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ENGEL BCI-ALGEBRAS: AN APPLICATION OF LEFT AND RIGHT COMMUTATORS

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Abstract. We introduce Engel elements in a BCI-algebra by using left and right normed commutators, and some properties of these elements are studied. The notion of n-Engel BCI-algebra as a natural generalization of commutative BCI-algebras is introduced, and we discuss Engel BCI-algebra, which is defined by left and right normed commutators. In particular, we prove that any nilpotent BCI-algebra of type 2 is an Engel BCI-algebra, but solvable BCI-algebras are not Engel, generally. Also, it is proved that 1-Engel BCI-algebras are exactly the commutative BCI-algebras.

Keywords: (left and right) Engel element; commutator; Engel BCI-algebra

MSC 2020: 03G25, 06F35

1. Introduction and preliminaries

BCK-algebras and BCI-algebras are two important classes of logical algebras introduced by Iséki in 1966 which have been extensively investigated by several researchers (see [6]). From then, mathematicians have studied and developed many concepts in these algebraic structures. For instance, Lei and Xi showed that each p-semisimple BCI-algebra can be converted into an abelian group and, conversely, that each abelian group can be converted into a BCI-algebra (see [8]). Some properties of these structures were presented in [3], [5], [7], [8]. The first author introduced pseudo-commutator of two elements in a BCK-algebra (see [9]). After that, the present authors used this notion to define a solvable BCK-algebra and considered solvable BCK-algebras using commutators (see [10]). Then we gave a new definition for solvability, nilpotency, centralizer and pseudo-center in a BCI-algebra and considered their properties (see [11], [12], [13]).

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In this paper, we present a definition for the notion of Engel element in BCI-algebras based on commutators. We define also the notion of Engel set of a subset of a BCI-algebra, give several characterizations of it, and prove that the class of commutative BCI-algebras and 1-Engel BCI-algebras are equal. We also illustrate these notions by some examples. One of the most important concepts in the study of groups is the notion of nilpotency. Engel groups are certain generalized nilpotent groups that have received considerable attention in recent years (see [6], [7]). It is known that there exists a one-to-one correspondence between each p-semisimple BCI-algebra and an abelian group (see [9]). Since a group is 1-Engel if and only if it is abelian (see [7]), p-semisimple BCI-algebras have connections with 1-Engel groups. This is one of the main motivations for defining the Engel BCI-algebras. However, it is shown in group theory that the inverse of a right Engel element is left one. But it is still an open problem whether every right Engel element of a group is a left Engel element (see [1], [2]). We show that the answer to this question in BCI-algebras is negative. Due to the close relationship between p-semisimple BCI-algebras and abelian groups, perhaps this will be an incentive for finding a negative answer to this question in group theory. We use the notion of Engel BCI-algebra to develop other new concepts such as solvability and nilpotency of type 2, in BCK- and BCI-algebras, and to discuss further properties of these concepts. We can also investigate the variety and some subvarieties of this specific type of BCI-algebras.

By a BCI-algebra, we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms for all $x, y, z \in X$ (see [5], [7], [6]):

(BCI1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCI2) $(x * (x * y)) * y = 0$,
(BCI3) $x * x = 0$,
(BCI4) $x * y = y * x = 0$ implies $x = y$.

Recall that given a BCI-algebra X, the BCI-ordering $\leqslant$ on X is defined by $x \leqslant y$ if and only if $x * y = 0$ for any $x, y \in X$. If in a BCI-algebra $(X, *, 0)$ the condition $0 * x = 0$ for all $x \in X$ holds, then it is a BCK-algebra. A BCI-algebra X has the following properties for all $x, y, z \in X$:

(BCI5) $x * 0 = x$,
(BCI6) $(x * y) * z = (x * z) * y$,
(BCI7) $x \leqslant y$ implies $x * z \leqslant y * z$ and $z * y \leqslant z * x$,
(BCI8) $0 * (x * y) = (0 * x) * (0 * y)$,
(BCI9) $x * (x * (x * y)) = x * y$.

An element $x \in X$ is called positive if $0 \leqslant x$, i.e. $0 * x = 0$. The set of all positive elements of X is said to be the BCK-part of X. We say that an element x of X is minimal if $y \leqslant x$ (i.e. $y * x = 0$) implies $x = y$ for any $y \in X$. The set of all minimal elements of X is said to be the p-semisimple part of X. An element x is called maximal if $x \leqslant y$ implies $x = y$, for any $y \in X$. If there exists $n \in N$ such that $0 * x^n = 0$, then x is called nilpotent, where $0 * x^n = (\dots((0 * x) * x) * \dots) * x$ and x occurs n times. An ideal I of X is a subset of X such that

(i) $0 \in I$,

(ii) $x, y * x \in I$ imply $y \in I$ for any $x, y \in X$.

A subalgebra Y of X is a nonempty subset of X such that Y is closed under the BCI-operation $*$ on X. If Y is both an ideal and a subalgebra of X, we call it a *closed ideal* of X. If I is an ideal of a BCI-algebra X, then the relation θ defined by $(x, y) \in \theta$ if and only if $x * y \in I$ and $y * x \in I$ is a *congruence relation* on X. We usually write C_x for $\{y \in X\colon (x, y) \in \theta\}$. Moreover, $C_0 = I$ is a closed ideal. Assume that $X/I = \{C_x\colon x \in X\}$. Then $(X/I, *, C_0)$ is a BCI-algebra, where $C_x * C_y = C_{x*y}$ for all $x, y \in X$. Also $C_x \leqslant C_y$ if and only if $x \leqslant y$. Let $(X, *, 0)$ and $(Y, \cdot, 0)$ be two BCI-algebras. Then a map $f\colon X \longrightarrow Y$ is called a *homomorphism* if $f(x * y) = f(x) \cdot f(y)$ for all $x, y \in X$. A BCI-algebra X is called a *commutative* BCI-algebra if $x \leqslant y$ implies $x = x \wedge y$, where $x \wedge y = y * (y * x)$. In any commutative BCI-algebra X, for all $x, y \in X$ we have $(y \wedge x) * (x \wedge y) = 0 * (x * y)$. A BCI-algebra X is called *associative* if $x * (y * z) = (x * y) * z$ for all $x, y, z \in X$. A BCI-algebra X is called *p-semisimple* if $0 * (0 * x) = x$ for all $x \in X$. In any p-semisimple BCI-algebra X, the following statements hold for all $x, y \in X$ (see [5]):

