho)|d|^2\,\mathrm{d}x \\
&\leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int \kappa(\varrho)\nabla\varrho\otimes\nabla\varrho : \nabla u\,\mathrm{d}x + \varepsilon\|\sqrt{\varrho}\dot{u}\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2) \\
&\quad + C(\varepsilon)\|u\cdot\nabla u\|_{L^2}^2 \\
&\leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int \kappa(\varrho)\nabla\varrho\otimes\nabla\varrho : \nabla u\,\mathrm{d}x + \varepsilon\|\sqrt{\varrho}\dot{u}\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2) \\
&\quad + \delta\|\nabla u\|_{H^1}^2 + C(\varepsilon,\delta)(\|u\|_{L^r_\omega}^s + 1)\|\nabla u\|_{L^2}^2 \\
&\leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int \kappa(\varrho)\nabla\varrho\otimes\nabla\varrho : \nabla u\,\mathrm{d}x + \varepsilon\|\sqrt{\varrho}\dot{u}\|_{L^2}^2 + C(1+\|\nabla u\|_{L^2}^2) \\
&\quad + C_*\delta\|\sqrt{\varrho}\dot{u}\|_{L^2}^2 + C(\varepsilon,\delta)(\|u\|_{L^r_\omega}^s + 1)\|\nabla u\|_{L^2}^2,
\end{aligned} \tag{3.10}
$$

where we use Lemma 2.2 in the second inequality, and (3.9) is used to get the third one. Then choosing ε, δ small enough, we get

$$
\begin{aligned}
&\int \varrho|\dot{u}|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int \mu(\varrho)|d|^2\,\mathrm{d}x \\
&\qquad\leqslant \frac{\mathrm{d}}{\mathrm{d}t}\int \kappa(\varrho)\nabla\varrho\otimes\nabla\varrho : \nabla u\,\mathrm{d}x + C(1+\|\nabla u\|_{L^2}^2)(\|u\|_{L^r_\omega}^s + 1).
\end{aligned} \tag{3.11}
$$

By assumption (3.2) and the Cauchy-Schwarz inequality, it is easily seen that

$$
C\int |\kappa(\varrho)||\nabla\varrho\otimes\nabla\varrho : \nabla u|\,\mathrm{d}x \leqslant \frac{\mu}{4}\|\nabla u\|_{L^2}^2 + C. \tag{3.12}
$$

Taking this into account, we can conclude that (3.3) holds for all $0 \leqslant T < T^*$ from (3.11) and the Gronwall inequality. Therefore, we complete the proof of Lemma 3.2. □

To continue our proof, we will derive the estimate of $\sqrt{\varrho}u_t$ by using the compatibility condition (1.7) on the initial data. More precisely, we have the following lemma.

Lemma 3.3. *Under condition* (3.2), *it holds that for any* $0 < T < T^*$,

$$\sup_{0\leqslant t\leqslant T} \|\sqrt{\varrho}u_t\|_{L^2}^2 + \int_0^T \|\nabla u_t\|_{L^2}^2 \,\mathrm{d}t \leqslant C. \tag{3.13}$$

P r o o f. Differentiating the momentum equations $(1.1)_2$ with respect to t, along with the continuity equation $(1.1)_1$, we get

$$\begin{aligned} &\varrho u_{tt} + \varrho u\cdot\nabla u_t - \operatorname{div}(2\mu(\varrho)d_t) + \nabla P_t \\ &\quad = (u\cdot\nabla\varrho)(u_t + u\cdot\nabla u) - \varrho u_t\cdot\nabla u - \operatorname{div}(2\mu'(\varrho)(u\cdot\nabla\varrho)d) \\ &\qquad + \operatorname{div}(\kappa'(\varrho)(u\cdot\nabla\varrho)\nabla\varrho\otimes\nabla\varrho) + 2\operatorname{div}(\kappa(\varrho)\nabla(u\cdot\nabla\varrho)\otimes\nabla\varrho). \end{aligned} \tag{3.14}$$

