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THE PERIODIC PROBLEM FOR THE SECOND ORDER INTEGRO-DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DEVIATION

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Abstract. We study the question of the unique solvability of the periodic type problem for the second order linear integro-differential equation with distributed argument deviation

$$u''(t) = p_0(t)u(t) + \int_0^\omega p(t,s)u(\tau(t,s))\,\mathrm{d}s + q(t),$$

and on the basis of the obtained results by the a priori boundedness principle we prove the new results on the solvability of periodic type problem for the second order nonlinear functional differential equations, which are close to the linear integro-differential equations. The proved results are optimal in some sense.

Keywords: linear integro-differential equation; periodic problem; distributed deviation; solvability

MSC 2020: 34K06, 34K13, 34B15

1. Introduction

On the interval $I = [0, \omega]$, consider the second order linear integro-differential equation

$$u''(t) = p_0(t)u(t) + \int_0^\omega p(t,s)u(\tau(t,s))\,\mathrm{d}s + q(t), \tag{1.1}$$

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and nonlinear functional differential equation

$$u''(t) = F(u)(t) + q(t) \tag{1.2}$$

with the periodic type two point boundary conditions

$$u^{(i-1)}(\omega) - u^{(i-1)}(0) = c_i, \quad i = 1, 2, \tag{1.3}$$

where $c_1, c_2 \in \mathbb{R}$, $p_0, f, q \in L_\infty(I, \mathbb{R})$, $p \in L_\infty(I^2, \mathbb{R})$, $\tau\colon\ I^2 \to I$ is a measurable function, and $F\colon\ C'(I, \mathbb{R}) \to L_\infty(I, \mathbb{R})$ is a continuous operator. (The spaces $C'(I, \mathbb{R})$ and $L_\infty(I, \mathbb{R})$ are defined below.)

We will say that a function $u\colon\ I \to \mathbb{R}$ is a solution of problem (1.2), (1.3) if it is absolutely continuous together with its first derivative, satisfies equation (1.2) almost everywhere on I and satisfies conditions (1.3).

It is well-known that there are many subjects in physics and technology using mathematical methods that depend on the integro-differential equations. For these and for purely theoretical reasons ample interesting literature is devoted to the periodic problem for the integro-differential equations (see, e.g., [4], [6], [3], [10] and the references therein). Our work is motivated by some original results for the functional differential equations with argument deviation (see [1], [2], [9]), and the results of Nieto (see [10]), Erbe and Guo (see [4]), and Kuo-Shou Chiu (see [3]). Nieto in [10] studied linear equation (1.1) on the interval $I = [0, 2\pi]$ when $p_0 \equiv M$, $p(t,s) = Nk(t,s)$ and $\tau(t,s) \equiv s$, i.e. the equation of the form

$$u''(t) = Mu(t) + N[Ku](t) + q(t) \tag{1.4}$$

under conditions (1.3) with $c_1 = c_2 = 0$, where $[Ku](t) = \int_0^{2\pi} k(t,s)u(s)\,\mathrm{d}s$, $k \in L^2(I \times I)$, $M > 0$ and $N \in \mathbb{R}$. In this paper, different sufficient efficient conditions of the unique solvability of linear problem (1.4), (1.3) are established, and one of them, the condition $\|\tau\|_2 < 1$, is optimal, where $\tau(t,s) = \int_0^{2\pi} G(t,r)k(t,s)\,\mathrm{d}r$, and G is the Green's function of the periodic problem for the equation $v''(t) = Mv(t)$. On the basis of these results, the periodic problem for the nonlinear equation $u''(t) = f(t, u(t), [Ku](t))$ is studied even in the case when the kernel k changes its sign. In [6] the authors develop the monotone iterative method based on comparison results, which is applicable for problem (1.4), (1.3) only if K is Volterra operator with nonnegative kernel. A more general case is considered in paper [4], here the operator K is of the form $[Kx] = N[Tx] + N_1[Sx]$, where T is the integral operator of Volterra type and S is the integral operator of Fredholm type with nonnegative kernels. Chiu in [3] investigates the existence of periodic solutions for the systems of integro-differential equations with piecewise alternately retarded and advanced argument of generalized type. In the mentioned paper the author proves interesting results of the solvability and unique solvability, but these results do not take into account the effect of argument deviation.

In this paper we establish the theorems which in some sense complete and generalize the results of the works cited above as well as some other known results. We first describe some classes of unique solvability for linear problem (1.1), (1.3), and on the basis of these results, by the a priori boundedness principle, we prove the existence theorems for nonlinear problem (1.2), (1.3). The conditions we obtain take into account the effect of argument deviation, and in some sense are optimal (see Remarks 2.1, 2.3).

Interesting results follow from our main proposition for such special cases of equations (1.1) and (1.2) as are linear integro-differential equations with distributed delay (see Corollary 2.2), linear differential equations with argument deviation (see Corollary 2.3), or the nonlinear equation

$$u''(t) = f\left(t, u(t), \int_0^\omega V(u)(t,s)u(\tau(t,s))\,\mathrm{d}s\right) + q(t), \tag{1.5}$$

where $f\colon\ I\times\mathbb{R}^2\to\mathbb{R}$ is from the Carathéodory class, and $V\colon\ C'(I,\mathbb{R})\to L_\infty(I^2,\mathbb{R})$ is a continuous bounded operator.

Also our results allow to obtain conditions of unique solvability for a large class of the two point BVP for higher order functional differential equations. Here as an example of such problems we consider nth order linear functional differential equation with argument deviation

$$u^{(n)}(t) = p_1(t)u(\tau(t)) + q(t) \tag{1.6}$$

under the two point boundary conditions

$$u^{(i-1)}(\omega) - u^{(i-1)}(0) = c_i, \quad u^{(j-1)}(0) = c_j, \quad i = 1, 2,\ j = 3, \dots, n, \tag{1.7}$$

if $n \geqslant 3$, where $c_k \in \mathbb{R}$, $k = \overline{1,n}$, $p_1 \in L_\infty(I,\mathbb{R})$, and $\tau\colon\ I\to I$ is a measurable function.

