Hongli Wang; Jianwei Yang

Incompressible limit of a fluid-particle interaction model

Applications of Mathematics, Vol. 66 (2021), No. 1, 69–86

Persistent URL: http://dml.cz/dmlcz/148511

66 (2021) APPLICATIONS OF MATHEMATICS No. 1, 69–86

INCOMPRESSIBLE LIMIT OF A FLUID-PARTICLE INTERACTION MODEL

Hongli Wang, Jianwei Yang, Zhengzhou

Received September 30, 2019. Published online November 27, 2020.

Abstract. The incompressible limit of the weak solutions to a fluid-particle interaction model is studied in this paper. By using the relative entropy method and refined energy analysis, we show that, for well-prepared initial data, the weak solutions of the compressible fluid-particle interaction model converge to the strong solution of the incompressible Navier-Stokes equations as long as the Mach number goes to zero. Furthermore, the desired convergence rates are also obtained.

Keywords: incompressible limit; relative entropy method; fluid-particle interaction model; incompressible Navier-Stokes equation

MSC 2020: 35B25, 35G25, 35Q35

1. Introduction

In this paper, we consider a fluid-particle interaction model called the Navier-Stokes-Smoluchowski equations, which was first derived formally by Carrillo and Goudon (without considering the dynamic viscosity term) [6], describing the evolution of particles dispersed in a viscous compressible fluid. This kind of coupled system plays an important role in sedimentation analysis of disperse suspensions of particles in fluids, which can be found in many practical applications, such as diesel engines, rocket propulsors, biotechnology, medicine, chemical engineering, and mineral processes [27], [28], [5], [3], [26]. Let $\varrho = \varrho(x,t)$, $u = u(x,t)$ and $\eta = \eta(x,t)$ denote the total mass density, the velocity field and the density of particles in the mixture, respectively. They are functions of a three-dimensional position vector $x \in \mathbb{T}^3$ and of the time $t > 0$, where $\mathbb{T}^3 \subset \mathbb{R}^3$ is a torus. These variables satisfy

This work was partially supported by National Natural Science Foundation of China (No. 12061040).

DOI: 10.21136/AM.2020.0253-19

the primitive conservation equations governing fluid-particle flows in the bubbling regime. These equations express the conservation of mass, the balance of momentum, and the balance of particle densities often referred as the Smoluchowski equation:

$$
\begin{cases}
\partial_t \varrho + \operatorname{div}(\varrho u) = 0, \\
\partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) + \nabla P(\varrho, \eta) = \mu \Delta u + (\mu + \nu)\nabla \operatorname{div} u - (\eta + \varrho)\nabla \Phi, \\
\partial_t \eta + \operatorname{div}(\eta(u - \nabla \Phi)) = \Delta \eta,
\end{cases} \tag{1.1}
$$

where the parameters μ and ν are the viscosity coefficients satisfying $\mu > 0$, $2\mu + 3\nu \geqslant 0$ and $\Phi = \Phi(x)$ is the external potential, which describes the effects of gravity and buoyancy. In this paper, the pressure $P(\varrho, \eta)$ that we consider is given by

$$
P(\varrho, \eta) = a\varrho^\gamma + \eta, \tag{1.2}
$$

for constants $a > 0$ and $\gamma > 1$.

In [7], Carrillo-Karper-Trivisa showed that solutions exist globally in time and proved the large-time stabilization of the system (1.1) with some physical constraints towards a unique stationary state. Using the relative entropy method, Ballew-Trivisa [1] presented the global-in-time existence of suitable weak solutions for a fluid-particle interaction model. Furthermore, Ballew-Trivisa [2] established the existence of weakly dissipative solutions of model (1.1) and established a weak-strong uniqueness result via the relative entropy method, yielding that a weakly dissipative solution agrees with a classical solution with the same initial data when such a classical solution exists. Constantin-Masmoudi in [11] proved the existence of global weak solutions for a nonlinear Smoluchowski equation coupled with Navier-Stokes equations in 2D by using a deteriorating regularity estimate in the spirit of [8]. Chen-Ding-Wang [9] established the existence theory of global solutions in H^3 to the stationary profile and obtained the optimal convergence rates. For the related two-phase flow model, we also refer to [14], [15], [18] and their references.

The main purpose of this paper is to rigorously prove an incompressible limit in the framework of the global weak solutions to system (1.1) based on the following scaling:

$$
t \mapsto \varepsilon t, \quad u \mapsto \varepsilon u^\varepsilon, \quad \Phi \mapsto \varepsilon^2 \Phi^\varepsilon, \quad \mu \mapsto \varepsilon \mu, \quad \nu \mapsto \varepsilon \nu. \tag{1.3}
$$

Then the system (1.1) can be rewritten as

$$
\begin{cases}
\partial_t \varrho^\varepsilon + \operatorname{div}(\varrho^\varepsilon u^\varepsilon) = 0, \\
\partial_t(\varrho^\varepsilon u^\varepsilon) + \operatorname{div}(\varrho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \dfrac{1}{\varepsilon^2}\nabla(a(\varrho^\varepsilon)^\gamma + \eta) \\
\qquad\qquad = \mu \Delta u^\varepsilon + (\mu + \nu)\nabla \operatorname{div} u^\varepsilon - (\varrho^\varepsilon + \eta^\varepsilon)\nabla \Phi^\varepsilon, \\
\partial_t \eta^\varepsilon + \operatorname{div}(\eta^\varepsilon u^\varepsilon) - \varepsilon \operatorname{div}(\eta^\varepsilon \nabla \Phi^\varepsilon) = \dfrac{1}{\varepsilon}\Delta \eta^\varepsilon,
\end{cases} \tag{1.4}
$$

with the initial data

$$\varrho^\varepsilon(x,0)=\varrho_0^\varepsilon(x),\ \eta^\varepsilon(x,0)=\eta_0^\varepsilon(x),\ (\varrho^\varepsilon u^\varepsilon)(x,0)=M_0^\varepsilon(x)\quad \text{for } x\in\mathbb{T}^3, \tag{1.5}$$

where ε is the low Mach number. Here we use the superscript to emphasize the dependence of ε for each variable in (1.4). Formally, letting $\varepsilon\to 0$, we obtain from the momentum equation $(1.4)_2$ that n^ε and ϱ^ε converge to some functions $\varrho(t)$ and $\eta(t)$ depending only on t, respectively. Here, we expect them to be the positive constants ϱ^* and η^*. Thus, the equations of $(1.4)_1$ and $(1.4)_3$ yield to the limit $\operatorname{div} u^0=0$, which is the incompressible condition of a fluid. Hence, we formally obtain the following incompressible Navier-Stokes equations

$$\partial_t u^0+(u^0\cdot\nabla)u^0+\nabla\pi=\frac{\mu}{\varrho^*}\Delta u^0,\quad \operatorname{div} u^0=0, \tag{1.6}$$

where $\nabla\pi$ in (1.6) is the “limit” of

$$\frac{1}{\varepsilon^2}\nabla(a(\varrho^\varepsilon)^\gamma+\eta^\varepsilon)+(\varrho^\varepsilon+\eta^\varepsilon)\nabla\Phi^\varepsilon.$$

In the present paper, we shall apply the method of relative entropy (or the modulated energy) and refined energy analysis to study the incompressible limit ($\varepsilon\to 0$) of weak solutions to the system (1.4). Therefore, we shall prove the limit on any time interval on which the incompressible Navier-Stokes equations possess a regular solution. Furthermore, we will obtain the desired convergence rates.

Recently, in [7], Carrillo et al. obtain the global existence of weak solutions of the mode (1.4) under reasonable physical assumptions on the initial data, the physical domain, and the external potential. It is a straightforward modification of their arguments to obtain the global smooth solution to the problem (1.4) in $\mathbb{T}^3$. We omit the details here.

Proposition 1.1 ([7]). *Let us assume that the external potential Φ^ε and the initial data $(\varrho_0^\varepsilon,\eta_0^\varepsilon,M_0^\varepsilon(x))$ satisfy:*

$$\Phi^\varepsilon(x)\in W^{1,d}(\mathbb{T}^3),\quad \varrho_0^\varepsilon\in L^\gamma(\mathbb{T}^3),\quad \eta_0^\varepsilon\in L^2(\mathbb{T}^3),\quad M_0^\varepsilon\in L^{6/5}(\mathbb{T}^3), \tag{1.7}$$

where, $d=\max\{3,\gamma/(\gamma-1)\}$. Then, for any $T>0$, the system (1.4) admits a weak solution $(\varrho^\varepsilon,\eta^\varepsilon,u^\varepsilon)\colon \mathbb{T}^3\times(0,\infty)\to\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^3$ such that the following hold:

(1) $\varrho^\varepsilon\in L^\infty([0,T];L^\gamma(\mathbb{T}^3))$, $\sqrt{\varrho^\varepsilon}u^\varepsilon\in L^\infty([0,T];L^2(\mathbb{T}^3))$, $u^\varepsilon\in L^2([0,T];H^1(\mathbb{T}^3))$, $\eta^\varepsilon\in L^2([0,T];L^3(\mathbb{T}^3))\cap L^1([0,T];W^{1,3/2}(\mathbb{T}^3))$.