Multiplying (3.14) by u_t and integrating over Ω, we get after integartion by parts that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho|u_t|^2\,\mathrm{d}x + 2\int \mu(\varrho)|d_t|^2\,\mathrm{d}x = \int -2\varrho u\cdot\nabla u_t\cdot u_t\,\mathrm{d}x \\ &\quad + \int (u\cdot\nabla\varrho)(u\cdot\nabla u)\cdot u_t\,\mathrm{d}x - \int \varrho u_t\cdot\nabla u\cdot u_t\,\mathrm{d}x \\ &\quad + \int 2\mu'(\varrho)(u\cdot\nabla\varrho)d : \nabla u_t\,\mathrm{d}x - \int \kappa'(\varrho)(u\cdot\nabla\varrho)\nabla\varrho\otimes\nabla\varrho : \nabla u_t\,\mathrm{d}x \\ &\quad - \int 2\kappa(\varrho)\nabla(u\cdot\nabla\varrho)\otimes\nabla\varrho : \nabla u_t\,\mathrm{d}x =: \sum_{k=1}^{6} J_k. \end{aligned} \tag{3.15}$$

To proceed, we estimate the terms from J_1 to J_6. First

$$\begin{aligned} J_1 &= \int -2\varrho u\cdot\nabla u_t\cdot u_t\,\mathrm{d}x \leqslant C\|\varrho\|_{L^\infty}^{1/2}\|\sqrt{\varrho}u_t\|_{L^3}\|u\|_{L^6}\|\nabla u_t\|_{L^2} \\ &\leqslant C\|\sqrt{\varrho}u_t\|_{L^2}^{1/2}\|\sqrt{\varrho}u_t\|_{L^6}^{1/2}\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2} \\ &\leqslant C\|\sqrt{\varrho}u_t\|_{L^2}^{1/2}\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}^{3/2} \\ &\leqslant \frac{1}{12}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\varrho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4 \\ &\leqslant \frac{1}{12}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C\|\sqrt{\varrho}u_t\|_{L^2}^2. \end{aligned} \tag{3.16}$$

Similarly,

$$\begin{aligned} J_2 &= \int (u\cdot\nabla\varrho)(u\cdot\nabla u)\cdot u_t\,\mathrm{d}x \leqslant C\|\nabla\varrho\|_{L^\infty}\|\nabla u\|_{L^2}\|u\|_{L^6}^2\|u_t\|_{L^6} \\ &\leqslant C\|\nabla\varrho\|_{L^\infty}\|\nabla u\|_{L^2}^3\|\nabla u_t\|_{L^2} \leqslant \frac{1}{12}\underline{\mu}\|\nabla u_t\|_{L^2}^2 + C, \end{aligned} \tag{3.17}$$

$$
\begin{aligned}
(3.18) \qquad J_3 = & -\int \varrho u_t\cdot\nabla u\cdot u_t\,\mathrm{d}x \leqslant C\|\varrho\|_{L^\infty}^{1/2}\|u_t\|_{L^6}\|\sqrt{\varrho}u_t\|_{L^3}\|\nabla u\|_{L^2}\\
&\leqslant C\|\nabla u_t\|_{L^2}\|\sqrt{\varrho}u_t\|_{L^2}^{1/2}\|\sqrt{\varrho}u_t\|_{L^6}^{1/2}\leqslant C\|\sqrt{\varrho}u_t\|_{L^2}^{1/2}\|\nabla u_t\|_{L^2}^{3/2}\\
&\leqslant \frac{1}{12}\underline{\mu}\|\nabla u_t\|_{L^2}^2+C\|\sqrt{\varrho}u_t\|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
(3.19) \qquad J_4 = & \int 2\mu'(\varrho)(u\cdot\nabla\varrho)d:\nabla u_t\,\mathrm{d}x\\
&\leqslant C\|\mu'(\varrho)\|_{L^\infty}\|\nabla\varrho\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^3}\|\nabla u_t\|_{L^2}\\
&\leqslant C\|\nabla u\|_{L^2}^{3/2}\|\nabla u\|_{H^1}^{1/2}\|\nabla u_t\|_{L^2}\leqslant \frac{1}{12}\underline{\mu}\|\nabla u_t\|_{L^2}^2+C\|\nabla u\|_{H^1}^2,
\end{aligned}
$$

$$
\begin{aligned}
(3.20) \qquad J_5 = & \int \kappa'(\varrho)(u\cdot\nabla\varrho)\nabla\varrho\otimes\nabla\varrho:\nabla u_t\,\mathrm{d}x\\
&\leqslant C\|\kappa'(\varrho)\|_{L^\infty}\|\nabla\varrho\|_{L^\infty}^3\|u\|_{L^2}\|\nabla u_t\|_{L^2}\leqslant \frac{1}{12}\underline{\mu}\|\nabla u_t\|_{L^2}^2+C.
\end{aligned}
$$

