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COMPLETE F-MOMENT CONVERGENCE FOR WEIGHTED SUMS OF WOD ARRAYS WITH STATISTICAL APPLICATIONS

Xi Chen, Xinran Tao and Xuejun Wang

Complete f-moment convergence is much more general than complete convergence and complete moment convergence. In this work, we mainly investigate the complete f-moment convergence for weighted sums of widely orthant dependent (WOD, for short) arrays. A general result on Complete f-moment convergence is obtained under some suitable conditions, which generalizes the corresponding one in the literature. As an application, we establish the complete consistency for the weighted linear estimator in nonparametric regression models. Finally, some simulations are provided to show the numerical performance of theoretical results based on finite samples.

Keywords: widely orthant dependent arrays, weighted sums, complete f-moment convergence, complete convergence, nonparametric regression models, complete consistency

Classification: 60F15, 62G20

1. INTRODUCTION

Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. The convergence problems are important in probability limit theory and mathematical statistics. In recent years, the problem of convergence has been studied by many scholars, such as almost sure convergence, convergence in r-order moments, convergence in probability, convergence in distribution, and so on. Let's introduce the concept of complete convergence. The concept of complete convergence was introduced by Hsu and Robbins [8], which is stated as follows.

Definition 1.1. A sequence $\{X_n, n \geq 1\}$ of random variables converges completely to the constant C, if

$$\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty, \quad \text{for all } \ \varepsilon > 0. \tag{1}$$

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Chow [4] put forward the following concept of the complete moment convergence, which is much stronger than complete convergence. The definition of the complete moment convergence is as follows.

Definition 1.2. Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $r > 0$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be two selected sequences of positive constants. The sequence $\{X_n, n \geq 1\}$ is said to be complete rth moment convergence, if

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|X_n| - \varepsilon\}_+^r < \infty, \quad \text{for all} \quad \varepsilon > 0. \tag{2}$$

Recently, the complete f-moment convergence was proposed by Wu et al. [31], which is the extension of the complete moment convergence. The concept of complete f-moment convergence is as follows.

Definition 1.3. Let $\{X_n, n \geq 1\}$ be a sequence of random variables, $\{c_n, n \geq 1\}$ be a specified sequence of positive constants and $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing continuous function with $f(0) = 0$. Then we say that $\{X_n, n \geq 1\}$ converges f-moment completely, if

$$\sum_{n=1}^{\infty} c_n E f(\{|X_n| - \varepsilon\}_+) < \infty, \quad \text{for all} \quad \varepsilon > 0. \tag{3}$$

It is easy to know that the complete moment convergence implies complete convergence. Furthermore, it is easy to get that complete f-moment convergence implies complete moment convergence if $f(t) = t^r$, and complete convergence if $c_n = 1, n \geq 1$ and $f(t) = t$. Therefore, complete f-moment convergence is more general than complete convergence and complete moment convergence. Recently, Liang and Zhang [13] established the following complete convergence result for weighted sums of negatively associated (NA, for short) arrays.

Theorem 1.4. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of rowwise NA random variables, which are stochastically dominated by the random variable X. Let $\alpha \geq 0$. Assume (i) $\gamma > 0$ and $\beta > \max\{0, \gamma^{-1} - 1\}$; (ii) $\gamma > 1$, $0 \geq \beta > \max\{-1/2, \gamma^{-1} - 1\}$ and $EX_{ni} = 0$. If $E|X|^\gamma < \infty$, then

$$\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} P\left(\max_{1\leq k\leq n} |T_{nk}| > \varepsilon n^\theta\right) < \infty, \quad \text{for all} \quad \varepsilon > 0, \tag{4}$$

where $T_{nk} = \sum_{i=1}^{k} i^\alpha X_{ni}$ and $\theta = \alpha + \beta + 1$.

The main purpose of this paper is to generalize the result of complete convergence in Theorem 1.4 to the case of complete f-moment convergence, and NA array to widely orthant dependent (WOD, for short) array. Now, let us introduce the concept of WOD random variables, which was introduced by Wang et al. [27]. The concept of WOD random variables is presented as follows.