(PSBCI1) $x * (0 * y) = y * (0 * x)$,

(PSBCI2) $0 * (y * x) = x * y$,

(PSBCI3) $x * (x * y) = y$.

From now on, $(X, *, 0)$ or simply X is a BCI-algebra, unless otherwise specified.

Definition 1.1 ([11]). (i) Let $x_1, x_2, \dots, x_n$ be elements of X. Then the element $((x_1 \wedge x_2) * (x_2 \wedge x_1)) * (0 * (x_1 * x_2))$ of X is called a *pseudo-commutator* of x_1 and x_2 of weight 2 and denoted by $[x_1, x_2]$.

$$[x_1, x_2] = ((x_2 \wedge x_1) * (x_1 \wedge x_2)) * (0 * (x_1 * x_2)).$$

If $x_1 \wedge x_2 = [x_1, x_2] * (x_2 \wedge x_1)$ or $x_2 \wedge x_1 = [x_1, x_2] * (x_1 \wedge x_2)$, then $[x_1, x_2]$ is called the commutator of x_1 and x_2. In general, the element $[x_1, x_2, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n]$ is a *pseudo-commutator of weight* $n \geqslant 2$, where by convention $[x_1] = x_1$.

(ii) For nonempty subsets A and B of X, the commutator of A and B is

$$[A, B] = \{[a_{i1}, b_{j_1}] * [a_{i2}, b_{j_2}] * \dots * [a_{in}, b_{j_n}]\colon\ a_{ik} \in A,\ b_{j_l} \in B,\ n \in \mathbb{N}\}.$$

So, the commutator $[A, B]$ is the set of all finite $*$-products of commutators of kind $[a, b]$ with $a \in A$ and $b \in B$. When $A = B = X$, $[X, X]$ is denoted by X'.

Theorem 1.2 ([10], [4]). *Let X be a BCK-algebra and $x, y \in X$. Then*
(i) $[x, y] \leqslant x$ *and* $[x, y] \leqslant y$,
(ii) *if* $y \leqslant x$, *then* $[x, y] = 0$,
(iii) $[x, y] = y$ *if and only if* $y = 0$, *so* $y \neq 0$ *if and only if* $[x, y] < y$.

Theorem 1.3 ([11]). *X is commutative if and only if $X' = \{0\}$.*

Definition 1.4 ([12]). The set $\{x \in X \colon [x, y] = [y, x] = 0$ for all $y \in X\}$ is called the *pseudo-center* of X and is denoted by $Z(X)$.

Definition 1.5 ([13]). We define $C^1(X) = [X, X], \ldots, C^k(X) = [C^{k-1}(X), X]$ and $C_1(X) = [X, X], \ldots, C_k(X) = [C_{k-1}(X), C_{k-1}(X)]$. X is said to be *nilpotent of type* 2 (respectively, *solvable*) if there exists $n \in \mathbb{N}$ such that $C^n(X) = \{0\}$ (respectively, $C_n(X) = \{0\}$). The least such n is called the nilpotency class (respectively, derived length) of X.

Theorem 1.6 ([13]).
(i) *Suppose that X is nilpotent of type 2. Then the nilpotency class of X is less than or equal to n if and only if $[[x_1, x_2, \ldots, x_{n-1}], x_n] = 0$ for all $x_i \in X$.*
(ii) *There are proper inclusions of classes:*

$$\begin{aligned}\{\textit{commutative BCI-algebras}\} &\subsetneq \{\textit{nilpotent BCI-algebras of type } 2\}\\ &\subsetneq \{\textit{solvable BCI-algebras}\}\end{aligned}$$

(iii) *Any finite BCI-algebra is solvable.*

2. Engel elements in BCI-algebras

Suppose that x and y are elements of X. For the pair (x, y) and a non-negative integer n we define inductively the *n-Engel left normed commutator* $[x, {}_n y]$ as follows:

$$[x, {}_0 y] = x, \ \ldots, \ [x, {}_n y] = [[x, {}_{n-1} y], y].$$

Also the *n-Engel right normed commutator* $[{}_n x, y]$ of the pair (x, y) is defined by induction as follows:

$$[{}_0 x, y] = y, \ \ldots, \ [{}_n x, y] = [x, [{}_{n-1} x, y]].$$

Especially, $[x, {}_1 y] = [{}_1 x, y] = [x, y] = ((y \wedge x) * (x \wedge y)) * (0 * (x * y))$.

Definition 2.1. For a positive integer k, an element x of X is called a *right k-Engel element* of X whenever $[x, {}_k y] = 0$ for all $y \in X$. An element x of X is called a *right Engel element* if it is right k-Engel for some non-negative integer k. We denote by $R(X)$ and $R_k(X)$ the set of right Engel elements and right k-Engel elements, respectively. So

$$R_k(X) = \{x \in X \colon\ [x, {}_k y] = 0 \text{ for all } y \in X\}$$

and

$$R(X) = \bigcup_{k \in N} R_k(X).$$

Notice that the variable element y appears on the right of the bracket and that if n can be chosen independently of y, then x is a right n-Engel element of X. Left Engel elements are defined in a similar way.

