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SOME PROPERTIES OF WEAK BANACH-SAKS OPERATORS

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Abstract. We establish necessary and sufficient conditions under which weak Banach-Saks operators are weakly compact (respectively, L-weakly compact; respectively, M-weakly compact). As consequences, we give some interesting characterizations of order continuous norm (respectively, reflexive Banach lattice).

Keywords: weak Banach-Saks operator; weakly compact operator; L-weakly compact operator; M-weakly compact operator; order continuous norm, positive Schur property; reflexive Banach space

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1. Introduction

Recall from [3] that an operator T from a Banach space X into a Banach space Y is called *weak Banach-Saks* if every weakly null sequence (x_n) in X has a subsequence such that $(T(x_{n_k}))$ is Cesàro convergent in the norm of Y, that is, the sequence of arithmetic means $\Big(N^{-1}\sum\limits_{k=1}^{N} T(x_{n_k})\Big)_N$ is convergent for the norm of Y. A nonempty bounded subset A of a Banach lattice E is said to be L-*weakly compact* if for every disjoint sequence (x_n) in the solid hull of A, we have $\lim\limits_{n\to\infty}\|x_n\| = 0$. An operator T from a Banach space X into E is L-weakly compact if $T(B_X)$ is L-weakly compact in E, where B_X denotes the closed unit ball of X. An operator $T\colon E \to X$ from a Banach lattice E into a Banach space X is said to be M-*weakly compact* if for every disjoint sequence (x_n) in B_E we have $\lim\limits_{n\to\infty}\|T(x_n)\| = 0$, where B_E denotes the closed unit ball of E (see [11]).

Firstly, we note that a weak Banach-Saks operator is not necessary weakly compact (respectively, L-weakly compact; respectively, M-weakly compact). In fact, the

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identity operator Id_{l^1} of the Banach lattice l^1 (respectively, Id_{c_0}), is weak Banach-Saks (because l^1 and c_0 has the weak Banach-Saks property) but Id_{l^1} is not weakly compact (respectively, Id_{c_0} is neither L-weakly compact nor M-weakly compact).

The b-weak compactness of weak Banach-Saks operators has been widely studied (see [3]). Let us recall that an operator T from a Banach lattice E into a Banach space Y is said to be b-*weakly compact* if it maps each subset of E which is b-order bounded (i.e. order bounded in the topological bidual E'') into a relatively weakly compact subset in Y. In [9], the M-weak compactness and the L-weak compactness of a weakly compact operator was investigated, and in [4] Aqzzouz et al. characterize Banach lattices on which each Dunford-Pettis operator is M-weakly compact (respectively, L-weakly compact).

The main purpose of this work is to characterize Banach lattices on which each weak Banach-Saks operator is weakly compact (respectively, L-weakly compact; respectively, M-weakly compact).

The article is organized as follows. In the next section, we give all the common notations and definitions of operators and Banach lattice theory that we will need. After, in the main results section, we study the weak compactness of weak Banach-Saks operators in the first subsection, the L-weak compactness of weak Banach-Saks operators in the second subsection, and the M-weak compactness of weak Banach-Saks operators in the last subsection.

2. Preliminaries

The notion of Banach-Saks property and weak Banach-Saks property are introduced in [13]. Note that the last property is introduced in the first time in [8] with the name Banach-Saks-Rosenthal property.

A Banach space X is said to have the *Banach-Saks property* (respectively, weak Banach-Saks property) if every bounded sequence (x_n) (respectively, weakly null sequence) in X has a subsequence (x_{n_k}) which is Cesàro convergent. It is proved that a space with the Banach-Saks property must be reflexive, but not all reflexive spaces have the Banach-Saks property. However, a reflexive space with weak Banach-Saks property is Banach-Saks (see [7] and [12]).

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leqslant |y|$, we have $\|x\| \leqslant \|y\|$. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for every net $(x_\alpha)_\alpha$ such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 in the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. Note that if E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice.