(2) $\varrho^\varepsilon \geqslant 0$ *represents a renormalized solution of the equation* $(1.4)_1$ *on* $\mathbb{T}^3 \times (0,\infty)$, *namely for any test function* $\xi \in \mathcal{D}(\mathbb{T}^3 \times [0,T))$, $T > 0$, *and any* b, B *such that*

$$b \in L^\infty \cap C[0,\infty), \quad B(\varrho^\varepsilon) := B(1) + \int_1^{\varrho^\varepsilon} \frac{b(z)}{z^2}\,\mathrm{d}z, \tag{1.8}$$

the renormalized continuity equation

$$\int_0^\infty \int_{\mathbb{T}^3} [B(\varrho^\varepsilon)\partial_t\xi + B(\varrho^\varepsilon)u^\varepsilon \cdot \nabla\xi - b(\varrho^\varepsilon)\xi \operatorname{div} u^\varepsilon]\,\mathrm{d}x\,\mathrm{d}t = -\int_{\mathbb{T}^3} B(\varrho_0^\varepsilon)\xi(\cdot,0)\,\mathrm{d}x \tag{1.9}$$

holds.

(3) *The balance of momentum* $(1.4)_2$ *holds in the sense of distributions, i.e., for any* $v \in \mathcal{D}(\mathbb{T}^3 \times [0,T))$,

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{T}^3} \Big[\varrho^\varepsilon u^\varepsilon \cdot \partial_t v + \varrho^\varepsilon u^\varepsilon \otimes u^\varepsilon : \nabla v + \frac{1}{\varepsilon^2}(p(\varrho^\varepsilon) + \eta^\varepsilon) \operatorname{div} v\Big]\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_0^\infty \int_{\mathbb{T}^3} [\mu \nabla u^\varepsilon \nabla v + \nu \operatorname{div} u^\varepsilon \operatorname{div} v - (\varrho^\varepsilon + \eta^\varepsilon)\nabla\Phi^\varepsilon \cdot v]\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad - \int_\Omega M_0^\varepsilon \cdot v(\cdot,0)\,\mathrm{d}x. \end{aligned} \tag{1.10}$$

(4) $\eta^\varepsilon \geqslant 0$ *is a weak solution of* $(1.4)_3$, *i.e., for any* $\varphi \in \mathcal{D}(\mathbb{T}^3 \times [0,T))$,

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{T}^3} \Big(\eta^\varepsilon \partial_t\varphi + \eta^\varepsilon u^\varepsilon \cdot \nabla\varphi - \varepsilon\eta^\varepsilon \nabla\Phi^\varepsilon \cdot \nabla\varphi - \frac{1}{\varepsilon}\nabla\eta^\varepsilon \nabla\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad = -\int_{\mathbb{T}^3} \eta_0^\varepsilon \varphi(\cdot,0)\,\mathrm{d}x. \end{aligned} \tag{1.11}$$

(5) *The energy inequality*

$$\begin{aligned} E^\varepsilon(t) + \int_0^t \int_{\mathbb{T}^3} \Big(&\mu|\nabla u^\varepsilon|^2 + (\mu+\nu)|\operatorname{div} u^\varepsilon|^2 \\ &+ \frac{1}{\varepsilon^3}|2\nabla\sqrt{\eta^\varepsilon} + \varepsilon^2\sqrt{\eta^\varepsilon}\nabla\Phi^\varepsilon|^2\Big)\,\mathrm{d}x\,\mathrm{d}s \leqslant E^\varepsilon(0) \end{aligned} \tag{1.12}$$

holds with

$$E^\varepsilon(t) = \int \Big[\frac{1}{2}\varrho^\varepsilon|u^\varepsilon|^2 + \frac{a}{\varepsilon^2(\gamma-1)}(\varrho^\varepsilon)^\gamma + \frac{1}{\varepsilon^2}\eta^\varepsilon \ln\eta^\varepsilon + (\varrho^\varepsilon + \eta^\varepsilon)\Phi^\varepsilon\Big]\,\mathrm{d}x \tag{1.13}$$

and

$$E^\varepsilon(0) = \int \Big[\frac{1}{2}\varrho_0^\varepsilon|u_0^\varepsilon|^2 + \frac{a}{\varepsilon^2(\gamma-1)}(\varrho_0^\varepsilon)^\gamma + \frac{1}{\varepsilon^2}\eta_0^\varepsilon \ln\eta_0^\varepsilon + (\varrho_0^\varepsilon + \eta_0^\varepsilon)\Phi^\varepsilon\Big]\,\mathrm{d}x \tag{1.14}$$

for a.e. $t > 0$.

Without the adjunction of particle and external potential, system (1.4) reduces to the Navier-Stokes equations, whose incompressible limit or low Mach number limit was studied by many mathematicians, such as Klainerman and Majda [19], Beirão da Veiga [4], Lin [20], Lions and Masmoudi [22], [23], Hsiao-Ju-Li [16], Ou [24], Donatelli-Feireisl-Novotný [13], Vauchelet-Zatorska [25], Danchin-Mucha [12], Chen-Zhai [10], etc. Since the existence of solutions to the scaled fluid-particle interaction model (1.1) or (1.4) follows from [2] and [7] for any choices of the dimensionless parameters, various singular limits or various scaling limits can be explored. Ballew-Trivisa [1] rigorously established the low Mach number and low stratification limits of the system (1.1) for both bounded and unbounded domains in a weak sense. More recently, Huang-Huang-Wen [17] established the uniform stability of the local solution family, which yields a lifespan of the Navier-Stokes-Smoluchowski system and proves the local existence of strong solutions for the incompressible system with small initial data. Furthermore, they obtained the convergence rates in the case without external force.

The present paper considers the low Mach number limit of system (1.1) in the scaling form (1.4), which is different from that of [2] and [17]. Under some reasonable assumptions, we prove rigorously that the weak solutions of the fluid-particle interaction model (1.4) correspond to the strong solution of the incompressible Navier-Stokes equations in the time interval (provided the latter exists) by using the refined relative entropy method.

In this present paper, we denote by χ the characteristic function and C the generic positive constants independent of ε. To simplify our notation from now on, we denote by

$$\sigma_{\varrho^\varepsilon}(t) = \int_{\mathbb{T}^3} (|\varrho^\varepsilon - \varrho^*|^2 \chi_{(|\varrho^\varepsilon - \varrho^*| \leqslant \delta)} + |\varrho^\varepsilon - \varrho^*|^\gamma \chi_{(|\varrho^\varepsilon - \varrho^*| > \delta)}) \,\mathrm{d}x,$$
$$\sigma_{\eta^\varepsilon}(t) = \int_{\mathbb{T}^3} [\eta^\varepsilon \ln \eta^\varepsilon - \eta^* \ln \eta^* - (\ln \eta^* + 1)(\eta^\varepsilon - \eta^*)] \,\mathrm{d}x$$

and

$$\sigma_{\varrho^\varepsilon}(0) = \int_{\mathbb{T}^3} (|\varrho_0^\varepsilon - \varrho^*|^2 \chi_{(|\varrho_0^\varepsilon - \varrho^*| \leqslant \delta)} + |\varrho_0^\varepsilon - \varrho^*|^\gamma \chi_{(|\varrho_0^\varepsilon - \varrho^*| > \delta)}) \,\mathrm{d}x,$$
$$\sigma_{\eta^\varepsilon}(0) = \int_{\mathbb{T}^3} [\eta_0^\varepsilon \ln \eta_0^\varepsilon - \eta^* \ln \eta^* - (\ln \eta^* + 1)(\eta_0^\varepsilon - \eta^*)] \,\mathrm{d}x$$

for any $\delta \in (0,1)$.

The main result of this paper can be stated as follows.

Theorem 1.1. *Assume that* $\{(\varrho^\varepsilon, \eta^\varepsilon, u^\varepsilon)\}_{\varepsilon>0}$ *is a sequence of weak solutions to the fluid-particle interaction model* (1.4) *obtained in Theorem* 1.1, *satisfying the* *conditions:*

$$\frac{\sigma_{\varrho^\varepsilon}(0)+\sigma_{\eta^\varepsilon}(0)}{\varepsilon^2}+\|\sqrt{\varrho_0^\varepsilon}u_0^\varepsilon-\sqrt{\varrho^*}u_0^0\|_{L^2(\mathbb{T}^3)}^2\leqslant C\varepsilon. \tag{1.15}$$

Also assume that (u,π) *with* $l>\frac{3}{2}+3$ *is the smooth solution to the incompressible Navier-Stokes equations* (1.6) *with the initial data* $u_0^0\in H^l(\mathbb{T}^3)$ *such that* $\operatorname{div} u_0^0=0$, *satisfying*

$$\sup_{0\leqslant t\leqslant T}(\|u^0\|_{H^l(\mathbb{T}^3)}+\|\partial_t u^0\|_{H^{l-2}(\mathbb{T}^3)}+\|\nabla\pi\|_{H^{l-2}(\mathbb{T}^3)}+\|\partial_t\pi\|_{H^{l-3}(\mathbb{T}^3)})\leqslant C(T) \tag{1.16}$$

for any $0<T<T_*$ *and some positive constant* $C(T)$, *where* T_* *is the maximal existence time of* (u^0,π). *Then, for any* $0<t<T$, *we have*

$$\frac{\sigma_{\varrho^\varepsilon}+\sigma_{\eta^\varepsilon}}{\varepsilon^2}+\|\sqrt{\varrho^\varepsilon}u^\varepsilon-\sqrt{\varrho^*}u^0\|_{L^2(\mathbb{T}^3)}^2\leqslant C\varepsilon \tag{1.17}$$

and

$$\big\|\varrho^\varepsilon u^\varepsilon-\varrho^* u^0\big\|_{L^{2\gamma/(\gamma+1)}(\mathbb{T}^3)}^2\leqslant C\varepsilon. \tag{1.18}$$

Moreover, we have

$$\|\nabla(u^\varepsilon-u^0)\|_{L^2([0,T];L^2(\mathbb{T}^3))}^2\leqslant C\varepsilon. \tag{1.19}$$

Remark 1.1. Theorem 1.1 describes the incompressible limit of the compressible fluid-particle system (1.4) with well-prepared initial data. For the general initial data, the fast singular oscillation appears. It is more difficult to prove the incompressible limit in this case, which will be considered in future work.