Throughout the paper we use the following notations: $\mathbb{R} =]-\infty,\infty[$, $\mathbb{R}_+ = [0,\infty[$; $C(I;\mathbb{R})$ is the Banach space of continuous functions $u\colon\ I\to\mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)|\colon\ t\in I\}$; $C'(I;\mathbb{R})$ is the Banach space of the functions $u\colon\ I\to\mathbb{R}$ which are continuous together with their first derivatives with the norm $\|u\|_{C'} = \max\{|u(t)| + |u'(t)|\colon\ t\in I\}$; $L(I;\mathbb{R})$ is the Banach space of the Lebesgue integrable functions $p\colon\ I\to\mathbb{R}$ with the norm $\|p\|_L = \int_0^\omega |p(s)|\,\mathrm{d}s$; $L_\infty(I,\mathbb{R})$ is the space of the essentially bounded measurable functions $p\colon\ I\to\mathbb{R}$ with the norm $\|p\|_\infty = \operatorname{esssup}\{|p(t)|\colon$

$t \in I\}$; $L_\infty(I^2,\mathbb{R})$ is the set of such functions $p\colon I^2 \to \mathbb{R}$, that for any fixed $t \in I$, $p(t,\cdot) \in L(I,\mathbb{R})$ and $\int_0^\omega |p(\cdot,s)|\,\mathrm{d}s \in L_\infty(I,\mathbb{R})$. Also for arbitrary $p_0, p_1 \in L_\infty(I,\mathbb{R})$, $p \in L_\infty(I^2,\mathbb{R})$, and measurable $\tau\colon I^2 \to I$ we will use the notations

$$l_0(p_0,p)(t) = |p_0(t)| + \int_0^\omega |p(t,s)|\,\mathrm{d}s,$$
$$l_1(p,\tau) = \frac{2\pi}{\omega}\biggl(\int_0^\omega \biggl(\int_0^\omega |p(\xi,s)||\tau(\xi,s)-\xi|\,\mathrm{d}s\biggr)\,\mathrm{d}\xi\biggr)^{1/2}.$$

Definition 1.1. Let $\sigma \in \{-1,1\}$ and $\tau\colon I^2 \to I$ be measurable function. We will say that the pair of functions (h_0,h), where $h_0 \in L_\infty(I,\mathbb{R}_+)$ and $h \in L_\infty(I^2,\mathbb{R}_+)$ belong to the set P_τ^σ if for arbitrary measurable functions $p_0\colon I \to \mathbb{R}$ and $p\colon I^2 \to \mathbb{R}$ such that

$$0 \leqslant \sigma p_0(t) \leqslant h_0(t), \quad 0 \leqslant \sigma p(t,s) \leqslant h(t,s) \quad \text{for } t,s \in I, \tag{1.8}$$
$$p_0(t) + \int_0^\omega p(t,s)\,\mathrm{d}s \not\equiv 0, \tag{1.9}$$

the homogeneous problem

$$v''(t) = p_0(t)v(t) + \int_0^\omega p(t,s)v(\tau(t,s))\,\mathrm{d}s, \tag{1.10}$$
$$v^{(i-1)}(\omega) - v^{(i-1)}(0) = 0, \quad i=1,2, \tag{1.11}$$

has no nontrivial solution.

2. Statement of the main results

2.1. Linear problem.

Proposition 2.1. *Let* $\sigma \in \{-1,1\}$,

$$h_0 \in L_\infty(I,\mathbb{R}_+), \quad h \in L_\infty(I^2,\mathbb{R}_+), \quad h_0(t) + \int_0^\omega h(t,s)\,\mathrm{d}s \not\equiv 0, \tag{2.1}$$

and for almost all $t \in I$ *the inequality*

$$\frac{1-\sigma}{2} l_0(h_0,h)(t) + l_1(h,\tau) l_0^{1/2}(h_0,h)(t) < \frac{4\pi^2}{\omega^2} \tag{2.2}$$

holds. Then

$$(h_0,h) \in P_\tau^\sigma. \tag{2.3}$$

Theorem 2.1. *Let $\sigma \in \{-1, 1\}$, $\sigma p_0 \in L_\infty(I, \mathbb{R}_+)$, $\sigma p \in L_\infty(I^2, \mathbb{R}_+)$ and condition* (1.9) *be fulfilled. Moreover, let for almost all $t \in I$ the inequality*

$$\frac{1-\sigma}{2} l_0(p_0, p)(t) + l_1(p, \tau) l_0^{1/2}(p_0, p)(t) < \frac{4\pi^2}{\omega^2} \tag{2.4}$$

hold. Then problem (1.1), (1.3) *is uniquely solvable.*

R e m a r k 2.1. Condition (2.4) is optimal in the sense that for the equation

$$u''(t) = p_0(t)u(t) \quad \text{for } t \in [0, 2\pi] \tag{2.5}$$

when $p_0(t) \leqslant 0$, condition (2.4) transforms into the condition $|p_0(t)| < 1$, which is optimal, because if $p_0 \equiv -1$, then $\sin t$ is a nonzero solution of problem (2.5), (1.3) with $c_1 = c_2 = 0$.

From the last theorem it also follows the well known fact that if $p_0(t) \geqslant 0$, then problem (2.5), (1.3) with $c_1 = c_2 = 0$, has only the zero solution.

When in equation (1.1) the coefficients p_0 and p are nonnegative, then $1 - \sigma = 0$, and from Theorem 2.1 it follows:

Corollary 2.1. *Let*

$$p_0 \in L_\infty(I, \mathbb{R}_+), \quad p \in L_\infty(I^2, \mathbb{R}_+), \quad p_0(t) + \int_0^\omega p(t, s)\,\mathrm{d}s \not\equiv 0, \tag{2.6}$$

and for almost all $t \in I$ let the inequality

$$\int_0^\omega \int_0^\omega p(\xi, s)|\tau(\xi, s) - \xi|\,\mathrm{d}s\,\mathrm{d}\xi \Big(p_0(t) + \int_0^\omega p(t, s)\,\mathrm{d}s \Big) < \frac{4\pi^2}{\omega^2}$$

hold. Then problem (1.1), (1.3) *is uniquely solvable.*

Let now $p_0 \equiv 0$, $\tau(t, s) \equiv t - \nu(t, s)$, and

$$0 \leqslant \nu(t, s) \leqslant t \quad \text{for } t, s \in I. \tag{2.7}$$

Then equation (1.1) transforms into the integro-differential equation with distributed delay

$$u''(t) = \int_0^\omega p(t, s)u(t - \nu(t, s))\,\mathrm{d}s + q(t), \tag{2.8}$$

and from Corollary 2.1 it follows:

Corollary 2.2. *Let $p \in L_\infty(I^2, \mathbb{R}_+)$, $\int_0^\omega p(t, s)\,\mathrm{d}s \not\equiv 0$ and for almost all $t \in I$ let the inequality*

$$\int_0^\omega \int_0^\omega p(\xi, s)\nu(\xi, s)\,\mathrm{d}s\,\mathrm{d}\xi \int_0^\omega p(t, s)\,\mathrm{d}s < \frac{4\pi^2}{\omega^2}$$

hold. Then problem (2.8), (1.3) *is uniquely solvable.*

If $p_0 \equiv 0$ and $\tau(t,s) = \tau(t)$ for $t, s \in I$, then equation (1.1) transforms into equation (1.6) with $n = 2$, $p_1(t) = \int_0^\omega p(t,s)\,\mathrm{d}s$, and then from Corollary 2.1 it follows:

Corollary 2.3. *Let* $p_1 \in L_\infty(I, \mathbb{R}_+)$ *be such that for almost all* $t \in I$ *the inequality*

$$p_1(t)\int_0^\omega p_1(s)|\tau(s)-s|\,\mathrm{d}s < \frac{4\pi^2}{\omega^2}$$

holds. Then problem (1.6), (1.3) *when* $n = 2$, *is uniquely solvable.*

Corollary 2.4. *Let* $n \geqslant 3$ *and the function* $p_1 \in L_\infty(I, \mathbb{R}_+)$ *be such that for almost all* $t \in I$ *the inequality*

$$\int_0^\omega \int_0^t p_1(s)|\tau(s)-t|\,\mathrm{d}s\,\mathrm{d}t \int_0^\omega p_1(s)\,\mathrm{d}s \leqslant \frac{4\pi^2((n-3)!)^2}{\omega^{2(n-2)}}$$

holds. Then problem (1.6), (1.7) *is uniquely solvable.*

Remark 2.2. If in Corollaries 2.1–2.3 we assume that $\sigma p_0 = h_0$ and $\sigma p = h$, we get the sufficient efficient conditions which guarantee inclusion (2.3).

2.2. Nonlinear problem. Now we consider the theorems on the solvability of nonlinear problem (1.2), (1.3). First we will introduce here the definitions.

Definition 2.1. We will say that the *operator* F *belongs to Carathéodory's local class* and write $F \in K(C', L_\infty)$ if $F\colon C'(I,\mathbb{R}) \to L_\infty(I,\mathbb{R})$ is continuous operator, and for an arbitrary $r > 0$

$$\sup\{|F(x)(t)|\colon \|x\|_{C'} \leqslant r,\ x \in C'(I,\mathbb{R})\} \in L_\infty(I,\mathbb{R}_+).$$

Definition 2.2. Let $\sigma \in \{-1, 1\}$, inclusion (2.3) hold and the operators $V_0\colon C'(I,\mathbb{R}) \to L_\infty(I,\mathbb{R})$, $V\colon C'(I,\mathbb{R}) \to L_\infty(I^2,\mathbb{R})$ be continuous. Then we will say that $(V_0, V) \in E(h_0, h, P_\tau^\sigma)$ if for all $x \in C'(I,\mathbb{R})$ the conditions

$$0 \leqslant \sigma V_0(x)(t) \leqslant h_0(t), \quad 0 \leqslant \sigma V(x)(t,s) \leqslant h(t,s) \quad \text{for } t, s \in I \tag{2.9}$$

hold, and

$$\inf\{\|L(x,1)\|_L\colon x \in C'(I,\mathbb{R})\} > 0, \tag{2.10}$$

where

$$L(x,y)(t)=V_0(x)(t)y(t)+\int_0^\omega V(x)(t,s)y(\tau(t,s))\,\mathrm{d}s. \tag{2.11}$$

Also throughout the paper we assume that

$$\operatorname{sgn} x=\begin{cases} 1 & \text{if } x\geqslant 0,\\ -1 & \text{if } x<0.\end{cases} \tag{2.12}$$

Then the next theorem is true:

Theorem 2.2. *Let $\sigma\in\{-1,1\}$ and*

$$(V_0+\widetilde{V}_0,V)\in E(h_0,h,P_\tau^\sigma), \tag{2.13}$$

where $\sigma V_0(x)(t)\geqslant 0$, $\sigma\widetilde{V}_0(x)(t)\geqslant 0$ on I for all $x\in C'(I,\mathbb{R})$.

Moreover, let the constant $r_0>0$, the operator $F\in K(C',L_\infty)$ and the function $g_0\in L(I,\mathbb{R}_+)$ be such that the conditions

$$\begin{aligned} g_0(t)&\leqslant\sigma(F(x)(t)-L(x,x)(t))\operatorname{sgn} x(t)\\ &\leqslant|\widetilde{V}_0(x)(t)x(t)|+\eta(t,\|x\|_{C'})\quad\text{for } t\in I,\ \|x\|_{C'}\geqslant r_0,\end{aligned} \tag{2.14}$$

and

$$|c_2|\leqslant\int_0^\omega g_0(s)\,\mathrm{d}s-\left|\int_0^\omega q(s)\,\mathrm{d}s\right| \tag{2.15}$$

hold, where the function $\eta\colon I\times\mathbb{R}_+\to\mathbb{R}_+$ is summable in the first argument, nondecreasing in the second one, and admits to the condition

$$\lim_{\varrho\to\infty}\frac{1}{\varrho}\int_0^\omega\eta(s,\varrho)\,\mathrm{d}s=0. \tag{2.16}$$

Then problem (1.2), (1.3) *has at least one solution.*

Remark 2.3. Inequality (2.15) cannot be replaced by the inequality

$$|c_2|\leqslant\int_0^\omega g_0(s)\,\mathrm{d}s-\left|\int_0^\omega q(s)\,\mathrm{d}s\right|+\varepsilon, \tag{2.17}$$

no matter how small $\varepsilon>0$ would be. Indeed, if $F\equiv 0$, $q(t)\equiv\varepsilon\omega^{-1}$, $g_0\equiv 0$, $c_2=0$, then instead of (2.15), inequality (2.17) holds and all other conditions of Theorem 2.2 are fulfilled with $L(x,y)\equiv 0$, $\eta\equiv 0$, $\widetilde{V}_0\equiv h_0\equiv 1$, $\sigma=1$. Nevertheless, in that case, problem (1.2), (1.3) is not solvable.