Definition 1.5. The $\{X_n, n \geq 1\}$ are called widely upper orthant dependent (WUOD, for short) if there exists a sequence $g_U(n)$ such that for every $n \geq 1$ and all $x_i \in (-\infty, \infty)$,

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq g_U(n) \prod_{i=1}^{n} P(X_i > x_i); \tag{5}$$

they are called widely lower orthant dependent (WLOD, for short) if there exists a sequence $g_L(n)$ such that for every $n \geq 1$ and all $x_i \in (-\infty, \infty)$,

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq g_L(n) \prod_{i=1}^{n} P(X_i \leq x_i), \tag{6}$$

and they are widely orthant dependent (WOD, for short) if they are both WUOD and WLOD. An array of random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is said to be rowwise WOD if for every $n \geq 1, \{X_{ni}, i \geq 1\}$ is a sequence of WOD random variables.

The concept of WOD random variables was introduced by Wang et al. [27]. Since then, many scholars studied the probability limit properties of WOD random variables and obtained some interesting results; Wang et al. [37] presented some probability inequalities and moment inequalities for WOD random variables, further got the complete convergence for weighted sums of arrays of rowwise WOD random variables and gave some applications; Qiu and Hu [15] investigated the strong limit theorems for weighted sums of WOD random variables; Qiu and Chen [13] established a complete convergence result and a complete moment convergence result for weighted sums of WOD random variables under mild conditions; Shen et al. [20] provided some exponential probability inequalities to get the complete convergence for arrays of rowwise WNOD random variables; Wu et al. [32] investigated complete moment convergence for WOD random variables under some mild conditions; Li et al. [11] established a Bernstein-type inequality for WOD random variables, and obtain the rates of strong convergence for kernel estimators of density and hazard functions under some suitable conditions; He [17]established the strong consistency and complete consistency of the Priestley–Chao estimator in nonparametric regression model with WOD errors under some general conditions and obtained the rates of strong consistency and complete consistency; Lu et al. [14] studied the complete f-moment convergence for WOD random variables and gave some applications; Chen and Sung [3] obtained a Spitzer-type law of large numbers for WOD random variables. Shen and Wu [19] investigated the complete qth moment convergence and provided some sufficient conditions for sums of WOD random variables; Xi et al. [28] presented some convergence properties for partial sums of WOD random variables and gave some applications; Lang et al. [18] investigated the complete convergence for weighted sums of WOD random variables, and so on.

The definition of stochastic domination below will play an important role throughout the paper.

Definition 1.6. An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is said to be stochastically dominated by the random variable X, if there is a positive constant C such that

$$P(|X_{ni}| > x) \leq CP(|X| > x), \quad \text{for all } x \geq 0,\ i \geq 1 \text{ and } n \geq 1. \tag{7}$$

In the case, we write $\{X_{ni}, i \geq 1, n \geq 1\} \prec X$.

The organization of the paper is as follows. Some lemmas are stated in Section 2. Main results and their proofs are provided in Section 3. An application to nonparametric regression models and numerical simulations are shown in Section 4. Throughout this paper, C denotes a positive constant not depending on n, which may be different in various places. Let $I(A)$ be the indicator function of the set A. Denote $x_+ = xI(x \geq 0)$ and $\log x = \ln \max\{x, e\}$.

2. SOME LEMMAS

In this section, we will provide some lemmas which are needed for the proofs of our main results. The first one is a basic property for WOD random variables, which can be found in Wang at el. [27].

Lemma 2.1. Let $\{X_n, n \geq 1\}$ be a sequence of WOD with dominating coefficients $g(n), n \geq 1$, and $\{f_n(\cdot), n \geq 1\}$ be all non-decreasing functions (or non-increasing functions). Then $\{f_n(X_n), n \geq 1\}$ is also a sequence of WOD with dominating coefficients $g(n), n \geq 1$.

The following lemma can be found in Wu et al. [30].

Lemma 2.2. Let $\{Y_i, 1 \leq i \leq n\}$ and $\{Z_i, 1 \leq i \leq n\}$ be two sequences of random variables. Then for any $q > r > 0$, $\varepsilon > 0$ and $a > 0$,

$$E\left(\left|\sum_{i=1}^{n}(Y_i + Z_i)\right| - \varepsilon a\right)_+^r \leq C_r\left(\varepsilon^{-q} + \tfrac{r}{q-r}\right)a^{r-q}E\left(\left|\sum_{i=1}^{n} Y_i\right|^q\right) + C_r E\left(\left|\sum_{i=1}^{n} Z_i\right|^r\right),$$

$$\begin{aligned} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}(Y_i + Z_i)\right| - \varepsilon a\right)_+^r \leq\ & C_r\left(\varepsilon^{-q} + \frac{r}{q-r}\right)a^{r-q}E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} Y_i\right|^q\right) \\ & + C_r E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} Z_i\right|^r\right), \end{aligned}$$

where $C_r = 1$ if $0 < r \leq 1$, or $C_r = 2^{r-1}$ if $r > 1$.