Remark 1.2. We remark that the estimate in Theorem 1.1 is uniform with respect to μ and ν. Therefore, Theorem 1.1 is not only a stability result with respect to ε but also with respect to μ and ν. In fact, we can show that the combined incompressible and vanishing viscosity limit of the fluid-particle system (1.4) is the incompressible Euler equation.

Our proof of Theorem 1.1 relies heavily on the use of energy inequalities and relative entropy methods.

Let $(\varrho^\varepsilon, u^\varepsilon, \eta^\varepsilon)$ be a solution of the system (1.4) with initial data (1.7) and u^0 be the smooth solution of the incompressible Navier-Stokes equations (1.6) with the initial data u^0_0. Denote by $M^\varepsilon = \varrho^\varepsilon u^\varepsilon$, $M^* = \varrho^* u^0$,

$$U^\varepsilon = \begin{pmatrix} \varrho^\varepsilon \\ \varrho^\varepsilon u^\varepsilon \\ \eta^\varepsilon \end{pmatrix}, \quad U^* = \begin{pmatrix} \varrho^* \\ \varrho^* u^0 \\ \eta^* \end{pmatrix}.$$

Given an entropy

$$E(U^\varepsilon) = \frac{(M^\varepsilon)^2}{2\varrho^\varepsilon} + \frac{a}{\varepsilon^2(\gamma-1)}(\varrho^\varepsilon)^\gamma + \frac{1}{\varepsilon^2}\eta^\varepsilon \ln \eta^\varepsilon + (\varrho^\varepsilon + \eta^\varepsilon)\Phi^\varepsilon, \tag{2.1}$$

the relative entropy is defined by

$$H(U^\varepsilon|U^*) = E(U^\varepsilon) - E(U^*) - DE(U^*)(U^\varepsilon - U^*),$$

where D denotes the derivation with respect to the variables $(\varrho^\varepsilon, M^\varepsilon, \eta^\varepsilon)$. We can get that the relative entropy associated with (1.4) is as follows:

$$\begin{aligned}
H(U^\varepsilon|U^*) &= \frac{(M^\varepsilon)^2}{2\varrho^\varepsilon} + \frac{a}{\varepsilon^2(\gamma-1)}(\varrho^\varepsilon)^\gamma + \frac{1}{\varepsilon^2}\eta^\varepsilon \ln \eta^\varepsilon + (\varrho^\varepsilon + \eta^\varepsilon)\Phi^\varepsilon \\
&\quad - \frac{(M^*)^2}{2\varrho^*} - \frac{a}{\varepsilon^2(\gamma-1)}(\varrho^*)^\gamma - \frac{1}{\varepsilon^2}\eta^* \ln \eta^\varepsilon - (\varrho^* + \eta^*)\Phi^\varepsilon \\
&\quad - \begin{pmatrix} -\frac{|u^0|^2}{2} + \frac{a\gamma}{\varepsilon^2(\gamma-1)}(\varrho^*)^{\gamma-1} + \Phi^\varepsilon \\ u^0 \\ \varepsilon^{-2}(\ln \eta^* + 1) + \Phi^\varepsilon \end{pmatrix} \cdot \begin{pmatrix} \varrho^\varepsilon - \varrho^* \\ \varrho^\varepsilon u^\varepsilon - \varrho^* u^0 \\ \eta^\varepsilon - \eta^* \end{pmatrix} \\
&= \frac{1}{2}\varrho^\varepsilon |u^\varepsilon|^2 + \frac{a}{\varepsilon^2(\gamma-1)}(\varrho^\varepsilon)^\gamma + \frac{1}{\varepsilon^2}\eta^\varepsilon \ln \eta^\varepsilon + (\varrho^\varepsilon + \eta^\varepsilon)\Phi^\varepsilon \\
&\quad - \frac{1}{2}\varrho^* |u^0|^2 - \frac{a}{\varepsilon^2(\gamma-1)}(\varrho^*)^\gamma - \frac{1}{\varepsilon^2}\eta^* \ln \eta^\varepsilon - (\varrho^* + \eta^*)\Phi^\varepsilon \\
&\quad + \frac{1}{2}\varrho^\varepsilon |u^0|^2 - \frac{1}{2}\varrho^* |u^0|^2 - \frac{a\gamma}{\varepsilon^2(\gamma-1)}(\varrho^*)^{\gamma-1}\varrho^\varepsilon + \frac{a\gamma}{\varepsilon^2(\gamma-1)}(\varrho^*)^\gamma \\
&\quad - \varrho^\varepsilon \Phi^\varepsilon + \varrho^* \Phi^\varepsilon - \varrho^\varepsilon u^\varepsilon \cdot u^0 + \varrho^* |u^0|^2 - \frac{1}{\varepsilon^2}\eta^\varepsilon \ln \eta^* \\
&\quad - \frac{1}{\varepsilon^2}\eta^\varepsilon - \eta^\varepsilon \Phi^\varepsilon + \frac{1}{\varepsilon^2}\eta^* \ln \eta^* + \frac{1}{\varepsilon^2}\eta^* + \eta^* \Phi^\varepsilon \\
&= \frac{1}{2}\varrho^\varepsilon |u^\varepsilon - u^0|^2 + \frac{a}{\varepsilon^2(\gamma-1)}[(\varrho^\varepsilon)^\gamma - (\varrho^*)^\gamma - \gamma(\varrho^*)^{\gamma-1}(\varrho^\varepsilon - \varrho^*)] \\
&\quad + \frac{1}{\varepsilon^2}\eta^\varepsilon \ln \eta^\varepsilon - \frac{1}{\varepsilon^2}\eta^* \ln \eta^* - \frac{1}{\varepsilon^2}(\ln \eta^* + 1)(\eta^\varepsilon - \eta^*).
\end{aligned}$$

Denoting

$$\begin{aligned}E_1(\varrho) &= \frac{a}{\gamma-1}\varrho^\gamma, \quad H_1(\varrho^\varepsilon) = E_1(\varrho^\varepsilon) - E_1'(\varrho^*)(\varrho^\varepsilon-\varrho^*) - E_1(\varrho^*),\\ E_2(\eta) &= \eta\ln\eta, \qquad H_2(\eta^\varepsilon) = E_2(\eta^\varepsilon) - E_2'(\eta^*)(\eta^\varepsilon-\eta^*) - E_2(\eta^*),\end{aligned}$$

then the relative entropy can be rewritten as

$$H(U^\varepsilon|U^*) = \frac{1}{2}\varrho^\varepsilon|u^\varepsilon-u^0|^2 + \frac{1}{\varepsilon^2}H_1(\varrho^\varepsilon) + \frac{1}{\varepsilon^2}H_2(\eta^\varepsilon). \tag{2.2}$$

Therefore, we introduce the following form of the modulated energy:

$$\mathcal{H}^\varepsilon(t) = \int_{\mathbb{T}^3} H(U^\varepsilon|U^*)\,\mathrm{d}x = \int_{\mathbb{T}^3}\Big[\frac{1}{2}\varrho^\varepsilon|u^\varepsilon-u^0|^2 + \frac{1}{\varepsilon^2}H_1(\varrho^\varepsilon) + \frac{1}{\varepsilon^2}H_2(\eta^\varepsilon)\Big]\,\mathrm{d}x. \tag{2.3}$$

In the next section, we are going to prove Theorem 1.1. We shall employ the evolution equations and elaborated computations to prove the inequality

$$\mathcal{H}^\varepsilon(t) \leqslant C\int_0^t \mathcal{H}^\varepsilon(s)\,\mathrm{d}s + C\varepsilon. \tag{2.4}$$

The Gronwall lemma then implies the desired result.

3. Proof of Theorem 1.1

In this section we shall prove Theorem 1.1.