R e m a r k 2.4. Let $\sigma \in \{-1,1\}$, $(h_0,h) \in P_\tau^\sigma$,

$$V_0(x)(t) = p_0(t), \quad \widetilde{V}_0(x)(t) = \widetilde{p}_0(t), \tag{2.18}$$

where $\sigma p_0, \sigma\widetilde{p}_0 \in L_\infty(I,\mathbb{R}_+)$, and $V\colon\ C'(I,\mathbb{R}) \to L_\infty(I^2,\mathbb{R})$ be the continuous operator. Then due to Definition 2.2 it is obvious that inclusion (2.13) holds if

$$\begin{aligned} &0 < |p_0(t)|, \quad \sigma(p_0(t) + \widetilde{p}_0(t)) \leqslant h_0(t) && \text{for } t \in I,\\ &0 \leqslant \sigma V(y)(t,s) \leqslant h(t,s) && \text{for } t,s \in I,\ y \in C'(I,\mathbb{R}). \end{aligned} \tag{2.19}$$

Corollary 2.5. *Let $\sigma \in \{-1,1\}$, inclusion (2.3) hold, the functions $g_0, \sigma p_0, \sigma\widetilde{p}_0 \in L_\infty(I,\mathbb{R}_+)$, and the continuous operator $V\colon\ C'(I,\mathbb{R}) \to L_\infty(I^2,\mathbb{R})$ be such that inequalities (2.15), (2.19) are fulfilled. Moreover, let*

$$\begin{aligned} g_0(t) &\leqslant \sigma(f(t,x_1,x_2) - p_0(t)x_1 - x_2)\operatorname{sgn} x_1\\ &\leqslant |\widetilde{p}_0(t)x_1| + \eta(t,|x_1|) \quad \textit{for } t \in I,\ x_1,x_2 \in \mathbb{R}, \end{aligned} \tag{2.20}$$

where $\eta\colon\ I \times \mathbb{R}_+ \to \mathbb{R}_+$ be summable in the first argument, nondecreasing in the second one and admits to condition (2.16). Then problem (1.5), (1.3) has at least one solution.

E x a m p l e 2.1. The integro-differential equation with distributive delay

$$u''(t) = \alpha u(t) + \frac{\beta}{1+\|u\|_{C'}} \int_0^1 |u'(s)|u\Big(t - \frac{t}{1+s}\Big)\,\mathrm{d}s + q(t) \quad \text{for } t \in [0,1], \tag{2.21}$$

where $\alpha,\beta \in \mathbb{R}_+$ and $\alpha \neq 0$, under conditions (1.3) with $\omega = 1$, $c_2 = 0$, has at least one solution if $\int_0^1 q(s)\,\mathrm{d}s = 0$ and $\beta(\alpha+\beta) < 8\pi^2/\ln 2 \approx 113{,}91$. Indeed, in view of Corollary 2.2 the last inequality guarantees the validity of inclusion (2.3), and then all the assumptions of Corollary 2.5 with $\sigma = 1$, $p_0 \equiv h_0 \equiv \alpha$, $V(y)(t) = \beta|y'(t)|/(1+\|y\|_{C'})$, $h \equiv \beta$, $g_0 \equiv \widetilde{p}_0 \equiv q(t,\varrho) \equiv 0$ are fulfilled. The solvability of problem (2.21), (1.3) does not follow from the previously known results.

3. Auxiliary Propositions

Now consider the modification of the well known Wirtinger's inequality (see Theorem 258 in [5]).

Proposition 3.1. *Let $v'' \in L_\infty(I,\mathbb{R})$ and conditions (1.11) hold. Then*

$$\int_0^\omega v'^2(s)\,\mathrm{d}s \leqslant \Big(\frac{\omega}{2\pi}\Big)^2 \int_0^\omega (v''(s))^2\,\mathrm{d}s. \tag{3.1}$$

Lemma 3.1. *Let all the conditions of Proposition* 2.1 *and conditions* (1.8), (1.9) *hold. Then problem* (1.10), (1.11) *has only the trivial solution.*

P r o o f. On the contrary, assume that problem (1.10), (1.11) has nonzero solution v. If $v \equiv c$ (obviously $c \neq 0$), then $v'' \equiv 0$ and in view of (1.10) we get the contradiction with (1.9), i.e. $v \not\equiv \text{const}$. Therefore due to (1.11) the inequality $v' \not\equiv \text{const}$ holds and then there exist $t_*, t^* \in I$ such that $t_* < t^*$ and $v'(t^*) - v'(t_*) \neq 0$. Therefore from (1.10) by (1.8), Schwarz and Cauchy-Schwarz inequalities it follows that

$$\begin{aligned} 0 < |v'(t^*) - v'(t_*)| &\leqslant \int_0^\omega \Big| p_0(\xi)v(\xi) + \int_0^\omega p(\xi,s)v(\tau(\xi,s))\,\mathrm{d}s \Big|\,\mathrm{d}\xi \\ &\leqslant \Big(\int_0^\omega |p_0(\xi)|\,\mathrm{d}\xi \int_0^\omega |p_0(\xi)|v^2(\xi)\,\mathrm{d}\xi \Big)^{1/2} \\ &\quad + \Big(\int_0^\omega\!\!\int_0^\omega |p(\xi,s)|\,\mathrm{d}s\,\mathrm{d}\xi \int_0^\omega\!\!\int_0^\omega |p(\xi,s)|v^2(\tau(\xi,s))\,\mathrm{d}s\,\mathrm{d}\xi \Big)^{1/2} \\ &\leqslant \delta^{1/2} \Big(\int_0^\omega |p_0(\xi)|\,\mathrm{d}\xi + \int_0^\omega\!\!\int_0^\omega |p(\xi,s)|\,\mathrm{d}s\,\mathrm{d}\xi \Big)^{1/2}, \end{aligned}$$

where $\delta = \int_0^\omega \delta_0(\xi)\,\mathrm{d}\xi$, $\delta_0(\xi) = \sigma\big(\int_0^\omega p(\xi,s)v^2(\tau(\xi,s))\,\mathrm{d}s + p_0(\xi)v^2(\xi)\big)$, and then

$$\delta > 0. \tag{3.2}$$

Analogously from (1.10) by (1.8), Schwarz and Cauchy-Schwarz inequalities we get

$$\begin{aligned} \int_0^\omega (v''(\xi))^2\,\mathrm{d}\xi &\leqslant \int_0^\omega \Big(|p_0(\xi)|^{1/2}(|p_0(\xi)|v^2(\xi))^{1/2} \\ &\quad + \Big(\int_0^\omega |p(\xi,s)|\,\mathrm{d}s \Big)^{1/2} \Big(\int_0^\omega |p(\xi,s)|v^2(\tau(\xi,s))\,\mathrm{d}s \Big)^{1/2} \Big)^2 \mathrm{d}\xi \\ &\leqslant \int_0^\omega l_0(h_0,h)(\xi)\delta_0(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{3.3}$$