Definition 2.2. For a positive integer k, an element x of X is called a *left k-Engel element* of X whenever $[y, {}_k x] = 0$ for all $y \in X$. Also, x is said to be a *left Engel element* of X if it is left k-Engel for some non-negative integer k. We denote by $L(X)$ and $L_k(X)$ the set of left Engel elements and left k-Engel elements, respectively. So

$$L_k(X) = \{x \in X : [y, {}_k x] = 0 \text{ for all } y \in X\}$$

and

$$L(X) = \bigcup_{k \in N} L_k(X).$$

In Definition 2.2, the variable y is on the left of bracket. Also, since $[x, 0] = [0, x] = 0$, for every $x \in X$, $0 \in R(X) \cap L(X)$.

Definition 2.3. An element x of X that is both the left and right Engel element is said to be an *Engel element.* The set of all Engel elements of X is denoted by $\mathrm{En}(X)$. Obviously, 0 is an Engel element in any BCI-algebra.

E x a m p l e 2.4. Let $X = \{0, a, b, c, d\}$ be a BCI-algebra in which $*$ is defined by the following table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	a	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

The pseudo-commutators of elements of X are given by the following table:

$[\cdot,\cdot]$	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	0	0	0
d	0	0	0	0	0

By routine calculations, we obtain $[a, {}_2d] = [[a,d],d] = [a,d] = a$, $[a, {}_3d] = [[a, {}_2d], d] = [a,d] = a$, $[b, {}_2d] = [[b,d],d] = [b,d] = b$ and $[b, {}_3d] = [[b, {}_2d], d] = [b,d] = b$. For every positive integer n, we have $[a, {}_nd] = a$, $[b, {}_nd] = b$ and hence $a, b \notin R(X)$. Therefore $R(X) = \{0, c, d\}$ and $L(X) = \{0, a, b, c\}$. Thus, $\mathrm{En}(X) = \{0, c\}$.

Lemma 2.5. *Let $x, y \in X$. Then for each $n \in \mathbb{N}$ the following assertions hold:*
(i) $[x * y, x] = 0$ *and* $[x, x * y] = 0$.
(ii) $[x, {}_ny]$ *and* $[{}_nx, y]$ *are positive elements of X.*
(iii) $[{}_2x, y] = 0$.
(iv) *If X is a BCK-algebra with $|X| \geqslant 2$, then $[x, {}_ny]$ and $[{}_nx, y]$ are not maximal elements.*

Proof. (i)

$$\begin{aligned}
[x*y, x] &= (((x*y)*((x*y)*x))*(x*(x*(x*y))))*(0*((x*y)*x)) \\
&= (((x*y)*((x*x)*y))*(x*y))*((0*(x*y))*(0*x)) \\
&= (((x*y)*(0*y))*(x*y))*((0*(0*x))*(x*y)) \\
&= (((x*y)*(x*y))*(0*y))*((0*(0*x))*(x*y)) \\
&= (0*(0*y))*((0*(0*x))*(x*y)) \\
&= (0*((0*(0*x))*(x*y)))*(0*y) \\
&= ((0*x)*(0*(x*y)))*(0*y) \\
&= ((0*x)*((0*x)*(0*y)))*(0*y) = 0, \\
[x, x*y] &= ((x*(x*(x*y)))*((x*y)*((x*y)*x)))*(0*(x*(x*y))) \\
&= ((x*y)*((x*y)*((x*y)*x)))*(0*(x*(x*y))) \\
&= ((x*y)*x)*((0*x)*(0*(x*y))) \\
&= ((x*x)*y)*((0*x)*((0*x)*(0*y))) \\
&= (0*y)*(0*((0*x)*(0*y))*x) \\
&= (0*((0*((0*x)*(0*y)))*x))*y \\
&= (((0*x)*(0*y))*(0*x))*y \\
&= (((0*x)*(0*x))*(0*y))*y = (0*(0*y))*y = 0.
\end{aligned}$$

(ii) Suppose that $x, y \in X$ and $n \in \mathbb{N}$. We will proceed by induction on n. If $n = 1$, then

$$\begin{aligned}
0 * [x, y] &= 0 * (((x * (x * y)) * (y * (y * x))) * (0 * (x * y))) \\
&= ((0 * (x * (x * y))) * (0 * (y * (y * x)))) * (0 * ((0 * (x * y)))) \\
&= (((0 * x) * (0 * (x * y))) * ((0 * y) * (0 * (y * x)))) \\
&\quad * (0 * ((0 * x) * (0 * y))) \\
&= (((0 * x) * ((0 * x) * (0 * y))) * ((0 * y) * ((0 * y) * (0 * x)))) \\
&\quad * (0 * ((0 * x) * (0 * y))) \\
&= (((0 * x) * ((0 * x) * (0 * y))) * ((0 * y) * ((0 * y) * (0 * x)))) \\
&\quad * ((0 * (0 * x)) * (0 * (0 * y))) \\
&= (((0 * x) * ((0 * x) * (0 * y))) * ((0 * (0 * x)) * (0 * (0 * y)))) \\
&\quad * ((0 * y) * ((0 * y) * (0 * x))) \\
&= (((0 * x) * ((0 * (0 * x)) * (0 * (0 * y)))) * ((0 * x) * (0 * y))) \\
&\quad * ((0 * y) * ((0 * y) * (0 * x))) \\
&= ((0 * (0 * (0 * x)) * (0 * (0 * (0 * y))) * x) * ((0 * x) * (0 * y))) \\
&\quad * ((0 * y) * ((0 * y) * (0 * x))) \\
&= (((0 * (0 * (0 * x)) * (0 * (0 * (0 * y)))) * ((0 * x) * (0 * y))) * x) \\
&\quad * ((0 * y) * ((0 * y) * (0 * x))) \\
&= ((((0 * x) * (0 * y)) * ((0 * x) * (0 * y))) * x) \\
&\quad * ((0 * y) * ((0 * y) * (0 * x))) \\
&= (0 * x) * ((0 * y) * ((0 * y) * (0 * x))) \\
&= (0 * ((0 * y) * ((0 * y) * (0 * x)))) * x \\
&= (0 * (((0 * ((0 * y) * (0 * x))))) * y) * x \\
&= (((0 * (0 * (0 * y))) * (0 * (0 * (0 * x)))) * (0 * y)) * x \\
&= (((0 * y) * (0 * x)) * (0 * y)) * x \\
&= (((0 * y) * (0 * y)) * (0 * x)) * x \\
&= (0 * (0 * x)) * x = 0.
\end{aligned}$$