A Banach lattice E is said to have the positive Schur property if every weakly convergent sequence to 0 in E^+ is norm convergent to zero. For example, the Banach space l^1 has the positive Schur property. A Banach lattice E is called a KB-*space* whenever every increasing norm bounded sequence of E^+ is norm convergent. As an example, each reflexive Banach lattice is a KB-space. A subset A of a vector lattice E is called *order bounded*, if A is included in an order interval of E. A linear mapping T from a vector lattice E into a vector lattice F is order bounded if it carries order bounded subsets of E into order bounded subsets of F. We will use the term operator $T\colon E \to F$ between two Banach lattices to mean a bounded linear mapping. Operator T is positive if $T(x) \geqslant 0$ in F whenever $x \geqslant 0$ in E. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T\colon E \to F$ between two Banach lattices is positive, then its adjoint $T'\colon F' \to E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. For terminologies concerning Banach lattice theory and positive operators we refer the reader to the excellent book of Aliprantis-Burkinshaw (see [1]).

3. Main results

3.1. Weak compactness of weak Banach-Saks operators. The following result gives necessary conditions under which each weak Banach-Saks operator is weakly compact.

Theorem 3.1. *Let E be a Banach lattice and X a Banach space. If each weak Banach-Saks operator $T\colon E \to X$ is weakly compact, then one of the following assertions is valid:*

(1) *the norm of E' is order continuous;*
(2) *X is reflexive.*

Proof. We have to show that if the norm of E' is not order continuous, then X is reflexive. Using the Eberlein-Smulian's theorem (see [1], Theorem 3.40) it suffices to show that every sequence (x_n) in the closed unit ball of X has a subsequence which converges weakly to an element of X. For this, we construct an operator $T\colon l^1 \to X$ (for more details about this operator, see the proof of Theorem 2.1 in [6]) such that $T(e_n) = x_n$, where (e_n) is the standard basis of l^1. As the class of weak Banach-Saks operators is a two sided ideal, the operator T is weak Banach-Saks (because E contains a sublattice isomorphic to l^1) and hence by our hypothesis, T is weakly compact. So, (x_n) has a subsequence which converges weakly to an element of X. This ends the proof. □

Remark 3.2. The second necessary condition in Theorem 2.1 is sufficient, but the first one is not sufficient. Indeed, the identity operator Id_{c_0} of the Banach lattice c_0 is weak Banach-Saks and the norm of $(c_0)' = l^1$ is order continuous, but Id_{c_0} is not weakly compact.

Let us recall from Theorem 4.70 of [1] that a Banach lattice E is reflexive if and only if E and E' are both KB-spaces. So, whenever E is a KB-space, we can derive from the above theorem the following characterization.

Theorem 3.3. *Let E be a* KB-*space and let X be a Banach space. The following assertions are equivalent:*

(1) *each operator $T\colon E \to X$ is weakly compact;*
(2) *each weak Banach-Saks operator $T\colon E \to X$ is weakly compact;*
(3) *one of the following assertions holds:*
    (a) *E is reflexive;*
    (b) *X is reflexive.*

As consequence of Theorem 3.3, we derive the following characterizations.

Corollary 3.4. *Let E be a* KB-*space and X a non reflexive Banach space. The following assertions are equivalent:*

(1) *each weak Banach-Saks operator $T\colon E \to X$ is weakly compact;*
(2) *the norm of E' is order continuous.*

Proof. It suffices to note that E is reflexive if and only if E is a KB-space and the norm of E' is order continuous. □

Corollary 3.5. *Let X be a Banach space. The following assertions are equivalent:*

(1) *each operator $T\colon l^1 \to X$ is weakly compact;*
(2) *X is reflexive.*

Proof. It suffices to note that l^1 is a KB-space but not reflexive. □

Whenever E and F are two Banach lattices, we obtain the following characterization.