Now note that in view of (1.10), for δ the representation is true:

$$\delta = \sigma \int_0^\omega v(\xi)v''(\xi)\,\mathrm{d}\xi + \int_0^\omega\!\!\int_0^\omega |p(\xi,s)|v(\tau(\xi,s)) \Big(\int_\xi^{\tau(\xi,s)} v'(\eta)\,\mathrm{d}\eta \Big)\,\mathrm{d}s\,\mathrm{d}\xi. \tag{3.4}$$

Due to (3.1) and (3.3), by integration by parts and boundary conditions (1.11) we find that

$$\begin{aligned} \sigma \int_0^\omega v(\xi)v''(\xi)\,\mathrm{d}\xi = -\sigma \int_0^\omega v'^2(\xi)\,\mathrm{d}\xi &\leqslant \frac{1-\sigma}{2} \int_0^\omega v'^2(\xi)\,\mathrm{d}\xi \\ &\leqslant \frac{1-\sigma}{2} \Big(\frac{\omega}{2\pi}\Big)^2 \int_0^\omega l_0(h_0,h)(\xi)\delta_0(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{3.5}$$

Also by using the Schwarz, Cauchy-Schwarz, (3.1) and (3.3) inequalities we have:

$$
\begin{aligned}
(3.6)\quad & \int_0^\omega\int_0^\omega |p(\xi,s)|v(\tau(\xi,s))\biggl(\int_\xi^{\tau(\xi,s)} v'(\eta)\,\mathrm{d}\eta\biggr)\,\mathrm{d}s\,\mathrm{d}\xi \\
&\leqslant \int_0^\omega\int_0^\omega |p(\xi,s)v(\tau(\xi,s))||\tau(\xi,s)-\xi|^{1/2}\,\mathrm{d}s\,\mathrm{d}\xi\biggl(\int_0^\omega v'^2(\eta)\,\mathrm{d}\eta\biggr)^{1/2} \\
&\leqslant \frac{\omega}{2\pi}\biggl(\int_0^\omega\int_0^\omega |p(\xi,s)|v^2(\tau(\xi,s))\,\mathrm{d}s\,\mathrm{d}\xi\int_0^\omega\int_0^\omega |p(\xi,s)||\tau(\xi,s)-\xi|\,\mathrm{d}s\,\mathrm{d}\xi\biggr)^{1/2} \\
&\quad\times\biggl(\int_0^\omega (v''(\eta))^2\,\mathrm{d}\eta\biggr)^{1/2} \\
&\leqslant \Bigl(\frac{\omega}{2\pi}\Bigr)^2\biggl(\delta\int_0^\omega l_0(h_0,h)(\xi)\delta_0(\xi)\,\mathrm{d}\xi\biggr)^{1/2} l_1(p,\tau).
\end{aligned}
$$

Therefore from (3.2) and (3.4) by estimates (3.5), (3.6), we get

$$
\begin{aligned}
(3.7)\qquad 0<\delta\leqslant \frac{\omega^2}{4\pi^2}\biggl(&\frac{1-\sigma}{2}\int_0^\omega l_0(h_0,h)(\xi)\delta_0(\xi)\,\mathrm{d}\xi \\
&+\biggl(\delta\int_0^\omega l_0(h_0,h)(\xi)\delta_0(\xi)\,\mathrm{d}\xi\biggr)^{1/2} l_1(h,\tau)\biggr).
\end{aligned}
$$

Let now $M_0=\|h_0\|_\infty$, $M_1=\bigl\|\int_0^\omega h(t,s)\,\mathrm{d}s\bigr\|_\infty$, $M=M_0+M_1$ and $N=\frac{1}{2}(1-\sigma)M+l_1(h,\tau)M^{1/2}$. Then due to condition (2.2) either

$$
l_0(h_0,h)(t)<M \quad\text{and}\quad N=\frac{4\pi^2}{\omega^2},\quad\text{or}\quad N<\frac{4\pi^2}{\omega^2}.
$$

Therefore if $N=4\pi^2/\omega^2$ ($N<4\pi^2/\omega^2$), then from (3.7) we get $\delta<N\delta\omega^2/4\pi^2=\delta$ ($\delta\leqslant N\delta\omega^2/4\pi^2<\delta$). Thus, in both cases we get the contradiction $\delta<\delta$. Therefore our assumption is invalid and v is the trivial solution of problem (1.10), (1.11). □

Lemma 3.2. *Let $\sigma\in\{-1,1\}$, $\tau\colon I^2\to I$ be measurable functions and $(V_1,V)\in E(h_0,h,P_\tau^\sigma)$. Then there exists such positive number ϱ_0 that for an arbitrary $x\in C'(I,\mathbb{R})$ and $q\in L_\infty(I,\mathbb{R})$, any solution u of the equation*

$$
(3.8)\qquad u''(t)=V_1(x)(t)u(t)+\int_0^\omega V(x)(t,s)u(\tau(t,s))\,\mathrm{d}s+q(t)
$$

under boundary conditions (1.3), *admits the estimate*

$$
(3.9)\qquad \|u\|_{C'}\leqslant\varrho_0(\mu(u)+|c_1|+|c_2|+\|q\|_L)\quad\text{if } \mu(u)=\min\{|u(t)|\colon t\in I\}.
$$

To prove this lemma, we need Lemma 3.3, which can be proved analogously as Lemma 1.1 of [7], in which $v_0, v_{0k} \in C(I,\mathbb{R})$, $k \in \mathbb{N}$.