Thus $[x, y]$ is a positive element of X. Assume that the hypothesis is true for n. So, $0 * [x, {}_n y] = 0 = 0 * [{}_n x, y]$. Hence $0 * [x, {}_{n+1} y] = 0 * [[x, {}_n y], y] = [0 * [x, {}_n y], 0 * y] = [0, 0 * y] = 0$ and $0 * [{}_{n+1} x, y] = [0 * x, 0 * [{}_n x, y]] = [0 * x, 0] = 0$. Hence the result holds for any positive integer n.

(iii) We first show that $[x,y]*x = 0*(x \wedge y)$.

$$\begin{aligned}
[x,y]*x &= (((x*(x*y))*(y*(y*x)))*(0*(x*y)))*x \\
&= (((x*(x*y))*(y*(y*x)))*x)*(0*(x*y)) \\
&= (((x*(x*y))*x)*(y*(y*x)))*(0*(x*y)) \\
&= (((x*x)*(x*y))*(y*(y*x)))*(0*(x*y)) \\
&= ((0*(x*y))*(y*(y*x)))*(0*(x*y)) \\
&= ((0*(x*y))*(0*(x*y)))*(y*(y*x)) \\
&= 0*(y*(y*x)).
\end{aligned}$$

Therefore, for every $x, y \in X$, we obtain

$$\begin{aligned}
[{}_2x,y] &= [x,[x,y]] \\
&= ((x*(x*[x,y]))*([x,y]*([x,y]*x)))*(0*(x*[x,y])) \\
&= ((x*(x*[x,y]))*([x,y]*(0*(y*(y*x)))))*((0*x)*(0*[x,y])) \\
&= ((x*(x*[x,y]))*([x,y]*(0*(y*(y*x)))))*((0*x)*0) \\
&= ((x*(x*[x,y]))*(0*x))*([x,y]*(0*(y*(y*x)))) \\
&= ((x*(0*x))*(x*[x,y]))*([x,y]*(0*(y*(y*x)))) \\
&\leqslant ([x,y]*(0*x))*([x,y]*(0*(y*(y*x)))) \\
&\leqslant (0*(y*(y*x)))*(0*x) \\
&= 0*((y*(y*x))*x) = 0*0 = 0.
\end{aligned}$$

(iv) We proceed by induction on n. For $n = 1$, we show that $[x,y]$ is not a maximal element of X. Suppose that there exist x and y in X such that $[x,y]$ is a maximal element of X. Then $[x,y] = y$, as $[x,y] \leqslant y$. Hence $y = 0$. Therefore $[x,y] = [x,0] = 0$. This is a contradiction, because, if 0 is a maximal element of X, then from $0 \leqslant x$ we deduce $X = \{0\}$.

Now assume that for $n \in \mathbb{N}$, $[x, {}_ny]$ and $[{}_nx, y]$ are not maximal elements of X. Since $[x, {}_{n+1}y] = [[x, {}_ny], y] \leqslant [x, {}_ny]$, $[x, {}_{n+1}y]$ is not maximal. Also $[{}_{n+1}x, y] = [x, [{}_nx, y]] \leqslant [{}_nx, y]$. Since $[{}_nx, y]$ is not a maximal element of X, $[{}_{n+1}x, y]$ is not either. Hence the result holds for $n+1$ in both cases. □

Lemma 2.6.

(i) *If X is a p-semisimple BCI-algebra, then any $x \in X$ is an Engel element of X.*

(ii) *Let X be commutative. Then any $x \in X$ is an Engel element of X.*

(iii) *Let X be associative. Then any $x \in X$ is an Engel element of X.*

P r o o f. (i) Let X be a p-semisimple BCI-algebra. Then for all $x, y \in X$

$$\begin{aligned}[x, {}_1y] = [{}_1x, y] = [x, y] &= ((x*(x*y))*(y*(y*x)))*(0*(x*y)) \\ &= (y*x)*(0*(x*y)) = (y*x)*(y*x) = 0.\end{aligned}$$

Hence x is an Engel element of X. (The last two equalities are obtained from the properties of (PBCI2) and (PBCI3)).

(ii) Since $[x, y] = 0$ for all $x, y \in X$ in any commutative BCI-algebra X. Therefore, every element of X is an Engel element.