Lemma 3.1. *Let* $(\varrho^\varepsilon,\ \eta^\varepsilon,\ u^\varepsilon)$ *be the weak solution to system* (1.4) *with initial data* (1.7) *on* $[0,T]$ *with* $T<T_\star$. *Then, we have the following properties:*

$$\sqrt{\varrho^\varepsilon}u^\varepsilon \text{ is bounded in } L^\infty([0,T];L^2(\mathbb{T}^3)), \tag{3.1}$$

$$\frac{1}{\varepsilon^2}H_1(\varrho^\varepsilon) \text{ is bounded in } L^\infty([0,T];L^1(\mathbb{T}^3)), \tag{3.2}$$

$$\frac{1}{\varepsilon^2}H_2(\eta^\varepsilon) \text{ is bounded in } L^\infty([0,T];L^1(\mathbb{T}^3)). \tag{3.3}$$

Proof. From the energy inequality (1.12), the first and the third equations in (1.4), we have for almost all $t\in[0,T]$,

$$\begin{aligned}&\int_{\mathbb{T}^3}\Big[\frac{1}{2}\varrho^\varepsilon|u^\varepsilon|^2 + \frac{1}{\varepsilon^2}H_1(\varrho^\varepsilon) + \frac{1}{\varepsilon^2}H_2(\eta^\varepsilon) + (\varrho^\varepsilon+\eta^\varepsilon)\Phi^\varepsilon\Big]\,\mathrm{d}x\\ &\quad + \int_0^t\int_{\mathbb{T}^3}\Big(\mu|\nabla u^\varepsilon|^2 + (\mu+\nu)|\operatorname{div} u^\varepsilon|^2 + \frac{1}{\varepsilon^3}|2\nabla\sqrt{\eta^\varepsilon} + \varepsilon^2\sqrt{\eta^\varepsilon}\nabla\Phi^\varepsilon|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant \int\Big[\frac{1}{2}\varrho_0^\varepsilon|u_0^\varepsilon|^2 + \frac{1}{\varepsilon^2}H_1(\varrho_0^\varepsilon) + \frac{1}{\varepsilon^2}H_2(\eta_0^\varepsilon) + (\varrho_0^\varepsilon+\eta_0^\varepsilon)\Phi^\varepsilon\Big]\,\mathrm{d}x \leqslant C,\end{aligned} \tag{3.4}$$

which implies that Lemma 3.1 holds. □

Lemma 3.2. *Let $(\varrho^\varepsilon,\eta^\varepsilon,u^\varepsilon)$ be the weak solution to system (1.4) on $[0,T]$ with $T<T_\star$. Then there exists a constant $C>0$ such that for all $\varepsilon\in(0,1)$ and $\gamma>1$,*

$$\|\varrho^\varepsilon-\varrho^*\|_{L^\infty([0,T];L^\lambda(\mathbb{T}^3))}+\|\sqrt{\eta^\varepsilon}-\sqrt{\eta^*}\|_{L^\infty([0,T];L^2(\mathbb{T}^3))}\leqslant C\varepsilon, \tag{3.5}$$

where, $\lambda=\min\{2,\gamma\}$. Moreover, we have

$$\|\sqrt{\varrho^\varepsilon}-\sqrt{\varrho^*}\|_{L^\infty([0,T];L^2(\mathbb{T}^3))}\leqslant C\varepsilon. \tag{3.6}$$

Proof. In view of the Lemma 5.3 in [21], there exist two positive constants $c_1\in(0,1)$ and $c_2\in(1,\infty)$ independent of ϱ^ε such that the following inequality

$$c_1\sigma_{\varrho^\varepsilon}\leqslant\int_{\mathbb{T}^3}H_1(\varrho^\varepsilon)\,\mathrm{d}x\leqslant c_2\sigma_{\varrho^\varepsilon}, \tag{3.7}$$

holds for any fixed $\delta\in(0,1)$.

(1) If $1\leqslant\gamma<2$, in view of (3.2), we have

$$\begin{aligned}\|\varrho^\varepsilon-\varrho^*\|^\gamma_{L^\gamma(\mathbb{T}^3)}&\leqslant C\Big(\int_{\mathbb{T}^3}|\varrho^\varepsilon-\varrho^*|^2\chi_{(|\varrho^\varepsilon-\varrho^*|\leqslant\delta)}\,\mathrm{d}x\Big)^{\gamma/2}\\&\quad+\int_{\mathbb{T}^3}|\varrho^\varepsilon-\varrho^*|^\gamma\chi_{(|\varrho^\varepsilon-\varrho^*|>\delta)}\,\mathrm{d}x\\&\leqslant C(\sigma_{\varrho^\varepsilon}^{\gamma/2}+\sigma_{\varrho^\varepsilon})\\&\leqslant C\Big[\Big(\int_{\mathbb{T}^3}H_1(\varrho^\varepsilon)\,\mathrm{d}x\Big)^{\gamma/2}+\int_{\mathbb{T}^3}H_1(\varrho^\varepsilon)\,\mathrm{d}x\Big]\\&\leqslant C(\varepsilon^\gamma+\varepsilon^2)\leqslant C\varepsilon^\gamma.\end{aligned} \tag{3.8}$$

(2) If $\gamma\geqslant2$, we get

$$\|\varrho^\varepsilon-\varrho^*\|^\gamma_{L^\gamma(\mathbb{T}^3)}\leqslant C\int_{\mathbb{T}^3}H_1(\varrho^\varepsilon)\,\mathrm{d}x\leqslant C\varepsilon^2, \tag{3.9}$$

and

$$\|\varrho^\varepsilon-\varrho^*\|^2_{L^2(\mathbb{T}^3)}\leqslant C\int_{\mathbb{T}^3}H_1(\varrho^\varepsilon)\,\mathrm{d}x\leqslant C\varepsilon^2. \tag{3.10}$$

From (3.8)–(3.10), one obtains

$$\|\varrho^\varepsilon-\varrho^*\|_{L^\gamma(\mathbb{T}^3)}\leqslant C\varepsilon^{\lambda/\gamma}\quad\text{and}\quad\|\varrho^\varepsilon-\varrho^*\|_{L^\lambda(\mathbb{T}^3)}\leqslant C\varepsilon, \tag{3.11}$$

where, $\lambda=\min\{2,\gamma\}$.

It is easy to verify that the following inequality

$$y\ln y-(\ln y_0+1)(y-y_0)-y_0\ln y_0\geqslant(\sqrt{y}-\sqrt{y_0})^2$$

holds for all $y>0$, where $y_0>0$ is a constant.

Using (3.3) and the above inequality, we have

$$\|\sqrt{\eta^\varepsilon}-\sqrt{\eta^*}\|^2_{L^2(\mathbb{T}^3)}=\int_{\mathbb{T}^3}|\sqrt{\eta^\varepsilon}-\sqrt{\eta^*}|^2\,\mathrm{d}x\leqslant\int_{\mathbb{T}^3}H_2(\eta^\varepsilon)\,\mathrm{d}x\leqslant C\varepsilon^2. \tag{3.12}$$

Then, one obtains

$$\|\varrho^\varepsilon-\varrho^*\|_{L^\lambda(\mathbb{T}^3)}+\|\sqrt{\eta^\varepsilon}-\sqrt{\eta^*}\|_{L^2(\mathbb{T}^3)}\leqslant C\varepsilon, \tag{3.13}$$

which implies that the inequality (3.5) holds.

For a fixed positive constant a, using the following two elementary inequalities

$$|\sqrt{x}-\sqrt{a}|^2\leqslant M|x-a|^\gamma,\quad |x-a|\geqslant\delta,\ \gamma\geqslant1$$

and

$$|\sqrt{x}-\sqrt{a}|^2\leqslant M|x-a|^2,\quad x\geqslant0$$

for some positive constant M, and $0<\delta<1$, it is easy to obtain that

$$\begin{aligned}\int_{\mathbb{T}^3}|\sqrt{\varrho^\varepsilon}-\sqrt{\varrho^*}|^2\,\mathrm{d}x&=\int_{\mathbb{T}^3}|\sqrt{\varrho^\varepsilon}-\sqrt{\varrho^*}|^2\chi_{(|\varrho^\varepsilon-\varrho^*|\leqslant\delta)}\,\mathrm{d}x\\&\quad+\int_{\mathbb{T}^3}|\sqrt{\varrho^\varepsilon}-\sqrt{\varrho^*}|^2\chi_{(|\varrho^\varepsilon-\varrho^*|>\delta)}\,\mathrm{d}x\\&\leqslant M\int_{\mathbb{T}^3}|\varrho^\varepsilon-\varrho^*|^2\chi_{(|\varrho^\varepsilon-\varrho^*|\leqslant\delta)}\,\mathrm{d}x\\&\quad+M\int_{\mathbb{T}^3}|\varrho^\varepsilon-\varrho^*|^\gamma\chi_{(|\varrho^\varepsilon-\varrho^*|>\delta)}\,\mathrm{d}x\\&\leqslant CM\int_{\mathbb{T}^3}H_1(\varrho^\varepsilon)\,\mathrm{d}x\leqslant C\varepsilon^2,\end{aligned}$$