Lemma 3.3. *Let $y, y_k \in L(I,\mathbb{R})$, $v_0, v_{0k} \in L_\infty(I,\mathbb{R})$, $k \in \mathbb{N}$,*

$$\lim_{k\to\infty} \|v_{0k} - v_0\|_\infty = 0, \quad \limsup_{k\to\infty} \|y_k\|_L < \infty,$$

and $\lim\limits_{k\to\infty} \int_0^t y_k(s)\,\mathrm{d}s = \int_0^t y(s)\,\mathrm{d}s$ uniformly on I. Then

$$\lim_{k\to\infty} \int_0^t y_k(s)v_{0k}(s)\,\mathrm{d}s = \int_0^t y(s)v_0(s)\,\mathrm{d}s \quad \text{uniformly on } t \in I.$$

P r o o f. Assume that Lemma 3.2 is not true. Then for an arbitrary natural k there exist operators

$$(V_{0k}, V_k) \in E(h_0, h, P_\tau^\sigma), \tag{3.10}$$

functions $x_k \in C'(I,\mathbb{R})$, $q_k \in L_\infty(I,\mathbb{R})$ and the numbers $c_{1k}, c_{2k} \in \mathbb{R}$ such that the problem

$$\begin{aligned} &u_k''(t) = V_{0k}(x_k)(t)u_k(t) + \int_0^\omega V_k(x_k)(t,s)u_k(\tau(t,s))\,\mathrm{d}s + q_k(t),\\ &u_k^{(i-1)}(\omega) - u_k^{(i-1)}(0) = c_{ik}, \quad i = 1,2, \end{aligned}$$

has such a solution u_k that $\|u_k\|_{C'} \geqslant k(\mu(u_k) + |c_{1k}| + |c_{2k}| + \|q_k\|_L)$. Then if we suppose that $v_k(t) = u_k(t)/\|u_k\|_{C'}$, $q_{0k}(t) = q_k(t)/\|u_k\|_{C'}$, we get

$$\|v_k\|_{C'} = 1, \quad \mu(v_k) + \|q_{0k}\|_L + \sum_{i=1}^2 |v_k^{(i-1)}(\omega) - v_k^{(i-1)}(0)| \leqslant \frac{1}{k}, \tag{3.11}$$

and almost everywhere on I the equality

$$v_k''(t) = V_{0k}(x_k)(t)v_k(t) + \int_0^\omega V_k(x_k)(t,s)v_k(\tau(t,s))\,\mathrm{d}s + q_{0k}(t) \tag{3.12}$$

holds. Therefore according to (3.10) and (3.11) we have

$$|v_k''(t)| \leqslant h_0(t) + \int_0^\omega h(t,s)\,\mathrm{d}s + |q_{0k}(t)| \quad \text{for } t \in I. \tag{3.13}$$

According to (3.11) and (3.13), the sequences $(v_k)_{k=1}^{\infty}$ and $(v_k')_{k=1}^{\infty}$ are uniformly bounded and equicontinuous on I. By the Arzelà-Ascoli lemma, without loss of generality it can be assumed that these sequences are uniformly convergent on I. Suppose $v = \lim\limits_{k\to\infty} v_k$ and $v \in C'(I,\mathbb{R})$. Also due to (3.11), conditions (1.11) hold, and

$$\lim_{k\to\infty} \|v_k - v\|_{C'} = 0, \quad \|v\|_{C'} = 1, \tag{3.14}$$

$$\mu(v) = 0. \tag{3.15}$$

Set $P_{0k}(t) = \int_0^t V_{0k}(x_k)(s)\,\mathrm{d}s$, $P_k(t,s) = \int_0^s V_k(x_k)(t,\xi)\,\mathrm{d}\xi$, then from inclusion (3.10) we get

$$\begin{aligned} P_{0k}(0) = 0, \quad 0 \leqslant \sigma(P_{0k}(t_2) - P_{0k}(t_1)) &\leqslant \int_{t_1}^{t_2} h_0(s)\,\mathrm{d}s, \\ P_k(t,0) = 0, \quad 0 \leqslant \sigma(P_k(t,s_2) - P_k(t,s_1)) &\leqslant \int_{s_1}^{s_2} h(t,s)\,\mathrm{d}s, \end{aligned} \tag{3.16}$$

for $0 \leqslant t_1 \leqslant t_2 \leqslant \omega$, $0 \leqslant s_1 \leqslant s_2 \leqslant \omega$, $t \in I$, and then the sequence $(P_{0k}(t))_{k=1}^{\infty}$, and for an arbitrary fixed $t_0 \in I$ sequence $(P_k(t_0,s))_{k=1}^{\infty}$, are uniformly bounded and equicontinuous on I. Then by the Arzelà-Ascoli lemma, without loss of generality it can be assumed that these sequences uniformly converge. Therefore if we denote the limits of these sequences by $P_0(t)$ and $P(t_0,s)$, we get

$$\lim_{k\to\infty} P_{0k}(t) = P_0(t), \quad \lim_{k\to\infty} P_k(t_0,s) = P(t_0,s), \tag{3.17}$$

uniformly on I, and then from (3.16) it follows that

$$\begin{aligned} 0 \leqslant \sigma(P_0(t_2) - P_0(t_1)) &\leqslant \int_{t_1}^{t_2} h_0(s)\,\mathrm{d}s, \\ 0 \leqslant \sigma(P(t_0,s_2) - P(t_0,s_1)) &\leqslant \int_{s_1}^{s_2} h(t_0,s)\,\mathrm{d}s. \end{aligned}$$

Consequently, the functions P_0 and $P(t,\cdot)$ are absolutely continuous, and there exist functions $p_0, p(t,\cdot) \in L(I,\mathbb{R})$ such that $P_0(t) = \int_0^t p_0(s)\,\mathrm{d}s$, $P(t,s) = \int_0^s p(t,\xi)\,\mathrm{d}\xi$, and inequalities (1.8) hold. Then for an arbitrary fixed $t_0 \in I$ due to (3.14), (3.17) and (1.8), by Lemma 3.3 with $y_k(s) = V_k(x_k)(t_0,s)$, $y(s) = p(t_0,s)$, and $v_{0k}(s) = v_k(\tau(t_0,s)), v_0(s) = v(\tau(t_0,s))$, we get

$$\lim_{k\to\infty} \int_0^{\omega} V_k(x_k)(t_0,s)v_k(\tau(t_0,s))\,\mathrm{d}s = \int_0^{\omega} p(t_0,s)v(\tau(t_0,s))\,\mathrm{d}s \quad \text{for } t_0 \in I. \tag{3.18}$$