(iii) Let X be associative. Then for each $x, y \in X$ we obtain

$$\begin{aligned}[x, {}_1y] = [{}_1x, y] = [x, y] & \\ &= ((x*(x*y))*(y*(y*x)))*(0*(x*y)) \\ &= (((x*x)*y)*((y*y)*x))*((0*x)*y) \\ &= ((0*y)*(0*x))*((0*x)*y) = (0*y)*((0*x)*((0*x)*y)) \\ &= (0*y)*(((0*x)*(0*x))*y) = (0*y)*(0*y) = 0.\end{aligned}$$

Therefore every element of X is an Engel element. □

However, it is still an open problem whether every right Engel element of a group is a left Engel element (see [1]), but Example 2.4 shows that the answer to this question in BCI-algebras is negative.

By the following example, it can be shown that the converses to parts (i), (ii), (iii) of Lemma 2.6 are not true.

E x a m p l e 2.7. Let $X = \{0, a, b, c\}$ be a BCI-algebra in which the operation $*$ is defined by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
c	c	a	a	0

By routine calculation, we obtain $[b, c] = a$ and $[x, y] = 0$ for other $x, y \in X$. On the other hand, $[b, {}_2c] = [[b, c], c] = [a, c] = 0$. So $\mathrm{En}(X) = X$. But X is not commutative, because $b = c \wedge b \neq b \wedge c = a$. Also X is not p-semisimple, because $0*(0*a) = 0 \neq a$. BCI-algebra X is not associative, because $b*(b*a) = a \neq (b*b)*a = 0$.

In the following remark, we study the relationship between an Engel element and positive, nilpotent, maximal and minimal elements in a BCI-algebra.

R e m a r k 2.8. By Lemma 2.5, for every $x, y \in X$ and $n \in \mathbb{N}$, the elements $[x, {}_n y]$ and $[{}_n y, x]$ are positive elements and nilpotent elements of order 1, but the converse is not true. For instance, the element d in Example 2.4 is positive (also nilpotent, because $0 * d = 0$), but it is not an Engel element of X and $d \neq [x, {}_n y]$ for all $x, y \in X$. In this example, d is a maximal element but is not Engel, and also c is an Engel element but is not a minimal element of X. In Example 2.7, b is an Engel element, but is neither a maximal nor a minimal element of X.

3. Engel sets, defined by left and right normed commutators

In this section, using the notion of commutator of two elements in a BCI-algebra, we introduce and study Engel sets, which are defined by left and right normed commutators and several properties of these sets are noticed.

Definition 3.1. Let X be any BCI-algebra. The sets $L(X)$ and $R(X)$ of all left (right, respectively) Engel elements are called its left (right, respectively) Engel set. Also for every integer number $k \geqslant 1$ the sets $L_k(X)$ and $R_k(X)$ of all left (right, respectively) k-Engel sets are called k-left (k-right, respectively) Engel set.

E x a m p l e 3.2. From Example 2.7 we obtain $L_1(X) = \{0, a, b\}$, $L_2(X) = L_3(X) = \ldots = X$. Also $R_1(X) = \{0, a, c\}$, $R_2(X) = R_3(X) = \ldots = X$.

Lemma 3.3. *For all $x \in X$, $0 * x \in L_1(X)$.*

P r o o f. Let $x \in X$. We show that for each $y \in X$, $[y, 0 * x] = 0$.

$$\begin{aligned}
[y, 0 * x] &= ((y * (y * (0 * x))) * ((0 * x) * ((0 * x) * y))) * (0 * (y * (0 * x))) \\
&\leqslant ((0 * x) * ((0 * x) * ((0 * x) * y))) * (0 * (y * (0 * x))) \\
&= ((0 * x) * y) * (0 * (y * (0 * x))) \\
&= ((0 * x) * y) * ((0 * y) * (0 * (0 * x))) \\
&= ((0 * x) * y) * ((0 * (0 * (0 * x))) * y) \\
&= ((0 * x) * y) * ((0 * x) * y) = 0.
\end{aligned}$$

Since 0 is a minimal element of X and $[y, 0 * x] \leqslant 0$, we deduce that $[y, 0 * x] = 0$. Hence $0 * x \in L_1(X)$. □

Lemma 3.4.

(i) $L_1(X) \subseteq L_2(X) \subseteq \ldots \subseteq L_n(X) \subseteq \ldots \subseteq L(X)$.

(ii) $R_1(X) \subseteq R_2(X) \subseteq \ldots \subseteq R_n(X) \subseteq \ldots \subseteq R(X)$.

Proof. (i) Let $x \in L_1(X)$. Then for all $y \in X$ we have $[y, x] = 0$. Hence $[y,_2 x] = [[y, x], x] = [0, x] = 0$. Therefore $x \in L_2(X)$ and hence $L_1(X) \subseteq L_2(X)$. Simple induction shows that $L_n(X) \subseteq L_{n+1}(X)$, for any positive integer n.

Obviously, $L_n(X) \subseteq L(X)$ for all $n \in \mathbb{N}$.

(ii) The proof is similar to the proof of (i). □

Corollary 3.5. *For all $x \in X$, we obtain $0 * x \in L(X)$.*

Theorem 3.6. *The following conditions on X are equivalent:*

(i) $L(X) = X$,

(ii) $R(X) = X$,

(iii) $\mathrm{En}(X) = X$.

Proof. (i) ⇒ (ii): Suppose that $X = L(X)$. Then $X = \{x \in X\colon$ for all $y \in X$ exists $n \in \mathbb{N}$ such that $[y, {}_n x] = 0\}$, i.e. for all $x \in X$ and for every $y \in X$ there exists $n \in \mathbb{N}$ such that $[y, {}_n x] = 0$. By substituting x for y and y for x for any $y \in X$, there exists a positive integer n such that $[x, {}_n y] = 0$ for all $x \in X$. Hence $X = R(X)$.