The following lemma is an important property for stochastic domination, which can be found in Adler and Rosalsky [1] and Adler et al. [2].

Lemma 2.3. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of random variables which is stochastically dominated by a random variable X. For any $a > 0$ and $b > 0$, the following two statements hold:

$$\begin{aligned} E|X_{ni}|^a I(|X_{ni}| \leq b) &\leq C_1[E|X|^a I(|X| \leq b) + b^a P(|X| > b)], \\ E|X_{ni}|^a I(|X_{ni}| > b) &\leq C_2 E|X|^a I(|X| > b), \end{aligned}$$

where C_1 and C_2 are positive constants. Thus, $E|X_{ni}|^a \leq CE|X|^a$, where C is a positive constant.

The following lemma is about the important properties for WOD random variables, which can be found in Wang et al. [37].

Lemma 2.4. Let $p > 1$ and $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for each $n \geq 1$. Then there exists positive constants $C_1(p)$ and $C_2(p)$ depending only on p such that

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq [C_1(p) + C_2(p)g(n)] \sum_{i=1}^{n} E|X_i|^p, \quad \text{for } 1 < p \leq 2,$$

and

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leq C_1(p) \sum_{i=1}^{n} E|X_i|^p + C_2(p)g(n) \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}, \quad \text{for } p > 2.$$

Using the method of Theorem 2.3.1 in Stout [23], we can obtain the following lemma from Lemma 2.4.

Lemma 2.5. Let $p > 1$ and $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for each $n \geq 1$.Then there exists positive constants $C_3(p)$ and $C_4(p)$ depending only on p such that

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq [C_3(p)(\log n)^p + C_4(p)g(n)(\log n)^p] \sum_{i=1}^{n} E|X_i|^p, \quad \text{for } 1 < p \leq 2,$$

and

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq C_3(p)(\log n)^p \sum_{i=1}^{n} E|X_i|^p + C_4(p)g(n)(\log n)^p \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2},$$
for $p > 2$.

3. MAIN RESULTS AND AND THEIR PROOFS

Throughout this section, let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise WOD random variables, with $g_{jU}(n)$ and $g_{jL}(n)$ being the dominating coeffients of jth row. Denote $g_j(n) = \max\{g_{jU}(n), g_{jL}(n)\}, j \geq 1$ and $g(n) = \max_{j\geq 1} g_j(n)$. Let X be some random variable, and α, β, γ be some constants. Denote $\theta = \alpha + \beta + 1$, $T_{nk} = \sum_{i=1}^{k} i^\alpha X_{ni}$ and $S_n = \max_{1\leq k\leq n} |T_{nk}| n^{-\theta}$ for $\alpha \geq 0$. Suppose

$$\sum_{n=1}^{\infty} g(n)n^{-\iota} < \infty, \quad \text{for any } \iota > 1. \tag{8}$$

Complete f-moment convergence relies on the selected functions, so we need the following functions f and h. Let

$$f:\mathbb{R}^+\to\mathbb{R}^+$$

be a continuous and increasing function with $f(0)=0$, and let

$$h:\mathbb{R}^+\to\mathbb{R}^+$$

be the inverse function of f. Then $h(f(t))=t, t\geq 0$. For some positive constants ν and δ, the function h satisfies the following condition

$$\int_{f(\delta)}^{\infty} h^{-\nu}(t)\,\mathrm{d}t<\infty. \tag{9}$$

Theorem 3.1. Let the conditions (8) and (9) be fulfilled, $\alpha\geq 0$, $\nu>0$ and $\{X_{ni}, i\geq 1, n\geq 1\}\prec X$. Assume that $\gamma>1$, $\gamma>\nu$, $0\geq\beta>\max\{-1/2,\gamma^{-1}-1\}$ and $EX_{ni}=0$. If $E|X|^\eta<\infty$ for some $\eta>\gamma$, then for any $\varepsilon>0$,

$$\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}Ef(\{|S_n|-\varepsilon\}_+)<\infty. \tag{10}$$