Analogously due to (3.14), (3.17) and (1.8), from Lemma 3.3 it follows

$$\lim_{k\to\infty}\int_0^t V_{0k}(x_k)(s)v_k(s)\,\mathrm{d}s=\int_0^t p_0(s)v(s)\,\mathrm{d}s\quad\text{for } t\in I. \tag{3.19}$$

Therefore according to the definition of the set $E(h_0,h,P_\tau^\sigma)$ and conditions (3.10), (3.11), the functions $g_k(t)=\int_0^\omega V_k(x_k)(t,s)v_k(\tau(t,s))\,\mathrm{d}s$ are measurable and the inequality $|g_k(t)|\leqslant\int_0^\omega h(t,s)\,\mathrm{d}s$ holds. Thus, (3.18) by the Lebesgue's bounded convergence theorem implies that the function $g(t)=\int_0^\omega p(t,s)v(\tau(t,s))\,\mathrm{d}s$ is integrable and the equality

$$\lim_{k\to\infty}\int_0^t\int_0^\omega (V_k(x_k)(\xi,s)v_k(\tau(\xi,s))-p(\xi,s)v(\tau(\xi,s)))\,\mathrm{d}s\,\mathrm{d}\xi=0 \tag{3.20}$$

holds on I. Therefore if we integrate equation (3.12) from 0 to t, and pass to the limit as $k\to\infty$, due to conditions (3.11), (3.14), (3.19) and (3.20) we find that v is a solution of problem (1.10), (1.11). Let $p_0(t)+\int_0^\omega p(t,s)\,\mathrm{d}s\equiv 0$, then $v''\equiv 0$ and conditions (1.11), (3.15) yield $v\equiv 0$. If $p_0(t)+\int_0^\omega p(t,s)\,\mathrm{d}s\not\equiv 0$, then conditions (1.8) and the inclusion $(h_0,h)\in P_\tau^\sigma$ implies that $v\equiv 0$. Thus, in both cases we get the contradiction with the second equality of (3.14), which proves our lemma. $\square$

The following definition is the modification of Definition 3 in paper [8].

Definition 3.1. Let the operator L be defined by equality (2.11), and

$$\widetilde{L}(x,y)(t)=L(x,y)(t)+\widetilde{V}_0(x)(t)y(t). \tag{3.21}$$

Then we say that the *pair of the operator* $\widetilde{L}$ *and boundary condition* (1.11) *belongs to the Opial class* O_0^2 if: for arbitrary $p_0\in L_\infty(I,\mathbb{R})$ and $p\in L_\infty(I^2,\mathbb{R})$, for which there exists such sequence $x_k\in C'(I,\mathbb{R})$, $k\in\mathbb{N}$ that for all $y\in C'(I,\mathbb{R})$ the equality

$$\lim_{k\to\infty}\widetilde{L}(x_k,y)(t)=p_0(t)y(t)+\int_0^\omega p(t,s)y(\tau(t,s))\,\mathrm{d}s \tag{3.22}$$

holds on I, problem (1.10), (1.11) has only the zero solution.

Lemma 3.4. *Let inclusion* (2.13) *hold and the operator* $\widetilde{L}(x,y)$ *be defined by equality* (3.21). *Then the pair of the operator* $\widetilde{L}(x,y)$ *and boundary condition* (1.11) *belongs to the Opial class* O_0^2.

P r o o f. From inclusion (2.13) we get that

$$0\leqslant\sigma V_0(x_k)(t)+\sigma\widetilde{V}_0(x_k)(t)\leqslant h_0(t),\quad 0\leqslant\sigma V(x_k)(t,s)\leqslant h(t,s) \tag{3.23}$$

for $t, s \in I$, and

$$\inf\{\|L(x,1)+\widetilde{V}_0(x)\|_L\colon\ x \in C'(I,\mathbb{R})\} > 0. \tag{3.24}$$

But (3.22)–(3.24) results in (1.8) and (1.9), and then due to inclusion (2.3), problem (1.10), (1.11) has only the zero solution. □

Now consider the following modifications of Corollary 1 of paper [8].

Lemma 3.5. *Let the pair of the operator* $\widetilde{L}$ *and conditions* (1.11) *belong to the Opial class* O_0^2, *where* $\widetilde{L}$ *is defined by equality* (3.21), $F \in K(C', L_\infty)$, *and there exist a positive number* ϱ_1 *such that for arbitrary* $\lambda \in (0,1)$ *every solution* u *of the problem*

$$u''(t) = \lambda\widetilde{L}(u,u)(t) + (1-\lambda)(F(u)(t)+q(t)), \tag{3.25}$$

$$u^{(i-1)}(\omega) - u^{(i-1)}(0) = \lambda c_i, \quad i = 1,2, \tag{3.26}$$

admits to the estimate

$$\|u(t)\|_{C'} \leqslant \varrho_1. \tag{3.27}$$

Then problem (1.2), (1.3) *has at least one solution.*

4. Proof of main results

Proof of Proposition 2.1. Follows from Lemma 3.1 and Definition 2.1. □

Proof of Theorem 2.1. In view of the fact that linear problem (1.1), (1.3) has Fredholm's property, the proof immediately follows from Proposition 2.1 with $h(t,s) \equiv \sigma p(t,s)$ and $h_0(t) \equiv \sigma p_0(t)$. □

Proof of Corollary 2.4. By integration by parts, we can rewrite the homogeneous problem corresponding to problem (1.6), (1.7) as (1.10), (1.11) with $p_0 \equiv 0$, $\tau(t,s) \equiv \tau(s)$ and

$$p(t,s) = \begin{cases} \dfrac{(t-s)^{n-3}}{(n-3)!}p_1(s) & \text{for } 0 \leqslant s \leqslant t \leqslant \omega, \\ 0 & \text{for } 0 \leqslant t < s \leqslant \omega. \end{cases}$$

Therefore $\int_0^\omega |p(t,s)|\,\mathrm{d}s \leqslant ((b-a)^{n-3}/(n-3)!)\int_0^t p_1(s)\,\mathrm{d}s$, and from Corollary 2.1 our corollary immediately follows. □

P r o o f of Theorem 2.2. Let $\lambda \in (0,1)$ be an arbitrary fixed number and u be a solution of problem (3.25), (3.26). Then it is obvious that u is also the solution of the equation

$$u''(t) = L(u,u)(t) + \lambda \widetilde{V}_0(u)(t)u(t) + (1-\lambda)(F(u)(t) - L(u,u)(s) + q(t)). \tag{4.1}$$