(ii) ⇒ (iii): Let $R(X) = X$ and $x \in X = R(X)$. Then, by the definition of $R(X)$, for all $y \in X$, there exists $n \in \mathbb{N}$ such that $[x, {}_n y] = 0$. So, for each $y \in X$ there exists $n \in \mathbb{N}$ such that $[y, {}_n x] = 0$. Thus $x \in R(X) \cap L(X)$ and hence $X \subseteq E(X)$. Since $E(X) \subseteq X$, we have $X = E(X)$.

(iii) ⇒ (i): If $\mathrm{En}(X) = X$, then $R(X) \cap L(X) = X$. So $R(X) = L(X) = X$. □

Corollary 3.7. *$L_n(X) = X$ if and only if $R_n(X) = X$.*

Proof. $X = L_n(X)$ if and only if $\{x \in X\colon$ for all $y \in X,\ [y, {}_n x] = 0\} = X$ if and only if for all $x, y \in X$, $[y, {}_n x] = 0$ if and only if for any $x, y \in X$, $[x, {}_n y] = 0$ if and only if $X = R_n(X)$. □

Definition 3.8. Let X be any BCI-algebra. We define

$$(X)L = \{x \in X\colon \text{ for all } y \in X \text{ exists } n \in \mathbb{N},\ [{}_n x, y] = 0\}$$

and

$$(X)R = \{x \in X\colon \text{ for all } y \in X \text{ exists } n \in \mathbb{N},\ [{}_n y, x] = 0\}.$$

Also, for every integer number $k \geqslant 1$

$$(X)L_k = \{x \in X\colon\ [{}_k x, y] = 0 \text{ for all } y \in X\}$$

and

$$(X)R_k = \{x \in X\colon\ [{}_k y, x] = 0 \text{ for all } y \in X\}.$$

E x a m p l e 3.9. Let X be the BCI-algebra of Example 2.7. Since $[b,c]=a$ and $[x,y]=0$ for other $x,y\in X$, $[_2b,c]=[b,[b,c]]=[b,a]=0$ by routine calculation, we have $(X)L_1=\{0,a,c\}$, $(X)L_2=(X)L_3=\ldots=X$. Also $(X)R_1=\{0,a,b\}$, $(X)R_2=(X)R_3=\ldots=X$.

Corollary 3.10. $(X)L_2=(X)L_3=\ldots=X$.

P r o o f. Since $[_2x,y]=0$ for every $x,y\in X$, $(X)L_2=\{x\in X\colon [_2x,y]=0$ for all $y\in X\}=X$. Therefore, $(X)L_2=(X)L_3=\ldots=X$. □

Theorem 3.11. *$(X)L_n=X$ if and only if $(X)R_n=X$ and $(X)L=X$ if and only if $(X)R=X$.*

P r o o f. The proof is similar to proof of Theorem 3.6 and Corollary 3.7. □

R e m a r k 3.12. Whereas, for each $y\in X$, $[y,0*x]=0$, $0*x\in(X)R_1$ and $0*x\in(X)R$.

Lemma 3.13. *Let A, B be two nonempty subsets of X and $A\subseteq B$. Then $R(B)\subseteq R(A)$, $L(B)\subseteq L(A)$, $R_n(B)\subseteq R_n(A)$, and $L_n(B)\subseteq L_n(A)$ for all $n\geqslant 0$.*

P r o o f. Let A and B be nonempty subsets of X and $A\subseteq B$. Suppose that $x\in R(B)$. Then for each $y\in B$ there exists $n\in\mathbb{N}$ such that $[x,_ny]=0$. Hence for any $y\in A$ we have $[x,_ny]=0$. Therefore $R(B)\subseteq R(A)$. Proofs of the remaining cases are similar. □

R e m a r k 3.14. The BCK-part of X, namely the set $\{x\in X\colon 0*x=0\}$, is a subset of $L_1(X)$, $L(X)$, $(X)R_1$ and $(X)R$.

4. Engel BCI-algebras

In this section, we introduce the concept of Engel BCI-algebras and study it in detail.

Definition 4.1. A BCI-algebra X in which all elements are Engel is said to be an Engel BCI-algebra. We call the number $\sup\{n\colon [x,_ny]=0,\ (x,y)\in X\times X\}$ the Engel degree of X.

Also a subset S of X is called an Engel set if, for all $x,y\in S$, there is a non-negative integer n such that $[x,_ny]=0$.

Example 4.2.

(1) Let $X = \{0, a, b, c, d\}$ be a BCI-algebra in which the operation $*$ is defined by the following table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

By routine calculation, $\mathrm{En}(X) = \{0, a, b, c\}$. Therefore X is an Engel BCI-algebra.

(2) The BCI-algebra in Example 2.4 is not Engel, because $a, b, d \notin \mathrm{En}(X)$.

Definition 4.3. X is called n-Engel if $X = L_n(X)$, or equivalently, $X = R_n(X)$.

Hence an n-Engel BCI-algebra is a BCI-algebra X such that $[x, {}_n y] = 0$ for all $x, y \in X$. That is, every element is an n-Engel element, i.e. the BCI-algebras in which all elements are left Engel are Engel. For a given positive integer n, a BCI-algebra X is n-Engel if all elements are left n-Engel elements. Obviously a 0-Engel BCI-algebra has order 1. We now consider four special classes of Engel BCI-algebras: p-semisimple, associative, commutative and nilpotent of type 2.