which implies that (3.6) holds. □

Lemma 3.3. *For all $t\in[0,T]$, we have*

$$\mathcal{H}^\varepsilon(t)\leqslant C\varepsilon. \tag{3.14}$$

P r o o f. In view (2.3) and (3.4), we get

$$
\begin{aligned}
(3.15)\qquad &\mathcal{H}^\varepsilon(t)+\int_0^t\int_{\mathbb{T}^3}(\mu|\nabla u^\varepsilon|^2+(\mu+\nu)|\operatorname{div}u^\varepsilon|^2)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_0^t\int_{\mathbb{T}^3}\Big(\frac{1}{\varepsilon^3}|2\nabla\sqrt{\eta^\varepsilon}+\varepsilon^2\sqrt{\eta^\varepsilon}\nabla\Phi^\varepsilon|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\\
&=\int_{\mathbb{T}^3}\Big[\frac12\varrho^\varepsilon|u^\varepsilon|^2+\frac{1}{\varepsilon^2}H_1(\varrho^\varepsilon)+\frac{1}{\varepsilon^2}H_2(\eta^\varepsilon)\Big]\,\mathrm{d}x\\
&\quad-\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\cdot u^0\,\mathrm{d}x+\frac12\int_{\mathbb{T}^3}\varrho^\varepsilon|u^0|^2\\
&\quad+\int_0^t\int_{\mathbb{T}^3}(\mu|\nabla u^\varepsilon|^2+(\mu+\nu)|\operatorname{div}u^\varepsilon|^2\\
&\quad+\frac{1}{\varepsilon^3}|2\nabla\sqrt{\eta^\varepsilon}+\varepsilon^2\sqrt{\eta^\varepsilon}\nabla\Phi^\varepsilon|^2)\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant\int\Big[\frac12\varrho_0^\varepsilon|u_0^\varepsilon-u_0^0|^2+\frac{1}{\varepsilon^2}H_1(\varrho_0^\varepsilon)+\frac{1}{\varepsilon^2}H_2(\eta_0^\varepsilon)\Big]\,\mathrm{d}x\\
&\quad-\int_{\mathbb{T}^3}(\varrho^\varepsilon+\eta^\varepsilon)\Phi^\varepsilon\,\mathrm{d}x+\int_{\mathbb{T}^3}(\varrho_0^\varepsilon+\eta_0^\varepsilon)\Phi^\varepsilon\,\mathrm{d}x-\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\cdot u^0\,\mathrm{d}x\\
&\quad+\int_{\mathbb{T}^3}\varrho_0^\varepsilon u_0^\varepsilon\cdot u_0^0\,\mathrm{d}x+\frac12\int_{\mathbb{T}^3}\varrho^\varepsilon|u^0|^2\,\mathrm{d}x-\frac12\int_{\mathbb{T}^3}\varrho_0^\varepsilon|u_0^0|^2\\
&=\mathcal{H}^\varepsilon(0)-\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\cdot u^0\,\mathrm{d}x+\int_{\mathbb{T}^3}\varrho_0^\varepsilon u_0^\varepsilon\cdot u_0^0\,\mathrm{d}x\\
&\quad-\int_{\mathbb{T}^3}(\varrho^\varepsilon+\eta^\varepsilon)\Phi^\varepsilon\,\mathrm{d}x+\int_{\mathbb{T}^3}(\varrho_0^\varepsilon+\eta_0^\varepsilon)\Phi^\varepsilon\,\mathrm{d}x\\
&\quad+\frac12\int_{\mathbb{T}^3}(\varrho^\varepsilon-\varrho^*)|u^0|^2\,\mathrm{d}x-\frac12\int_{\mathbb{T}^3}(\varrho_0^\varepsilon-\varrho^*)|u_0^0|^2\,\mathrm{d}x\\
&\quad+\frac{\varrho^*}{2}\int_{\mathbb{T}^3}|u^0|^2\,\mathrm{d}x-\frac{\varrho^*}{2}\int_{\mathbb{T}^3}|u_0^0|^2\,\mathrm{d}x.
\end{aligned}
$$

We use u^0 (the solution of incompressible Navier-Stokes equations (1.6)) as a test function in the weak formulation of momentum equation $(1.4)_2$ to yield the following equality for almost all $t\in[0,T]$:

$$
\begin{aligned}
(3.16)\qquad &-\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\cdot u^0\,\mathrm{d}x+\int_{\mathbb{T}^3}\varrho_0^\varepsilon u_0^\varepsilon\cdot u_0^0\,\mathrm{d}x\\
&=-\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\cdot\partial_s u^0\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\otimes u^\varepsilon:\nabla u^0\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\mu\int_0^t\int_{\mathbb{T}^3}\nabla u^\varepsilon:\nabla u^0\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{T}^3}(\varrho^\varepsilon+\eta^\varepsilon)\nabla\Phi^\varepsilon\cdot u^0\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}
$$

where we have used the fact that $\operatorname{div}u^0=0$.

Taking the L^2 inner product of the equation (1.6) by u^0, we have

$$\frac{\varrho^*}{2}\int_{\mathbb{T}^3}|u^0|^2\,\mathrm{d}x-\frac{\varrho^*}{2}\int_{\mathbb{T}^3}|u_0^0|^2\,\mathrm{d}x=-\mu\int_0^t\int_{\mathbb{T}^3}|\nabla u^0|^2\,\mathrm{d}x\,\mathrm{d}s. \tag{3.17}$$

Inserting (3.16)–(3.17) into (3.15) and using (1.6), we have

$$\begin{aligned}
&\mathcal{H}^\varepsilon(t)+\int_0^t\int_{\mathbb{T}^3}(\mu|\nabla u^\varepsilon|^2+(\mu+\nu)|\operatorname{div}u^\varepsilon|^2)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\int_0^t\int_{\mathbb{T}^3}\Big(\frac{1}{\varepsilon^3}|2\nabla\sqrt{\eta^\varepsilon}+\varepsilon^2\sqrt{\eta^\varepsilon}\nabla\Phi^\varepsilon|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\\
&\leqslant\mathcal{H}^\varepsilon(0)-\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\cdot\partial_s u^0\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\otimes u^\varepsilon:\nabla u^0\,\mathrm{d}x\,\mathrm{d}s-\int_{\mathbb{T}^3}(\varrho^\varepsilon+\eta^\varepsilon)\Phi^\varepsilon\,\mathrm{d}x\\
&\quad+\int_{\mathbb{T}^3}(\varrho_0^\varepsilon+\eta_0^\varepsilon)\Phi^\varepsilon\,\mathrm{d}x+\int_0^t\int_{\mathbb{T}^3}(\varrho^\varepsilon+\eta^\varepsilon)\nabla\Phi^\varepsilon\cdot u^0\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\frac12\int_{\mathbb{T}^3}(\varrho^\varepsilon-\varrho^*)|u^0|^2\,\mathrm{d}x-\frac12\int_{\mathbb{T}^3}(\varrho_0^\varepsilon-\varrho^*)|u_0^0|^2\,\mathrm{d}x\\
&\quad+\mu\int_0^t\int_{\mathbb{T}^3}\nabla u^\varepsilon:\nabla u^0\,\mathrm{d}x\,\mathrm{d}s-\mu\int_0^t\int_{\mathbb{T}^3}|\nabla u^0|^2\,\mathrm{d}x\,\mathrm{d}s\\
&=\mathcal{H}^\varepsilon(0)+\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\cdot(u^0\cdot\nabla)u^0\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\cdot\nabla\pi\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-\frac{\mu}{\varrho^*}\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\cdot\Delta u^0\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\otimes u^\varepsilon:\nabla u^0\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-\int_{\mathbb{T}^3}(\varrho^\varepsilon+\eta^\varepsilon)\Phi^\varepsilon\,\mathrm{d}x+\int_{\mathbb{T}^3}(\varrho_0^\varepsilon+\eta_0^\varepsilon)\Phi^\varepsilon\,\mathrm{d}x\\
&\quad+\int_0^t\int_{\mathbb{T}^3}(\varrho^\varepsilon+\eta^\varepsilon)\nabla\Phi^\varepsilon\cdot u^0\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\frac12\int_{\mathbb{T}^3}(\varrho^\varepsilon-\varrho^*)|u^0|^2\,\mathrm{d}x-\frac12\int_{\mathbb{T}^3}(\varrho_0^\varepsilon-\varrho^*)|u_0^0|^2\,\mathrm{d}x\\
&\quad+\mu\int_0^t\int_{\mathbb{T}^3}\nabla u^\varepsilon:\nabla u^0\,\mathrm{d}x\,\mathrm{d}s-\mu\int_0^t\int_{\mathbb{T}^3}|\nabla u^0|^2\,\mathrm{d}x\,\mathrm{d}s\\
&=\mathcal{H}^\varepsilon(0)+\sum_{k=1}^5\mathcal{I}_k,
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
\mathcal{I}_1&=\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\cdot(u^0\cdot\nabla)u^0\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\otimes u^\varepsilon:\nabla u^0\,\mathrm{d}x\,\mathrm{d}s,\\
\mathcal{I}_2&=\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\cdot\nabla\pi\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_3 = & -\frac{\mu}{\varrho^*}\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\cdot\Delta u^0\,\mathrm{d}x\,\mathrm{d}s \\
& +\mu\int_0^t\int_{\mathbb{T}^3}\nabla u^\varepsilon:\nabla u^0\,\mathrm{d}x\,\mathrm{d}s-\mu\int_0^t\int_{\mathbb{T}^3}|\nabla u^0|^2\,\mathrm{d}x\,\mathrm{d}s, \\
\mathcal{I}_4 = & -\int_{\mathbb{T}^3}(\varrho^\varepsilon+\eta^\varepsilon)\Phi^\varepsilon\,\mathrm{d}x+\int_{\mathbb{T}^3}(\varrho_0^\varepsilon+\eta_0^\varepsilon)\Phi^\varepsilon\,\mathrm{d}x \\
& +\int_0^t\int_{\mathbb{T}^3}(\varrho^\varepsilon+\eta^\varepsilon)\nabla\Phi^\varepsilon\cdot u^0\,\mathrm{d}x\,\mathrm{d}s, \\
\mathcal{I}_5 = & \frac{1}{2}\int_{\mathbb{T}^3}(\varrho^\varepsilon-\varrho^*)|u^0|^2\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{T}^3}(\varrho_0^\varepsilon-\varrho^*)|u_0^0|^2\,\mathrm{d}x.
\end{aligned}$$