Theorem 3.6. *Let E and F be two Banach lattices such that E is a* KB-*space. The following assertions are equivalent:*

(1) *each weak Banach-Saks operator* $T\colon E \to F$ *is weakly compact;*
(2) *each positive weak Banach-Saks operator* $T\colon E \to F$ *is weakly compact;*
(3) *one of the following assertions holds:*
    (a) E *is reflexive;*
    (b) F *is reflexive.*

P r o o f. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are straightforward. For (2) $\Rightarrow$ (3) we assume that E is not reflexive and we show that F is reflexive. Since F is a Banach lattice, a solid set $A \subset F$ is relatively weakly compact if and only if every sequence of positive elements of A has a subsequence which is weakly convergent to some element x of A (the Eberlein-Smulian's theorem in a Banach lattice). Hence, it suffices to show that every sequence (x_n) of a positive elements in the closed unit ball of F has a subsequence which converges weakly to an element of F, that is F is reflexive. To this end, we proceed as in the proof of Theorem 3.1. □

R e m a r k 3.7. The assumption "E is a KB-space" is essential in Theorem 3.6. For instance, it follows from Theorem 6.7 of [14] that each positive operator $T\colon c_0 \to (l^\infty)'$ is weakly compact. However, neither c_0 nor $(l^\infty)'$ is reflexive.

On the other hand, we observe that if E is a Banach lattice, the second power of a weak Banach-Saks operator $T\colon E \to E$ is not necessary weakly compact. In fact, the identity operator Id_{l^1} is weak Banach-Saks but its second power $(\mathrm{Id}_{l^1})^2 = \mathrm{Id}_{l^1}$ is not weakly compact.

In the following, we give a necessary and sufficient conditions for which the second power of a weak Banach-Saks operator is always weakly compact. Note that there exists a KB-space such that the norm of its topological dual is not order continuous. In fact, the Banach lattice l^1 is a KB-space, but the norm of $(l^1)' = l^\infty$ is not order continuous.

Theorem 3.8. *Let* E *be a* KB-*space. The following assertions are equivalent:*

(1) *for all positive operators* S *and* T *from* E *into* E *with* $0 \leqslant S \leqslant T$ *and* T *weak Banach-Saks,* S *is weakly compact;*
(2) *each positive weak Banach-Saks operator* $T\colon E \to E$ *is weakly compact;*
(3) *for each positive weak Banach-Saks operator* $T\colon E \to E$*, the second power* T^2 *is weakly compact;*
(4) *the norm of* E' *is order continuous.*

P r o o f. (1) $\Rightarrow$ (2): Let $T\colon E \to E$ be a positive weak Banach-Saks operator. Since $0 \leqslant T \leqslant T$, then by our hypothesis T is weakly compact.

(2) $\Rightarrow$ (3): By our hypothesis, T is weakly compact and hence T^2 is weakly compact.

(3) $\Rightarrow$ (4): By way of contradiction, suppose that the norm of E' is not order continuous. We have to construct a positive weak Banach-Saks operator such that its second power is not weakly compact.

Since the norm of E' is not order continuous, it follows from Theorem 2.4.14 and Proposition 2.3.11 of [11] that E contains a complemented copy of l^1 and that there exists a positive projection $P\colon E \to l^1$.

Consider the operator $T = i \circ P$, where i is the canonical injection of l^1 in E. Clearly the operator T is weak Banach-Saks but it is not weakly compact. Otherwise, the operator $P \circ T \circ i = \mathrm{Id}_{l_1}$ would be weakly compact, and this is impossible. Hence, the operator $T^2 = T$ is not weakly compact.

(4) $\Rightarrow$ (1): Since E is a KB-space and the norm of E' is order continuous, Theorem 4.70 of [1] implies that E is reflexive and hence each operator $T\colon E \to E$ is weakly compact. □

We end this paragraph by proving a necessary condition for which a positive weak Banach-Saks operator is compact. In fact, we have the following result:

Theorem 3.9. *Let E and F be two Banach lattices. If each positive weak Banach-Saks operator $T\colon E \to F$ is compact, then one of the following assertions holds:*

(1) *the norm of E' is order continuous;*
(2) *F is finite dimensional.*

Proof. We suppose that the two statements are not true and we construct a positive weak Banach-Saks operator $T\colon E \to F$ which is not compact.

As the norm of E' is not order continuous, it follows from Theorem 4.69 of [1] that E contains a sublattice which is isomorphic to l^1 and there exists a positive projection P from E onto l^1.