Also, from inclusion (2.13) and inequality (2.10) it is clear that the function $\delta\colon \mathbb{R}_+ \to \mathbb{R}$ defined by the equality

$$\delta(r) \equiv \inf\{\|L(x,1)\|_L + \|\widetilde{V}_0(x)\|_L\colon \ \|x\|_{C'} \geqslant r, \ x \in C'(I,\mathbb{R})\}$$

is positive and nondecreasing, and therefore there exists $r_1 > r_0$ such that

$$r_1\delta(r_1) > 2|c_2|. \tag{4.2}$$

Now show that

$$\mu(u) = \min\{|u(t)|\colon \ t \in I\} \leqslant r_1. \tag{4.3}$$

Assume on the contrary that $|u(t)| > r_1$ on I. Then $\operatorname{sgn} u(t) = \operatorname{sgn} u(0)$, and by (2.9), (2.14), (2.15), (3.26), (4.1) and (4.2) we get

$$\begin{aligned}
|c_2| &\geqslant \sigma c_2 \lambda \operatorname{sgn} u(0) = \sigma(u'(\omega) - u'(0)) \operatorname{sgn} u(0) = \sigma \int_0^\omega u''(s) \operatorname{sgn} u(s)\,\mathrm{d}s \\
&= \int_0^\omega (|L(u,u)(s)| + \lambda|\widetilde{V}_0(u)(s)u(s)|)\,\mathrm{d}s \\
&\quad + (1-\lambda)\sigma \int_0^\omega (F(u)(s) - L(u,u)(s) + q(s)) \operatorname{sgn} u(s)\,\mathrm{d}s \\
&\geqslant \lambda r_1\delta(r_1) + (1-\lambda)\Big(\int_0^\omega g_0(s)\,\mathrm{d}s + \sigma \operatorname{sgn} u(0) \int_0^\omega q(s)\,\mathrm{d}s\Big) \\
&\geqslant \lambda r_1\delta(r_1) + (1-\lambda)|c_2| > |c_2|.
\end{aligned}$$

The obtained contradiction $|c_2| > |c_2|$ proves that (4.3) holds.

Let now ϱ_0 be a number defined in Lemma 3.2. Then due to condition (2.16) there exists such a constant $\varrho_1 > r_0$ that the inequality

$$\varrho_0\Big(\omega + r_1 + |c_1| + |c_2| + \|q\|_L + \int_0^\omega \eta(s,\varrho)\,\mathrm{d}s\Big) < \varrho \quad \text{for } \varrho \geqslant \varrho_1 \tag{4.4}$$

holds. Assume that $\|u\|_{C'} \geqslant \varrho_1$, and note that in view of nonnegativity of the operator $\sigma\widetilde{V}_0(u)(t)$ we have $\sigma\widetilde{V}_0(u)(t) = |\widetilde{V}_0(u)(t)|$. Therefore on account of (2.12), condition (2.13) and nonnegativity of the function η, we get that u is a solution of the equation

$$u''(t) = L(u,u)(t) + (\lambda + (1-\lambda)\nu(t))\widetilde{V}_0(u)(t)u(t) + \eta_1(t, \|u\|_{C'}), \tag{4.5}$$

where

$$\begin{aligned} \eta_1(t, \|u\|_{C'}) &= \sigma(1-\lambda)(1+\eta(t,\|u\|_{C'}))\nu(t)\operatorname{sgn} u(t) + (1-\lambda)q(t), \\ \nu(t) &= \frac{\sigma(F(u)(t) - L(u,u)(t))\operatorname{sgn} u(t)}{|\widetilde{V}_0(u)(t)u(t)| + \eta(t,\|u\|_{C'}) + 1}, \end{aligned}$$

and due to condition (2.14) the estimations

$$0 \leqslant \nu(t) < 1, \quad |\eta_1(t,\|u\|_{C'})| \leqslant 1 + \eta(t,\|u\|_{C'}) + |q(t)| \tag{4.6}$$

are valid on I. Now note that according to conditions (2.13), (4.6) and the nonnegativity of the operators $\sigma\widetilde{V}_0(u)(t)$ and $\sigma V_0(u)(t)$, the estimation

$$0 \leqslant \sigma(V_0(u)(t) + (\lambda + (1-\lambda)\nu(t))\widetilde{V}_0(u)(t)) \leqslant \sigma(V_0(u)(t) + \widetilde{V}_0(u)(t)) \leqslant h_0(t)$$

is satisfied on I. Consequently, due to inclusion (2.13), for arbitrary $\lambda \in (0,1)$ the inclusion $(V_1, V) \in E(h_0, h, P_\tau^\sigma)$, where $V_1(x)(t) = V_0(x)(t) + (\lambda + (1-\lambda)\nu(t))\widetilde{V}_0(x)(t)$, is valid too. Then from (4.5) by Lemma 3.2 due to inequality (4.6) we get the estimation

$$\|u\|_{C'} \leqslant \varrho_0\biggl(\mu(u) + |c_1| + |c_2| + \int_0^\omega (|q(s)| + \eta(s,\|u\|_{C'}) + 1)\,\mathrm{d}s\biggr),$$

which in view of (4.3) contradicts with inequality (4.4), i.e. our assumption is invalid and estimation (3.27) holds.

On the other hand, from Lemma 3.4 due to inclusion (2.13) it follows that the pair of the operator $\widetilde{L}$ and conditions (1.11) belongs to the Opial class O_0^2, and therefore all the assumptions of Lemma 3.5 are fulfilled, from which the solvability of problem (1.1), (1.3) follows. □

P r o o f of Corollary 2.5. Assume that the operators V_0, $\widetilde{V}_0$ are defined by (2.18). Then due to Remark 2.4 in view of (2.19), inclusion (2.13) holds. Also, from (2.20) the validity of conditions (2.14) follows, with

$$\begin{aligned} F(x)(t) &= f\biggl(t, x(t), \int_0^\omega V(x)(t,s)x(\tau(t,s))\,\mathrm{d}s\biggr), \\ L(x,y)(t) &= p_0(t)y(t) + \int_0^\omega V(x)(t,s)y(\tau(t,s))\,\mathrm{d}s. \end{aligned}$$

Therefore all the assumptions of Theorem 2.2 are fulfilled from which validity of our corollary immediately follows. □

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