Now, we begin to treat the integrals $\mathcal{I}_k(k=1,2,3,4,5)$ and $\mathcal{H}^\varepsilon(0)$ term by term. In fact, we do not need to deal with $\mathcal{I}_5$, which will be canceled later.

For $\mathcal{I}_1$, we rewrite and estimate it in the following:

$$\begin{aligned}
\mathcal{I}_1 = & \int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^0\otimes u^\varepsilon\cdot u^0\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\otimes u^\varepsilon:\nabla u^0\,\mathrm{d}x\,\mathrm{d}s, \\
= & -\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon(u^\varepsilon-u^0)\otimes(u^\varepsilon-u^0):\nabla u^0\,\mathrm{d}x\,\mathrm{d}s \\
& +\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^0\otimes u^0:\nabla u^0\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\otimes u^0]:\nabla u^0\,\mathrm{d}x\,\mathrm{d}s \\
\leqslant & \, C\int_0^t\mathcal{H}^\varepsilon(s)\,\mathrm{d}s+\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^0\otimes u^0:\nabla u^0\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^\varepsilon\otimes u^0:\nabla u^0\,\mathrm{d}x\,\mathrm{d}s \\
= & \, C\int_0^t\mathcal{H}^\varepsilon(s)\,\mathrm{d}s+\mathcal{I}_{11}+\mathcal{I}_{12}.
\end{aligned}$$

Using the Lemma 3.2 and the inequality (1.16), we get that

$$\begin{aligned}
\mathcal{I}_{11} = & \int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^0\otimes u^0:\nabla u^0\,\mathrm{d}x\,\mathrm{d}s \\
= & \frac{1}{2}\int_0^t\int_{\mathbb{T}^3}\varrho^\varepsilon u^0\cdot\nabla|u^0|^2\,\mathrm{d}x\,\mathrm{d}s \\
= & \frac{1}{2}\int_0^t\int_{\mathbb{T}^3}(\varrho^\varepsilon-\varrho^*)u^0\cdot\nabla|u^0|^2\,\mathrm{d}x\,\mathrm{d}s \\
\leqslant & \, C\|\varrho^\varepsilon-\varrho^*\|_{L^\infty([0,T];L^\lambda(\mathbb{T}^3))}\|u^0\cdot\nabla|u^0|^2\|_{L^\infty([0,T];L^{\lambda/(\lambda-1)}(\mathbb{T}^3))}\leqslant C\varepsilon,
\end{aligned}$$

where we noticed that

$$\int_0^t\int_{\mathbb{T}^3}u^0\cdot\nabla|u^0|^2\,\mathrm{d}x\,\mathrm{d}s=-\int_0^t\int_{\mathbb{T}^3}\operatorname{div}u^0\cdot|u^0|^2\,\mathrm{d}x\,\mathrm{d}s=0$$

due to $\operatorname{div} u^0 = 0$. From $(1.4)_1$ and the inequality (1.16), we get, by integration by parts, that

$$\begin{aligned}
\mathcal{I}_{12} = & -\int_0^t \int_{\mathbb{T}^3} \varrho^\varepsilon u^\varepsilon \otimes u^0 : \nabla u^0 \,\mathrm{d}x\,\mathrm{d}s = -\frac{1}{2}\int_0^t \int_{\mathbb{T}^3} \varrho^\varepsilon u^\varepsilon \cdot \nabla |u^0|^2 \,\mathrm{d}x\,\mathrm{d}s \\
= & -\frac{1}{2}\int_0^t \int_{\mathbb{T}^3} \partial_s(\varrho^\varepsilon - \varrho^*) \cdot |u^0|^2 \,\mathrm{d}x\,\mathrm{d}s = -\frac{1}{2}\int_{\mathbb{T}^3} (\varrho^\varepsilon - \varrho^*)|u^0|^2 \,\mathrm{d}x \\
& + \frac{1}{2}\int_{\mathbb{T}^3} (\varrho_0^\varepsilon - \varrho^*)|u_0^0|^2 \,\mathrm{d}x + \frac{1}{2}\int_0^t \int_{\mathbb{T}^3} (\varrho^\varepsilon - \varrho^*)\partial_s |u^0|^2 \,\mathrm{d}x\,\mathrm{d}s \\
\leqslant & -\mathcal{I}_5 + C\|\varrho^\varepsilon - \varrho^*\|_{L^\infty([0,T];L^\lambda(\mathbb{T}^3))} \|\partial_s |u^0|^2\|_{L^\infty([0,T];L^{\lambda/(\lambda-1)}(\mathbb{T}^3))} \\
\leqslant & -\mathcal{I}_5 + C\varepsilon.
\end{aligned}$$

So, we have

$$\mathcal{I}_1 + \mathcal{I}_5 \leqslant C\int_0^t \mathcal{H}^\varepsilon(s)\,\mathrm{d}s + C\varepsilon. \tag{3.19}$$

For $\mathcal{I}_2$, using Lemma 3.2, the Young inequality, the continuity equations $(1.4)_1$ and integrating by parts, we get that

$$\begin{aligned}
\mathcal{I}_2 = & \int_0^t \int_{\mathbb{T}^3} \varrho^\varepsilon u^\varepsilon \cdot \nabla \pi \,\mathrm{d}x\,\mathrm{d}s = -\int_0^t \int_{\mathbb{T}^3} \operatorname{div}(\varrho^\varepsilon u^\varepsilon)\pi \,\mathrm{d}x\,\mathrm{d}s \\
= & \int_0^t \int_{\mathbb{T}^3} \partial_s(\varrho^\varepsilon - \varrho^*)\pi \,\mathrm{d}x\,\mathrm{d}s = \int_{\mathbb{T}^3} (\varrho^\varepsilon - \varrho^*)\pi \,\mathrm{d}x \\
& - \int_{\mathbb{T}^3} (\varrho_0^\varepsilon - \varrho^*)\pi_0 \,\mathrm{d}x - \int_0^t \int_{\mathbb{T}^3} (\varrho^\varepsilon - \varrho^*)\partial_s \pi \,\mathrm{d}x\,\mathrm{d}s \\
\leqslant & \|\varrho^\varepsilon - \varrho^*\|_{L^\lambda(\mathbb{T}^3)} \|\pi\|_{L^{\lambda/(\lambda-1)}(\mathbb{T}^3)} + \|\varrho_0^\varepsilon - \varrho^*\|_{L^\lambda(\mathbb{T}^3)} \|\pi_0\|_{L^{\lambda/(\lambda-1)}(\mathbb{T}^3)} \\
& + \|\varrho^\varepsilon - \varrho^*\|_{L^\infty([0,T];L^\lambda(\mathbb{T}^3))} \|\partial_s \pi\|_{L^\infty([0,T];L^{\lambda/(\lambda-1)}(\mathbb{T}^3))} \leqslant C\varepsilon.
\end{aligned} \tag{3.20}$$