Proposition 4.4. *Both the p-semisimple BCI-algebra and the associative BCI-algebra are Engel BCI-algebras.*

Theorem 4.5. *X is 1-Engel if and only if X is commutative.*

Proof. Let X be an 1-Engel BCI-algebra. Also suppose that $x, y \in X$ and $x \leqslant y$. Then $[x, y] = [{}_1x, y] = [x,{}_1 y] = 0$, for all $x, y \in X$. But $[x, y] = ((x * (x * y)) * (y * (y * x))) * (0 * (x * y)) = ((x * 0) * (y * (y * x))) = x * (y * (y * x)) = 0$. So $x \leqslant (y * (y * x))$. By BCI2, $y * (y * x) \leqslant x$, and thus $y * (y * x) = x \wedge y = x$. Therefore X is commutative.

Conversely, if X is a commutative BCI-algebra, then for all $x, y \in X$, $(y \wedge x) * (x \wedge y) = 0 * (x * y)$, which implies that $[x, y] = ((y \wedge x) * (x \wedge y)) * (0 * (x * y)) = 0$. Therefore X is 1-Engel BCI-algebra. □

Theorem 4.6. *Any nilpotent BCI-algebra of type 2 is an Engel BCI-algebra.*

Proof. Let X be a nilpotent BCI-algebra of type 2 and the nilpotency class of X be n. By Theorem 1.6, $[[x_1, x_2, \ldots, x_{n-1}], x_n] = 0$ for all $x_i \in X$. Hence $[x, {}_n y] = 0$ for all $x, y \in X$. Therefore X is Engel. □

In Example 2.7 we saw that X is an Engel BCI-algebra, but that it is neither a p-semisimple nor a commutative, nor an associative BCI-algebra. In the following example we show that the concepts of Engel, solvable, and nilpotent BCI-algebras of type 2 are different.

Example 4.7. Let $X = \{0, a, b, c, d\}$ and let the operation "$*$" on X be given by the following table:

$*$	0	a	b	c	d
0	0	0	0	c	c
a	a	0	0	c	c
b	b	b	0	c	c
c	c	c	c	0	0
d	d	c	c	a	0

Routine calculation shows that $(X, *, 0)$ is a BCI-algebra and $C^1(X) = C_1(X) = \{0, a\}$, $C_2(X) = \{0\}$ and $C^2(X) = C^3(X) = C^4(X) = \ldots = \{0, a\}$. Thus X is solvable, but X is not nilpotent of type 2. Also, for each $n \in \mathbb{N}$, we obtain $[a, {}_n b] = a$. So, X is not Engel.

Lemma 4.8. *The following conditions on X are equivalent:*

(i) $X' = \{0\}$,

(ii) X *is* 1-*Engel,*

(iv) $X = Z(X)$.

Proof. (i) $\Rightarrow$ (ii): If $X' = \{0\}$, then for each $x, y \in X$ we have $[x, y] = 0$. So X is 1-Engel.

(ii) $\Rightarrow$ (iii): Let X be 1-Engel. Then $[x, y] = 0$ for every $x, y \in X$. Hence $Z(X) = \{x \in X \colon\ [x, y] = [y, x] = 0 \text{ for all } y \in X\} = X$.

(iii) $\Rightarrow$ (i): Let $X = Z(X)$. Then $\{x \in X \colon\ [x, y] = [y, x] = 0 \text{ for all } y \in X\} = X$. Hence $[x, y] = 0$ for every $x, y \in X$. Therefore $X' = \{0\}$. $\square$

It is clear that $R_0(X) = L_0(X) = \{0\}$, $R_1(X) = L_1(X) = Z(X)$. So x is left 1-Engel or right 1-Engel element if and only if x is in the pseudo-center, and X is 1-Engel if and only if X is commutative.

Theorem 4.9. *For every $m \geqslant n$, if X is an n-Engel BCI-algebra, then X is m-Engel.*

Proof. Suppose that X is an n-Engel BCI-algebra. Then $[x, {}_n y] = 0$ for all $x, y \in X$. But for each $m \geqslant n$ and $x, y \in X$, we obtain $[x, {}_m y] = [[x, {}_n y], {}_{m-n} y] = [0, {}_{m-n} y] = 0$. Hence X is m-Engel. $\square$

R e m a r k 4.10. Commutative BCI-algebras of order $n \geqslant 2$ are 1-Engel but are not 0-Engel, because the single 0-Engel BCI-algebra is $\{0\}$. Therefore the converse of Theorem 4.9 is generally not true.

Theorem 4.11. *Let $(X, *, 0)$ and $(Y, \cdot, 0)$ be two BCI-algebras. If $f\colon X \longrightarrow Y$ is a homomorphism from X to Y and X is an Engel BCI-algebra, then $f(X)$ is an Engel BCI-algebra.*

P r o o f. Suppose that $t_1, t_2 \in f(X)$. Then for some $x, y \in X$ we have $t_1 = f(x)$ and $t_2 = f(y)$. Since X is Engel, there exists $n \in \mathbb{N}$ such that $[x, {}_n y] = 0$. Hence $[t_1, {}_n t_2] = [f(x), {}_n f(y)] = f([x, {}_n y]) = f(0) = 0$. So $f(X)$ is Engel. □

In general, the homomorphic image of an Engel BCI-algebra is not a BCI-algebra. Thus, Engel BCI-algebras do not form a variety (for more details see Example 5.8 of [11]).