On the other hand, if F is infinite dimensional, then by Lemma 2.3 from [5] there exists a disjoint norm bounded sequence (y_n) of F^+ which does not converge to zero in norm. We consider the positive operator S from l^1 into F defined by $S((\lambda_n)_n) = \sum\limits_{n=1}^{\infty} \lambda_n y_n$ for each $(\lambda_n) \in l^1$. Since $S(e_n) = y_n$ for each n, where (e_n) is the standard basis sequence of l^1, the operator S is not compact.

Now, consider the composed operator $T = S \circ P\colon E \to l^1 \to F$. Note that T is weak Banach-Saks. However, the operator T is not compact. If not, the operator $S = T \circ i$ would be compact, where i is the inclusion operator of l^1 in E. This is a contradiction and so the proof is done. □

Remark 3.10. The second necessary condition in Theorem 3.9 is sufficient, but the first one is not sufficient. In fact, for $p > 1$, the operator $T_p\colon X_p \to c_0$ mentioned in [10] is weak Banach-Saks but not compact. However, the norm of $(X_p)'$ is order continuous.

As consequence of Theorem 3.9, we obtain the following result.

Corollary 3.11. *Let E and F be Banach lattices. If the norm of the topological dual E' of E is not order continuous, then each positive weak Banach-Saks operator from E into F is compact if and only if F is finite-dimensional.*

3.2. L-weak compactness of weak Banach-Saks operators. In the following result, we give necessary conditions under which each positive weak Banach-Saks operator is L-weakly compact:

Theorem 3.12. *Let E and F be two Banach lattices such that the norm of F is order continuous. If each positive weak Banach-Saks operator T from E into F is* L*-weakly compact, then one of the following conditions holds:*

(1) *F is a* KB*-space;*
(2) *E' has the positive Schur property.*

Proof. We suppose that F is not a KB-space. Then Theorem 2.4.12 from [11] implies that F contains a sublattice isomorphic to c_0, and since the norm of F is order continuous, it follows from Corollary 2.4.3 of [11] that there exists a positive projection $P\colon F \to c_0$ and $i\colon c_0 \to F$ the canonical embedding.

Applying Theorem 3.1 from [9], it suffices to show that each disjoint weakly null sequence $(x'_n)_n \subset (E')^+$ is norm convergent to 0.

Let $(x'_n)_n \subset (E')^+$ be a disjoint weakly null sequence. Consider the operator $T\colon E \to c_0$ defined by

$$T(x) = (x'_n(x))_n.$$

Let $S = i \circ T\colon E \to c_0 \to F$. Since $S = i \circ \mathrm{Id}_{c_0} \circ T$, S is a positive weak Banach-Saks operator, hence by our hypothesis it is L-weakly compact.

Theorem 5.64 of [1] implies that $S'\colon F' \to l^1 \to E'$ is M-weakly compact. Hence, $S' \circ P' = T'$ is M-weakly compact. As $T'\colon l^1 \to E'$ satisfies $T'((\lambda_n)_n) = \sum\limits_{n=1}^{\infty} \lambda_n x'_n$, we have $T'(e_n) = x'_n$ for all $n \in \mathbb{N}$, where (e_n) is the standard basis of l^1 and so we conclude that $\|x'_n\| \to 0$. That is, E' has the positive Schur property. $\square$

Now we give a necessary conditions under which each weak Banach-Saks operator is L-weakly compact.

Theorem 3.13. *Let E and F be two Banach lattices. If each positive weak Banach-Saks operator $T\colon E\to F$ is* L-*weakly compact, then one of the following assertions is valid:*

(1) $E=\{0\}$;

(2) *F is finite dimensional;*

(3) *the norms of E' and F are order continuous.*

P r o o f. The proof follows along the lines of the proof of Theorem 3.3 of [4]. We prove separately the two following assertions:

(a) If the norm of E' is not order continuous, then F is finite-dimensional.

(b) If the norm of F is not order continuous, then $E=\{0\}$.