Proof. Note that

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}Ef\left(\{|S_n|-\varepsilon\}_+\right) &= \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}\int_0^\infty P(|S_n|>\varepsilon+h(t))\,\mathrm{d}t\\
&= \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}\int_0^{f(\delta)} P(|S_n|>\varepsilon+h(t))\,\mathrm{d}t\\
&\quad+\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}\int_{f(\delta)}^{\infty} P(|S_n|>\varepsilon+h(t))\,\mathrm{d}t\\
&:= I_1+I_2.
\end{aligned} \tag{11}$$

Thus, we only need to prove $I_1<\infty$ and $I_2<\infty$. Firstly, we will deal with I_2. By Markov's inequality and (9), it can be obtained that

$$\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}\int_{f(\delta)}^{\infty} P(|S_n|>\varepsilon+h(t))\,\mathrm{d}t\\
&\leq \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu}E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^\alpha X_{ni}\right|-\varepsilon n^\theta\right)_+^\nu.
\end{aligned} \tag{12}$$

Note that $\gamma(\beta+1)>1$, and $\frac{\alpha\gamma-\alpha-\beta}{(\gamma-1)\theta}<1$ by $\gamma>1$. We choose q such that $\max\{\frac{1}{\gamma(\beta+1)},\frac{\alpha\gamma-\alpha-\beta}{(\gamma-1)\theta}\}<q<1$. Define

$$X_{ni}^{(1)}=-n^{\theta q}I(i^\alpha X_{ni}<-n^{\theta q})+i^\alpha X_{ni}I(|X_{ni}|\leq n^{\theta q})+n^{\theta q}I(i^\alpha X_{ni}>n^{\theta q});$$

$$X_{ni}^{(2)}=(i^\alpha X_{ni}-n^{\theta q})I(n^{\theta q}<i^\alpha X_{ni}\leq n^{\theta q}+n^\theta)+n^\theta I(i^\alpha X_{ni}>n^\theta+n^{\theta q});$$

$$X_{ni}^{(3)} = (i^\alpha X_{ni} + n^{\theta q})I(-n^\theta - n^{\theta q} \le i^\alpha X_{ni} < -n^{\theta q}) - n^\theta I(i^\alpha X_{ni} < -n^\theta - n^{\theta q});$$
$$X_{ni}^{(4)} = (i^\alpha X_{ni} - n^{\theta q} - n^\theta)I(i^\alpha X_{ni} > n^\theta + n^{\theta q});$$
$$X_{ni}^{(5)} = (i^\alpha X_{ni} + n^{\theta q} + n^\theta)I(i^\alpha X_{ni} < -n^\theta - n^{\theta q}).$$

We can easily see that, $\{X_{ni}^{(1)}, i \ge 1, n \ge 1\}$, $\{X_{ni}^{(2)}, i \ge 1, n \ge 1\}$, $\{X_{ni}^{(3)}, i \ge 1, n \ge 1\}$, $\{X_{ni}^{(4)}, i \ge 1, n \ge 1\}$ and $\{X_{ni}^{(5)}, i \ge 1, n \ge 1\}$ are all arrays of rowwise WOD random variables by Lemma 2.1, and

$$T_{nk} = \sum_{i=1}^{k} \sum_{l=1}^{5} X_{ni}^{(l)}.$$

First, we prove that

$$D_n^{(1)} := n^{-\theta} \max_{1\le k\le n} \left| \sum_{i=1}^{k} EX_{ni}^{(1)} \right| \to 0, \quad n \to \infty.$$

By $\gamma > 1$, we have that

$$\begin{aligned} D_n^{(1)} &\le n^{-\theta} \sum_{i=1}^{n} E|X_{ni}^{(1)}| \le 2n^{-\theta} \sum_{i=1}^{n} E|i^\alpha X_{ni}| I(|i^\alpha X_{ni}| > n^{\theta q}) \\ &\le 2n^{-\theta} \sum_{i=1}^{n} n^{-\theta q(\gamma-1)} E|i^\alpha X_{ni}|^\gamma I(|i^\alpha X_{ni}| > n^{\theta q}) \\ &\le Cn^{\alpha\gamma+1-[(\gamma-1)q+1]\theta} E|X|^\gamma \to 0, \quad n \to \infty. \end{aligned} \tag{13}$$