Proposition 4.12. *Let I be an ideal of X. Then I and X/I are Engel sets if and only if X is an Engel BCI-algebra.*

P r o o f. Let X be an Engel BCI-algebra. Since $I \subseteq X$, and any subset of an Engel BCI-algebra is an Engel set by definition, it follows that, I is an Engel set. Let x, y be elements of X. Then $C_x, C_y \in X/I$. Since X is Engel, there exists $n \in \mathbb{N}$ such that $[x, {}_n y] = 0$. We claim that $[C_x, {}_n C_y] = C_{[x, {}_n y]}$. We will proceed by induction on n. If $n = 1$, then

$$\begin{aligned} C_{[x,y]} &= C_{((y\wedge x)*(x\wedge y))*(0*(x*y))} = (C_{(y\wedge x)} * C_{(x\wedge y)}) * C_{0*(x*y)} \\ &= (C_{x*(x*y)} * C_{y*(y*x)}) * C_{0*(x*y)} \\ &= ((C_x * (C_x * C_y)) * (C_y * (C_y * C_x))) * (C_0 * (C_x * C_y)) = [C_x, C_y]. \end{aligned}$$

Now, let $[C_x, {}_{n-1} C_y] = C_{[x, {}_{n-1} y]}$. Therefore $C_{[x, {}_n y]} = C_{[[x, {}_{n-1} y], y]} = [C_{[x, {}_{n-1} y]}, C_y] = [[C_x, {}_{n-1} C_y], C_y] = [C_x, {}_n C_y]$. Hence the above claim holds for every positive integer n. Thus $[C_x, {}_n C_y] = C_{[x, {}_n y]} = C_0$. So X/I is Engel.

Conversely, let I and X/I be Engel sets. If x is an arbitrary element of X, then $C_x \in X/I$. Therefore, for every $y \in X$, there exists a positive integer n such that $[C_x, {}_n C_y] = C_0$. But $[C_x, {}_n C_y] = C_{[x, {}_n y]}$, so $C_{[x, {}_n y]} = C_0$. Hence $[x, {}_n y] \in I$. Since I is Engel, there exists $m \in N$ such that $[[x, {}_n y], {}_m y] = 0$. Whence $[x, {}_{n+m} y] = 0$. Then X is Engel. □

By Lemma 3.13 if X is Engel, then any subalgebra of X is Engel too. Also, the intersection of any two Engel subalgebras of X is Engel. Since commutative BCI-algebras form a variety, 1-Engel BCI-algebras also form a variety. The quotient of an Engel BCI-algebra is an Engel BCI-algebra.

Proposition 4.13. *The product of two Engel BCI-algebras is again an Engel BCI-algebra.*

P r o o f. Let X and Y be two Engel BCI-algebras of degrees r and s, respectively. For every (x, y) and $(a, b) \in X \times Y$ and $n \in \mathbb{N}$ we show by induction that $[(x, y), {}_n(a, b)] = ([x, {}_n a], [y, {}_n b])$. In the case $n = 1$, we obtain

$$\begin{aligned}
[(x, y), (a, b)] &= \big(((x, y) * ((x, y) * (a, b))) * ((a, b) * ((a, b) * (x, y)))\big) \\
&\quad * \big((0, 0) * ((x, y) * (a, b))\big) \\
&= \big((x * (x * a), y * (y * b)) * (a * (a * x), b * (b * y))\big) \\
&\quad * \big(0 * (x * a), 0 * (y * b)\big) \\
&= \big((x * (x * a) * (a * (a * x))), ((y * (y * b)) * (b * (b * y)))\big) \\
&\quad * \big(0 * (x * a), 0 * (y * b)\big) \\
&= \big((x * (x * a) * (a * (a * x)) * (0 * (x * a))), \\
&\quad ((y * (y * b)) * (b * (b * y))) * (0 * (y * b))\big) \\
&= ([x, a], [y, b]).
\end{aligned}$$

Now inductively assume that $[(x, y),_{n-1} (a, b)] = ([x,_{n-1} a], [y,_{n-1} b])$. Then

$$\begin{aligned}
[(x, y), {}_n(a, b)] &= [[(x, y), {}_{n-1}(a, b)], (a, b)] \\
&= [([x, {}_{n-1}a], [y, {}_{n-1}b]), (a, b)] \\
&= ([[x, {}_{n-1}a], a], [[y, {}_{n-1}b], b]) \\
&= ([x, {}_n a], [y, {}_n b]).
\end{aligned}$$

Since X and Y are Engel of degrees r and s, $[x, {}_r a] = 0$ and $[y, {}_s b] = 0$. By the assumption $n = \max\{r, s\}$, we have $[x, {}_n a] = 0$ and $[y, {}_n b] = 0$. Hence $[(x, y), {}_n(a, b)] = ([x, {}_n a], [y, {}_n b]) = (0, 0)$. Therefore $X \times Y$ is Engel of degree less than or equal to n. □

The above proposition can be generalized to arbitrary families of Engel BCI-algebras.

5. Conclusions

In the present paper, we have introduced the concepts of Engel elements and Engel sets in BCI-algebras and investigated some of their properties. To develop the theory of BCI-algebras, one of the most encouraging ideas could be investigating the Engel degree of BCI-algebras and finding a relation diagram between subclasses of BCI-algebras. For instance, 1-Engel BCI-algebras are strictly commutative BCI-algebras. It is hoped that this work contributes to further studies. Therefore, we think that the results presented in this paper and the forthcoming works can pave the way for a bright future of the theory of the BCI-algebras. The major goal of Engel theory in BCI-algebras can be stated as follows: to find conditions on X which will ensure that $L(X)$ and $R(X)$ are subalgebras, if possible. Some important problems for future work are:

(1) For which BCI-algebra X and for which positive integers n are $L(X)$, $R(X)$ and $R_n(X)$, $L_n(X)$ subalgebras of X. This question can be discussed for various classes of BCI-algebras and small positive integers n.

(2) What is the relationship between left and right Engel elements of X?

(3) When do the right n-Engel elements of X form a subalgebra?

(4) For which positive integers n does there exist a positive integer k such that $R_n(X) \subseteq L_k(X)$ for all BCI-algebras X?

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