Assume that (a) is false, i.e. the norm of E' is not order continuous and F is infinite dimensional. It follows from Theorem 3.1 of [4] that there exists a disjoint norm bounded sequence (y_n) of F^+ which does not converge in norm to zero. Also, since the norm of E' is not order continuous, it follows from Theorem 2.4.14 and Proposition 2.3.11 of [11] that E contains a sublattice isomorphic to l^1 and there exists a positive projection $P\colon E\to l^1$.

To finish the proof, we have to construct a positive weak Banach-Saks operator which is not L-weakly compact. Consider the operator $S\colon l^1\to F$ defined by

$$S((\lambda_n))=\sum_{n=1}^{\infty}\lambda_n y_n \quad \text{for each } (\lambda_n)\in l^1.$$

The operator S is well defined and it is a positive weak Banach-Saks, because l^1 has the weak Banach-Saks property. But S is not L-weakly compact. Otherwise, since $S(e_n)=y_n$ for all $n\geqslant 1$ where (e_n) is the standard basis of l^1 and (y_n) is a disjoint sequence, (y_n) is norm convergent to zero and this is false.

On the other hand, since the identity operator of the Banach lattice l^1 is weak Banach-Saks, the composed operator $T=S\circ P\colon E\to l^1\to F$ is a positive weak Banach-Saks, because $S\circ P=S\circ \mathrm{Id}_{l^1}\circ P$. However, T is not L-weakly compact. Otherwise, $T\circ i=S$ is L-weakly compact where $i\colon l^1\to E$ is the canonical embedding of l^1 into E, and this is a contradiction.

Now, assume that (b) is false, i.e. the norm of F is not order continuous and $E\neq\{0\}$. Choose $z\in E^+$ such that $\|z\|=1$. Hence, it follows from Theorem 39.3 of [15] that there exists $\varphi\in(E')^+$ such that $\varphi(z)=\|\varphi\|=1$.

On the other hand, since the norm of F is not order continuous, there exists some $y\in F^+$ and there exists a disjoint sequence $(y_n)\subset[0,y]$ which does not converge to zero in norm.

We consider the operator $T\colon E \to F$ defined as follows

$$T(x) = \varphi(x)y \quad \text{for each } x \in E.$$

It is clear that T is positive and compact (because its rank is one) and hence T is a positive weak Banach-Saks operator. But T is not L-weakly compact. In fact, since $\|z\| = 1$ and $T(z) = \varphi(z)y = y$ then, $y \in T(B_E)$. As $(y_n) \subset [0, y]$, we conclude that (y_n) is a disjoint sequence in the solid hull of $T(B_E)$. If T is L-weakly compact, then $\lim\limits_{n\to\infty} \|y_n\| = 0$ and this is a contradiction. □

Remark 3.14. The two necessary conditions (1) and (2) in Theorem 3.13 are sufficient, but the condition (3) is not. In fact, the identity operator of the Banach lattice c_0 is weak Banach-Saks, but it is not L-weakly compact. However, the norms of $(c_0)' = l^1$ and of c_0 are order continuous.

Recall from [2] that an operator T from a Banach lattice E into a Banach space X is said to be b-*weakly compact* if it carries each b-order bounded subset of E (i.e. order bounded in E'') into a relatively weakly compact subset of X.

Our following result gives sufficient conditions under which each weak Banach-Saks operator is L-weakly compact.

Theorem 3.15. *Let E and F be two Banach lattices such that the norm of E is order continuous. Then each weak Banach-Saks operator T from E into F is* L-*weakly compact if one of the following statements is valid:*

(1) $E = \{0\}$;
(2) *F is finite dimensional;*
(3) *E' has an order continuous norm and F has the positive Schur property.*

Proof. (1) Obvious.

(2) Since F is finite dimensional, it follows from Corollary 3.2 of [4] that T is L-weakly compact.

(3) Let $T\colon E \to F$ be a weak Banach-Saks operator. Since F has the positive Schur property, F is a KB-space and so it follows from Theorem 2.1 of [3] that T is b-weakly compact. As E' has order continuous norm, it follows from Theorem 2.14 of [6] that T is weakly compact. Since F has the positive Schur property, Theorem 3.4 of [9] implies that T is L-weakly compact. □

As consequence of Theorem 3.13 and Theorem 3.15, we give the following result.