For $\mathcal{I}_3$, using Lemma 3.2, Young inequality, and (3.6), we have

$$\begin{aligned}
\mathcal{I}_3 = & -\frac{\mu}{\varrho^*}\int_0^t \int_{\mathbb{T}^3} \varrho^\varepsilon u^\varepsilon \cdot \Delta u^0 \,\mathrm{d}x\,\mathrm{d}s \\
& + \mu \int_0^t \int_{\mathbb{T}^3} \nabla u^\varepsilon : \nabla u^0 \,\mathrm{d}x\,\mathrm{d}s - \mu \int_0^t \int_{\mathbb{T}^3} |\nabla u^0|^2 \,\mathrm{d}x\,\mathrm{d}s, \\
= & -\frac{\mu}{\varrho^*}\int_0^t \int_{\mathbb{T}^3} (\sqrt{\varrho^\varepsilon} - \sqrt{\varrho^*})\sqrt{\varrho^\varepsilon} u^\varepsilon \cdot \Delta u^0 \,\mathrm{d}x\,\mathrm{d}s \\
& - \frac{\mu}{\sqrt{\varrho^*}}\int_0^t \int_{\mathbb{T}^3} (\sqrt{\varrho^\varepsilon} - \sqrt{\varrho^*}) u^\varepsilon \cdot \Delta u^0 \,\mathrm{d}x\,\mathrm{d}s - \mu \int_0^t \int_{\mathbb{T}^3} u^\varepsilon \cdot \Delta u^0 \,\mathrm{d}x\,\mathrm{d}s \\
& + \mu \int_0^t \int_{\mathbb{T}^3} \nabla u^\varepsilon : \nabla u^0 \,\mathrm{d}x\,\mathrm{d}s - \mu \int_0^t \int_{\mathbb{T}^3} |\nabla u^0|^2 \,\mathrm{d}x\,\mathrm{d}s \\
\leqslant & C\|\sqrt{\varrho^\varepsilon} - \sqrt{\varrho^*}\|_{L^\infty([0,T];L^2(\mathbb{T}^3))}
\end{aligned} \tag{3.21}$$

$$
\begin{aligned}
&\times \|\sqrt{\varrho^\varepsilon}u^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{T}^3))}\|\Delta u^0\|_{L^1([0,T];L^\infty(\mathbb{T}^3))}\\
&+ C\|\sqrt{\varrho^\varepsilon}-\sqrt{\varrho^*}\|_{L^\infty([0,T];L^2(\mathbb{T}^3))}\\
&\times \|u^\varepsilon\|_{L^2([0,T];L^2(\mathbb{T}^3))}\|\Delta u^0\|_{L^{12}([0,T];L^\infty(\mathbb{T}^3))}\\
&+ 2\mu\int_0^t\int_{\mathbb{T}^3}\nabla u^\varepsilon : \nabla u^0\,\mathrm{d}x\,\mathrm{d}s - \mu\int_0^t\int_{\mathbb{T}^3}|\nabla u^0|^2\,\mathrm{d}x\,\mathrm{d}s\\
\leqslant{}& C\varepsilon + 2\mu\int_0^t\int_{\mathbb{T}^3}\nabla u^\varepsilon : \nabla u^0\,\mathrm{d}x\,\mathrm{d}s - \mu\int_0^t\int_{\mathbb{T}^3}|\nabla u^0|^2\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

For $\mathcal{I}_4$, in view of (1.7), we have

$$
\begin{aligned}
\mathcal{I}_4 ={}& -\int_{\mathbb{T}^3}(\varrho^\varepsilon+\eta^\varepsilon)\Phi^\varepsilon\,\mathrm{d}x + \int_{\mathbb{T}^3}(\varrho_0^\varepsilon+\eta_0^\varepsilon)\Phi^\varepsilon\,\mathrm{d}x\\
&+\int_0^t\int_{\mathbb{T}^3}(\varrho^\varepsilon+\eta^\varepsilon)\nabla\Phi^\varepsilon\cdot u^0\,\mathrm{d}x\,\mathrm{d}s\\
={}& -\int_{\mathbb{T}^3}(\varrho^\varepsilon-\varrho^*)\Phi^\varepsilon\,\mathrm{d}x + \int_{\mathbb{T}^3}(\varrho_0^\varepsilon-\varrho^*)\Phi^\varepsilon\,\mathrm{d}x\\
&+\int_0^t\int_{\mathbb{T}^3}(\varrho^\varepsilon-\varrho^*)\nabla\Phi^\varepsilon\cdot u^0\,\mathrm{d}x\,\mathrm{d}s - \varrho^*\underbrace{\int_0^t\int_{\mathbb{T}^3}\nabla\Phi^\varepsilon\cdot u^0\,\mathrm{d}x\,\mathrm{d}s}_{=0,\ \text{due to } \operatorname{div}u^0=0}\\
&-\int_{\mathbb{T}^3}(\eta^\varepsilon-\eta^*)\Phi^\varepsilon\,\mathrm{d}x + \int_{\mathbb{T}^3}(\eta_0^\varepsilon-\eta^*)\Phi^\varepsilon\,\mathrm{d}x\\
&+\int_0^t\int_{\mathbb{T}^3}(\eta^\varepsilon-\eta^*)\nabla\Phi^\varepsilon\cdot u^0\,\mathrm{d}x\,\mathrm{d}s - \eta^*\underbrace{\int_0^t\int_{\mathbb{T}^3}\nabla\Phi^\varepsilon\cdot u^0\,\mathrm{d}x\,\mathrm{d}s}_{=0,\ \text{due to } \operatorname{div}u^0=0}\\
\leqslant{}& (\|\varrho^\varepsilon-\varrho^*\|_{L^\lambda(\mathbb{T}^3)} + \|\varrho_0^\varepsilon-\varrho^*\|_{L^\lambda(\mathbb{T}^3)})\|\Phi^\varepsilon\|_{L^{\lambda/(\lambda-1)}(\mathbb{T}^3)}\\
&+C\|\varrho^\varepsilon-\varrho^*\|_{L^\infty([0,T];L^\lambda(\mathbb{T}^3))}\|\nabla\Phi^\varepsilon\|_{L^{\lambda/(\lambda-1)}(\mathbb{T}^3)}\\
&+\|\sqrt{\eta^\varepsilon}-\sqrt{\eta^*}\|_{L^2(\mathbb{T}^3)}\|\sqrt{\eta^\varepsilon}+\sqrt{\eta^*}\|_{L^6(\mathbb{T}^3)}\|\Phi^\varepsilon\|_{L^3(\mathbb{T}^3)}\\
&+\|\sqrt{\eta_0^\varepsilon}-\sqrt{\eta^*}\|_{L^2(\mathbb{T}^3)}\|\sqrt{\eta_0^\varepsilon}+\sqrt{\eta^*}\|_{L^6(\mathbb{T}^3)}\|\Phi^\varepsilon\|_{L^3(\mathbb{T}^3)}\\
&+\|u^0\|_{L^\infty([0,T]\times\mathbb{T}^3)}\|\nabla\Phi^\varepsilon\|_{L^3(\mathbb{T}^3)}\|\sqrt{\eta^\varepsilon}-\sqrt{\eta^*}\|_{L^\infty([0,T];L^2(\mathbb{T}^3))}\\
&\times\|\sqrt{\eta^\varepsilon}+\sqrt{\eta^*}\|_{L^1([0,T];L^6(\mathbb{T}^3))} \leqslant C\varepsilon.
\end{aligned}
\tag{3.22}
$$

Inserting (3.19)–(3.22) into (3.18), we get

$$
\begin{aligned}
\mathcal{H}^\varepsilon(t) &+ \int_0^t\int_{\mathbb{T}^3}(\mu|\nabla u^\varepsilon-\nabla u^0|^2 + (\mu+\nu)|\operatorname{div}u^\varepsilon|^2)\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_0^t\int_{\mathbb{T}^3}\Big(\frac{1}{\varepsilon^3}|2\nabla\sqrt{\eta^\varepsilon}+\varepsilon^2\sqrt{\eta^\varepsilon}\nabla\Phi^\varepsilon|^2\Big)\,\mathrm{d}x\,\mathrm{d}s\\
&\leqslant \mathcal{H}^\varepsilon(0) + C\int_0^t\mathcal{H}^\varepsilon(s)\,\mathrm{d}s + C\varepsilon.
\end{aligned}
\tag{3.23}
$$

Using the initial conditions (1.7) and the inequality (3.6), we have that $\mathcal{H}^\varepsilon(0) \leqslant C\varepsilon$, since

$$\begin{aligned}\int_{\mathbb{T}^3} \varrho_0^\varepsilon |u_0^\varepsilon - u_0|^2 \,\mathrm{d}x &\leqslant 2\int_{\mathbb{T}^3} |\sqrt{\varrho_0^\varepsilon} u_0^\varepsilon - \sqrt{\varrho^*} u_0|^2 \,\mathrm{d}x + 2\int_{\mathbb{T}^3} |(\sqrt{\varrho^*} - \sqrt{\varrho_0^\varepsilon}) u_0|^2 \,\mathrm{d}x \\ &\leqslant 2\int_{\mathbb{T}^3} |\sqrt{\varrho_0^\varepsilon} u_0^\varepsilon - \sqrt{\varrho^*} u_0|^2 \,\mathrm{d}x + C\int_{\mathbb{T}^3} |\sqrt{\varrho^*} - \sqrt{\varrho_0^\varepsilon}|^2 \,\mathrm{d}x \leqslant C\varepsilon.\end{aligned}$$