Define

$$D_n^{(2)} := n^{-\theta} \max_{1\le k\le n} \left| \sum_{i=1}^{k} EX_{ni}^{(2)} \right|.$$

Similar to the proof above, by $\gamma > 1$, it is easy to get that

$$\begin{aligned} D_n^{(2)} &= n^{-\theta} \sum_{i=1}^{n} EX_{ni}^{(2)} = n^{-\theta} \sum_{i=1}^{n} E[(i^\alpha X_{ni} - n^{\theta q})I(n^{\theta q} < i^\alpha X_{ni} \le n^{\theta q} + n^\theta) \\ &\quad + n^\theta I(i^\alpha X_{ni} > n^{\theta q} + n^\theta)] \\ &\le n^{-\theta} \sum_{i=1}^{n} E[|i^\alpha X_{ni}| I(|i^\alpha X_{ni}| > n^{\theta q}) + n^\theta I(|i^\alpha X_{ni}| > n^\theta)] \\ &\le Cn^{\alpha\gamma+1-[(\gamma-1)q+1]\theta} + Cn^{1-\gamma(\beta+1)} \to 0, \quad n \to \infty. \end{aligned} \tag{14}$$

Define

$$D_n^{(4)} := n^{-\theta} \max_{1\le k\le n} \left| \sum_{i=1}^{k} EX_{ni}^{(4)} \right|.$$

We get that

$$\begin{aligned} D_n^{(4)} &\le n^{-\theta} \sum_{i=1}^{n} E|X_{ni}^{(4)}| \le n^{-\theta} \sum_{i=1}^{n} E|i^\alpha X_{ni}| I(|i^\alpha X_{ni}| > n^\theta) \\ &\le C n^{-\theta} \sum_{i=1}^{n} E|i^\alpha X| I(|i^\alpha X| > n^\theta) \\ &\le C n^{1-\gamma(\beta+1)} \to 0, \quad n \to \infty. \end{aligned} \tag{15}$$

Similarly, we can also obtain that

$$D_n^{(3)} := n^{-\theta} \max_{1\le k\le n} \left| \sum_{i=1}^{k} EX_{ni}^{(3)} \right| \to 0, \quad n \to \infty,$$

$$D_n^{(5)} := n^{-\theta} \max_{1\le k\le n} \left| \sum_{i=1}^{k} EX_{ni}^{(5)} \right| \to 0, \quad n \to \infty.$$

By $EX_{ni} = 0$, Lemma 2.2 and C_r inequality, when $\tau > \nu \ge 1$, it follows by (13) that

$$\begin{aligned} I_2 &\le \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} E\left(\max_{1\le k\le n} \left| \sum_{i=1}^{k} \sum_{l=1}^{5} (X_{ni}^{(l)} - EX_{ni}^{(l)}) \right| - \varepsilon n^\theta \right)_+^\nu \\ &\le \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} E\left(\sum_{l=1}^{5} \max_{1\le k\le n} \left| \sum_{i=1}^{k} (X_{ni}^{(l)} - EX_{ni}^{(l)}) \right| - \varepsilon n^\theta \right)_+^\nu \\ &\le C \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} E\left(\max_{1\le k\le n} \left| \sum_{i=1}^{k} \left(X_{ni}^{(1)} - EX_{ni}^{(1)} \right) \right| + \sum_{l=2}^{5} \left| \sum_{i=1}^{n} X_{ni}^{(l)} \right| - \varepsilon n^\theta/2 \right)_+^\nu \\ &\le C \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} E\left(\max_{1\le k\le n} \left| \sum_{i=1}^{k} \left(X_{ni}^{(1)} - EX_{ni}^{(1)} \right) \right| \right. \\ &\quad \left. + \sum_{l=2}^{5} \left| \sum_{i=1}^{n} \left(X_{ni}^{(l)} - EX_{ni}^{(l)} \right) \right| - \varepsilon n^\theta/3 \right)_+^\nu \\ &\le C \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} E\left(\max_{1\le k\le n} \left| \sum_{i=1}^{k} (X_{ni}^{(1)} - EX_{ni}^{(1)}) \right|^\tau \right) \\ &\quad + C \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} E\left| \sum_{i=1}^{n} \left(X_{ni}^{(2)} - EX_{ni}^{(2)} \right) \right|^\tau \end{aligned}$$