Finally remarking that $3<q\leqslant 6$, by assumption (3.2), one has

$$
\begin{aligned}
(3.21) \qquad J_6 = & \int 2\kappa(\varrho)\nabla(u\cdot\nabla\varrho)\otimes\nabla\varrho:\nabla u_t\,\mathrm{d}x\\
&\leqslant C\|\kappa(\varrho)\|_{L^\infty}\|\nabla\varrho\|_{L^\infty}^2\|\nabla u\|_{L^2}\|\nabla u_t\|_{L^2}\\
&\quad + C\|\kappa(\varrho)\|_{L^\infty}\|\nabla\varrho\|_{L^\infty}\|\nabla^2\varrho\|_{L^3}\|u\|_{L^6}\|\nabla u_t\|_{L^2}\\
&\leqslant \frac{1}{12}\underline{\mu}\|\nabla u_t\|_{L^2}^2+C.
\end{aligned}
$$

It remains to estimate $\|\nabla u\|_{H^1}$, since it appears in the estimate of term J_4, see (3.19). Indeed, we can deduce from Lemma 2.3 that

$$
\begin{aligned}
(3.22) \qquad \|\nabla u\|_{H^1}+\|P\|_{H^1} &\leqslant C(1+\|\nabla\varrho\|_{L^\infty})\|F\|_{L^2}\\
&\leqslant C(1+\|\nabla\varrho\|_{L^\infty})\|\varrho u_t+\varrho u\cdot\nabla u+\operatorname{div}(\kappa(\varrho)\nabla\varrho\otimes\nabla\varrho)\|_{L^2}\\
&\leqslant C(\|\sqrt{\varrho}u_t\|_{L^2}+\|u\|_{L^6}\|\nabla u\|_{L^3}+\|\nabla\varrho\|_{L^6}^3+\|\nabla\varrho\|_{L^\infty}\|\nabla^2\varrho\|_{L^2})\\
&\leqslant C\|\sqrt{\varrho}u_t\|_{L^2}+\frac{1}{2}\|\nabla u\|_{H^1}+C,
\end{aligned}
$$

which implies

$$
(3.23) \qquad \|\nabla u\|_{H^1}\leqslant C\|\sqrt{\varrho}u_t\|_{L^2}+C.
$$

Combining all the estimates (3.16)–(3.21) and (3.23), we deduce that

$$
(3.24) \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\varrho|u_t|^2\,\mathrm{d}x+2\int\mu(\varrho)|d_t|^2\,\mathrm{d}x\leqslant\frac{1}{2}\underline{\mu}\|\nabla u_t\|_{L^2}^2+C(1+\|\sqrt{\varrho}u_t\|_{L^2}^2).
$$

Together with the fact that

$$2\int |d_t|^2\,\mathrm{d}x = \int |\nabla u_t|^2\,\mathrm{d}x,$$

we obtain (3.13) by applying the Gronwall inequality. Therefore, the proof of Lemma 3.3 is completed. $\square$

Lemma 3.4. *Under condition* (3.2), *it holds that for any* $0 < T < T^*$,

$$\sup_{0\leqslant t\leqslant T}(\|\varrho_t\|_{W^{1,q}} + \|u\|_{H^2} + \|P\|_{H^1}) + \int_0^T (\|u\|_{W^{2,q}}^2 + \|P\|_{W^{1,q}}^2)\,\mathrm{d}t \leqslant C. \tag{3.25}$$

P r o o f. As a direct consequence of Lemma 3.3 and (3.23), we can easily conclude that

$$\sup_{0\leqslant t\leqslant T}(\|u\|_{H^2} + \|P\|_{H^1}) \leqslant C. \tag{3.26}$$

And, by use of the continuity equation $(1.1)_1$, one deduces that

$$\begin{aligned}\|\varrho_t\|_{W^{1,q}} &\leqslant C(\|\varrho_t\|_{L^q} + \|\nabla\varrho_t\|_{L^q})\\ &\leqslant C(\|u\cdot\nabla\varrho\|_{L^q} + \|\nabla(u\cdot\nabla\varrho)\|_{L^q})\\ &\leqslant C(\|u\|_{L^\infty}\|\nabla\varrho\|_{L^q} + \|u\|_{L^\infty}\|\nabla^2\varrho\|_{L^q} + \|\nabla u\|_{L^6}\|\nabla\varrho\|_{L^{6q/(6-q)}})\\ &\leqslant C\|u\|_{H^2}\|\nabla\varrho\|_{W^{1,q}}.\end{aligned} \tag{3.27}$$