Thus the proof of Lemma 3.3 is completed due to the Gronwall inequality. □

We are now in a position to prove Theorem 1.1. Using the Lemma 3.3 and the Hölder inequality, we have that

$$\begin{aligned}\|\sqrt{\varrho^\varepsilon} u^\varepsilon - \sqrt{\varrho^*} u^0\|_{L^2(\mathbb{T}^3)}^2 &\leqslant 2\|\sqrt{\varrho^\varepsilon}(u^\varepsilon - u^0)\|_{L^2(\mathbb{T}^3)}^2 + 2\|(\sqrt{\varrho^*} - \sqrt{\varrho^\varepsilon}) u^0\|_{L^2(\mathbb{T}^3)}^2 \\ &\leqslant C\varepsilon + C\|\sqrt{\varrho^*} - \sqrt{\varrho^\varepsilon}\|_{L^2(\mathbb{T}^3)}^2 \leqslant C\varepsilon\end{aligned}$$

for any $t \in [0, T]$. Noticing that

$$\frac{\sigma_{\varrho^\varepsilon} + \sigma_{\eta^\varepsilon}}{\varepsilon^2} \leqslant C\int_0^t \int_{\mathbb{T}^3} \Big[\frac{1}{\varepsilon^2} H_1(\varrho^\varepsilon) + \frac{1}{\varepsilon^2} H_2(\eta^\varepsilon)\Big] \,\mathrm{d}x \,\mathrm{d}s,$$

we conclude that the inequality (1.17) holds. Using the Hölder inequality and that fact $1 < 2\gamma/(\gamma+1) < \gamma$, as $\varepsilon \to 0$,

$$\begin{aligned}&\|\varrho^\varepsilon u^\varepsilon - \varrho^* u^0\|_{L^{2\gamma/(\gamma+1)}(\mathbb{T}^3)}^2 \\ &\quad\leqslant 2\|\varrho^\varepsilon(u^\varepsilon - u^0)\|_{L^{2\gamma/(\gamma+1)}(\mathbb{T}^3)}^2 + 2\|(\varrho^\varepsilon - \varrho^*) u^0\|_{L^{2\gamma/(\gamma+1)}(\mathbb{T}^3)}^2 \\ &\quad\leqslant 2\|\sqrt{\varrho^\varepsilon}\|_{L^{2\gamma}(\mathbb{T}^3)}^2 \|\sqrt{\varrho^\varepsilon}(u^\varepsilon - u^0)\|_{L^2(\mathbb{T}^3)}^2 \\ &\qquad + 2\|\varrho^\varepsilon - \varrho^*\|_{L^\lambda(\mathbb{T}^3)}^2 \|u^0\|_{L^{2\lambda\gamma/(\lambda\gamma+\lambda-2\gamma)}(\mathbb{T}^3)}^2 \\ &\quad\leqslant C\varepsilon + C\varepsilon^2 \leqslant C\varepsilon.\end{aligned}$$

We note that $0 < \lambda\gamma + \lambda - 2\gamma < 2\lambda\gamma$ for the definition of λ in Lemma 3.2. So we conclude that (1.18) holds.

Moreover, from (3.23), we have

$$\mu \int_0^t \int_{\mathbb{T}^3} |\nabla u^\varepsilon - \nabla u^0|^2 \,\mathrm{d}x \,\mathrm{d}s \leqslant C\varepsilon, \tag{3.24}$$

which implies that the inequality (1.19) holds. Thus the proof of Theorem 1.1 is finished. □

References

[1] *J. Ballew, K. Trivisa*: Suitable weak solutions and low stratification singular limit for a fluid particle interaction model. Q. Appl. Math. *70* (2012), 469–494. zbl MR doi

[2] *J. Ballew, K. Trivisa*: Weakly dissipative solutions and weak-strong uniqueness for the Navier-Stokes-Smoluchowski system. Nonlinear Anal., Theory Methods Appl., Ser. A *91* (2013), 1–19. zbl MR doi

[3] *C. Baranger, L. Boudin, P.-E. Jabin, S. Mancini*: A modeling of biospray for the upper airways. ESAIM, Proc. *14* (2005), 41–47. zbl MR doi

[4] *H. Beirão da Veiga*: Singular limits in compressible fluid dynamics. Arch. Ration. Mech. Anal. *128* (1994), 313–327. zbl MR doi

[5] *S. Berres, R. Bürger, K. H. Karlsen, E. M. Tory*: Strongly degenerate parabolic-hyperbolic systems modeling polydisperse sedimentation with compression. SIAM J. Appl. Math. *64* (2003), 41–80. zbl MR doi

[6] *J. A. Carrillo, T. Goudon*: Stability and asymptotic analysis of a fluid-particle interaction model. Commun. Partial Differ. Equations *31* (2006), 1349–1379. zbl MR doi

[7] *J. A. Carrillo, T. Karper, K. Trivisa*: On the dynamics of a fluid-particle interaction model: The bubbling regime. Nonlinear Anal., Theory Methods Appl., Ser. A *74* (2011), 2778–2801. zbl MR doi

[8] *J.-Y. Chemin, N. Masmoudi*: About lifespan of regular solutions of equations related to viscoelastic fluids. SIAM J. Math. Anal. *33* (2001), 84–112. zbl MR doi

[9] *Y. Chen, S. Ding, W. Wang*: Global existence and time-decay estimates of solutions to the compressible Navier-Stokes-Smoluchowski equations. Discrete Contin. Dyn. Syst. *36* (2016), 5287–5307. zbl MR doi

[10] *Z.-M. Chen, X. Zhai*: Global large solutions and incompressible limit for the compressible Navier-Stokes equations. J. Math. Fluid Mech. *21* (2019), Article ID 26, 23 pages. zbl MR doi

[11] *P. Constantin, N. Masmoudi*: Global well-posedness for a Smoluchowski equation coupled with Navier-Stokes equations in 2D. Commun. Math. Phys. *278* (2008), 179–191. zbl MR doi

[12] *R. Danchin, P. B. Mucha*: Compressible Navier-Stokes system: Large solutions and incompressible limit. Adv. Math. *320* (2017), 904–925. zbl MR doi

[13] *D. Donatelli, E. Feireisl, A. Novotný*: On incompressible limits for the Navier-Stokes system on unbounded domains under slip boundary conditions. Discrete Contin. Dyn. Syst., Ser. B *13* (2010), 783–798. zbl MR doi

[14] *S. Evje, H. Wen, C. Zhu*: On global solutions to the viscous liquid-gas model with unconstrained transition to single-phase flow. Math. Models Methods Appl. Sci. *27* (2017), 323–346. zbl MR doi

[15] *E. Feireisl, M. Petcu*: Stability of strong solutions for a model of incompressible two-phase flow under thermal fluctuations. J. Differ. Equation *267* (2019), 1836–1858. zbl MR doi

[16] *L. Hsiao, Q. Ju, F. Li*: The incompressible limits of compressible Navier-Stokes equations in the whole space with general initial data. Chin. Ann. Math., Ser. B *30* (2009), 17–26. zbl MR doi

[17] *B. Huang, J. Huang, H. Wen*: Low Mach number limit of the compressible Navier-Stokes-Smoluchowski equations in multi-dimensions. J. Math. Phys. *60* (2019), Article ID 061501, 20 pages. zbl MR doi

[18] *F. Huang, D. Wang, D. Yuan*: Nonlinear stability and existence of vortex sheets for inviscid liquid-gas two-phase flow. Discrete Contin. Dyn. Syst. *39* (2019), 3535–3575. zbl MR doi

[19] *S. Klainerman, A. Majda*: Compressible and incompressible fluids. Commun. Pure Appl. Math. *35* (1982), 629–651. zbl MR doi

[20] *C.-K. Lin*: On the incompressible limit of the compressible Navier-Stokes equations. Commun. Partial Differ. Equations *20* (1995), 677–707. zbl MR doi

[21] *P.-L. Lions*: Mathematical Topics in Fluid Mechanics. Vol. 2: Compressible Models. Oxford Lecture Series in Mathematics and Its Applications 10. Clarendon Press, Oxford, 1998. zbl MR
[22] *P.-L. Lions, N. Masmoudi*: Incompressible limit for a viscous compressible fluid. J. Math. Pures Appl., IX. Sér *77* (1998), 585–627. zbl MR doi
[23] *N. Masmoudi*: Incompressible, inviscid limit of the compressible Navier-Stokes system. Ann. Inst. Henri Poincaré, Anal. Non linéaire *18* (2001), 199–224. zbl MR doi
[24] *Y. Ou*: Incompressible limits of the Navier-Stokes equations for all time. J. Differ. Equations *247* (2009), 3295–3314. zbl MR doi
[25] *N. Vauchelet, E. Zatorska*: Incompressible limit of the Navier-Stokes model with a growth term. Nonlinear Anal., Theory Methods Appl., Ser. A *163* (2017), 34–59. zbl MR doi
[26] *I. Vinkovic, C. Aguirre, S. Simoëns, M. Gorokhovski*: Large eddy simulation of droplet dispersion for inhomogeneous turbulent wall flow. Int. J. Multiphase Flow *32* (2006), 344–364. zbl doi
[27] *F. A. Williams*: Spray combustion and atomization. Phys. Fluids *1* (1958), 541–545. zbl doi
[28] *F. A. Williams*: Combustion Theory. CRC Press, Boca Raton, 1985. doi

Authors' addresses: *Hongli Wang*, School of Physics and Electronics, North China University of Water Resources and Electric Power, Zhengzhou 450045, Henan Province, P. R. China, e-mail: wanghongli@ncwu.edu.cn, *Jianwei Yang* (corresponding author), School of Mathematics and Statistics, North China University of Water Resources and Electric Power, Zhengzhou 450045, Henan Province, P. R. China, e-mail: yangjianwei@ncwu.edu.cn.