Corollary 3.16. *Let E be a nonzero Banach lattice with order continuous norm and F be a Banach lattice with the positive Schur property. Then the following statements are equivalent:*

(1) *each weak Banach-Saks operator* $T\colon E \to F$ *is* L-*weakly compact;*
(2) *each positive weak Banach-Saks operator* $T\colon E \to F$ *is* L-*weakly compact;*
(3) *One of the following is valid:*
   (a) E' *has an order continuous norm,*
   (b) F *is finite dimensional.*

3.3. M-weak compactness of weak Banach-Saks operator. Our following result gives necessary conditions under which each weak Banach-Saks operator is M-weakly compact.

Theorem 3.17. *Let* E *be a Banach lattice and* F *a nonzero Banach lattice. If each weak Banach-Saks operator* $T\colon E \to F$ *is* M-*weakly compact, then* E' *has an order continuous norm.*

Proof. Assume by way of contradiction that the norm of E' is not order continuous. To finish the proof, we have to construct a weak Banach-Saks operator $T\colon E \to F$ which is not M-weakly compact. Since the norm of E' is not order continuous, it follows from Theorem 2.4.14 and Proposition 2.3.11 of [11] that E contains a closed sublattice which is isomorphic to l^1 and there exists a positive projection $P\colon E \to l^1$.

On the other hand, as $F \neq \{0\}$, there exists a non-null element $y \in F^+$. Now, we consider the operator $S\colon l^1 \to F$ defined by

$$S((\lambda_n)) = \left(\sum_{n=1}^{\infty} \lambda_n\right) y \quad \text{for each } (\lambda_n) \in l^1.$$

It is clear that S is well defined. Also, S is weak Banach-Saks. Hence the operator

$$T = S \circ P\colon E \to l^1 \to F$$

is weak Banach-Saks but it is not M-weakly compact. In fact, if we denote by (e_n) the canonical basis of $l^1 \subset E$, the sequence (e_n) is disjoint and bounded in E. Moreover we have $T((e_n)) = y$ for each $n \geqslant 1$. Then, $\|T((e_n))\| \nrightarrow 0$ (because $y \neq 0$). So, T is not M-weakly compact and this proves the result. □

Remark 3.18. The necessary condition in Theorem 3.17 is not sufficient. In fact, the identity operator of the Banach lattice c_0 is weak Banach-Saks but is not M-weakly compact. However, the norm of $(c_0)' = l^1$ is order continuous.

In the following result, we give sufficient conditions under which each weak Banach-Saks is M-weakly compact:

Theorem 3.19. *Let E and F be two Banach lattices, $F \neq \{0\}$.*

(1) *If E is a* KB*-space and E' has the positive Schur property, then each weak Banach-Saks operator $T\colon E \to F$ is* M*-weakly compact.*

(2) *If the norms of E and E' are order continuous and F has the positive Schur property, then each order bounded weak Banach-Saks operator $T\colon E \to F$ is* M*-weakly compact.*

Proof. (1) Let $T\colon E \to F$ be a weak Banach-Saks operator. Since E is a KB-space, it follows from Theorem 2.1 of [3] that T is b-weakly compact. As the norms of E and E' are order continuous, Theorem 2.12 of [6] implies that T is weakly compact.

Now, since E' has the positive Schur property, it follows from Theorem 3.3 of [9] that T is M-weakly compact.

(2) Let $T\colon E \to F$ be an order bounded weak Banach-Saks operator. By Theorem 3.15, T is L-weakly compact, since F has the positive Schur property. Therefore, by [1], Theorem 5.67, T is M-weakly compact. □

The following characterization is a consequence of Theorem 3.19 and the proof of Theorem 3.17, since the operator $S\colon l^1 \to F$ is order bounded.

Corollary 3.20. *Let E and F, $F \neq \{0\}$, be two Banach lattices such that the norm of E is order continuous and F has the positive Schur property. Then each order bounded weak Banach-Saks operator $T\colon E \to F$ is* M*-weakly compact if and only if E' has an order continuous norm.*

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