By assumption (3.2) and (3.26), the boundedness of $\|\varrho_t\|_{W^{1,q}}$ is verified. Finally, apply (2.6) in Lemma 2.3 with $F = -\varrho u_t - \varrho u\cdot\nabla u - \operatorname{div}(\kappa(\varrho)\nabla\varrho\otimes\nabla\varrho)$ to get

$$\begin{aligned}\|\nabla u\|_{W^{1,q}} + \|P\|_{W^{1,q}} &\leqslant C(1+\|\nabla\varrho\|_{L^\infty})(\|\varrho u_t\|_{L^q} + \|\varrho u\cdot\nabla u\|_{L^q}\\ &\quad + \|\kappa(\varrho)|\nabla^2\varrho||\nabla\varrho|\|_{L^q} + \|\kappa'(\varrho)|\nabla\varrho|^3\|_{L^q})\\ &\leqslant C(\|\varrho u_t\|_{L^q} + \|\varrho u\cdot\nabla u\|_{L^q} + 1)\\ &\leqslant C(\|\sqrt{\varrho}u_t\|_{L^2}^{(6-q)/(2q)}\|\sqrt{\varrho}u_t\|_{L^6}^{(3q-6)/(2q)}\\ &\quad + \|\nabla u\|_{L^2}^{6(q-1)/(5q-6)}\|\nabla u\|_{W^{1,q}}^{(4q-6)/(5q-6)} + 1).\end{aligned} \tag{3.28}$$

By Young's inequality and the Sobolev embedding inequality, it can be easily seen that

$$\begin{aligned}\|\nabla u\|_{W^{1,q}}^2 + \|P\|_{W^{1,q}}^2 &\leqslant C\|\sqrt{\varrho}u_t\|_{L^2}^{(6-q)/q}\|\nabla u_t\|_{L^2}^{3(q-2)/q}\\ &\quad + C\|\nabla u\|_{L^2}^{12(q-1)/q} + C \leqslant C\|\sqrt{\varrho}u_t\|_{L^2}^{(6-q)/q}\|\nabla u_t\|_{L^2}^{3(q-2)/q} + C.\end{aligned} \tag{3.29}$$

Hence

$$\begin{aligned}(3.30)\qquad \int_0^T (\|\nabla u\|^2_{W^{1,q}} + \|P\|^2_{W^{1,q}})\,\mathrm{d}t &\leqslant C\int_0^T \|\sqrt{\varrho}u_t\|_{L^2}^{(6-q)/q}\|\nabla u_t\|_{L^2}^{3(q-2)/q}\,\mathrm{d}t + C\\ &\leqslant C\Big(\sup_{0\leqslant t\leqslant T}\|\sqrt{\varrho}u_t\|^2_{L^2}\Big)^{(6-q)/(2q)}\int_0^T\|\nabla u_t\|^2_{L^2}\,\mathrm{d}t + C \leqslant C,\end{aligned}$$

here the second inequality holds since $q \leqslant 6$. Therefore, we complete the proof of Lemma 3.4. $\square$

Proof of Theorem 1.3. In fact, in view of (3.3) and (3.25), it is easy to see that the functions $(\varrho, u)(x, t = T^*) = \lim_{t\to T^*}(\varrho, u)$ have the same regularities imposed on the initial data (1.6) at the time $t = T^*$. Furthermore,

$$\begin{aligned}-\operatorname{div}(2\mu(\varrho)d) + \nabla P + \operatorname{div}(\kappa(\varrho)\nabla\varrho\otimes\nabla\varrho)|_{t=T^*}\\ = \lim_{t\to T^*}\varrho^{1/2}(\varrho^{1/2}u_t + \varrho^{1/2}u\cdot\nabla u) := \varrho^{1/2}g|_{t=T^*}\end{aligned}$$

with $g = (\varrho^{1/2}u_t + \varrho^{1/2}u\cdot\nabla u)|_{t=T^*} \in L^2$ due to (3.13). Thus, the functions $(\varrho, u)|_{t=T^*}$ satisfy the compatibility condition (1.7) at time T^*. Therefore, we can take $(\varrho, u)|_{t=T^*}$ as the initial data and apply the local existence theorem (Theorem 1.2) to extend the local strong solution beyond T^*. This contradicts the definition of maximal existence time T^*, and thus, the proof of Theorem 1.3 is completed. $\square$

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