$$
\begin{aligned}
&+C\sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2-\theta\tau}E\left|\sum_{i=1}^{n}\left(X_{ni}^{(3)}-EX_{ni}^{(3)}\right)\right|^{\tau}\\
&+C\sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2-\theta\nu}E\left|\sum_{i=1}^{n}\left(X_{ni}^{(4)}-EX_{ni}^{(4)}\right)\right|^{\nu}\\
&+C\sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2-\theta\nu}E\left|\sum_{i=1}^{n}\left(X_{ni}^{(5)}-EX_{ni}^{(5)}\right)\right|^{\nu}\\
:=\ & I_{21}+I_{22}+I_{23}+I_{24}+I_{25}. \qquad (16)
\end{aligned}
$$

When $0<\nu<1$, similarly, we can also get that

$$
\begin{aligned}
I_2 \leq\ & \sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2-\theta\nu}E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}\sum_{l=1}^{5}(X_{ni}^{(l)}-EX_{ni}^{(l)})\right|-\varepsilon n^{\theta}\right)_{+}^{\nu}\\
\leq\ & C\sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2-\theta\nu}E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}\left(X_{ni}^{(1)}-EX_{ni}^{(1)}\right)\right|+\sum_{l=2}^{3}\left|\sum_{i=1}^{n}\left(X_{ni}^{(l)}-EX_{ni}^{(l)}\right)\right|\right.\\
&\left.+\sum_{l=4}^{5}\left|\sum_{i=1}^{n}X_{ni}^{(l)}\right|-\varepsilon n^{\theta}/3\right)_{+}^{\nu}\\
\leq\ & C\sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2-\theta\tau}E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}(X_{ni}^{(1)}-EX_{ni}^{(1)})\right|^{\tau}\right)\\
&+C\sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2-\theta\tau}E\left|\sum_{i=1}^{n}\left(X_{ni}^{(2)}-EX_{ni}^{(2)}\right)\right|^{\tau}\\
&+C\sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2-\theta\tau}E\left|\sum_{i=1}^{n}\left(X_{ni}^{(3)}-EX_{ni}^{(3)}\right)\right|^{\tau}\\
&+C\sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2-\theta\nu}E\left|\sum_{i=1}^{n}X_{ni}^{(4)}\right|^{\nu}+C\sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2-\theta\nu}E\left|\sum_{i=1}^{n}X_{ni}^{(5)}\right|^{\nu}\\
:=\ & I_{21}+I_{22}+I_{23}+I_{24}^{'}+I_{25}^{'}. \qquad (17)
\end{aligned}
$$

Taking $\tau>\max\{2,\gamma,\frac{2\gamma(\beta+1)}{2\beta+1},\frac{2\gamma(\beta+1)}{\theta[2-(2-\gamma)q]-(\alpha\gamma+1)},\frac{\gamma(\beta+1)}{(1-q)\theta},\frac{2\gamma(\beta+1)}{\gamma(\beta+1)-1}\}$, we obtain by Lemma 2.5 and C_r inequality that,

$$
\begin{aligned}
I_{21} =\ & C\sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2-\theta\tau}E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}\left(X_{ni}^{(1)}-EX_{ni}^{(1)}\right)\right|^{\tau}\right)\\
\leq\ & C\sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2-\theta\tau}(\log n)^{\tau}\sum_{i=1}^{n}E\left|\left(X_{ni}^{(1)}-EX_{ni}^{(1)}\right)\right|^{\tau}\\
&+C\sum_{n=1}^{\infty}n^{\gamma(\beta+1)-2-\theta\tau}g(n)(\log n)^{\tau}\left(\sum_{i=1}^{n}E\left(X_{ni}^{(1)}-EX_{ni}^{(1)}\right)^{2}\right)^{\frac{\tau}{2}}
\end{aligned}
$$

$$\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}(\log n)^{\tau}\sum_{i=1}^{n}E\left|X_{ni}^{(1)}\right|^{\tau} \\
&\quad +C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}g(n)(\log n)^{\tau}\left(\sum_{i=1}^{n}E\left(X_{ni}^{(1)}\right)^{2}\right)^{\frac{\tau}{2}} \\
&:= I_{211}+I_{212}. \qquad (18)
\end{aligned}$$

For I_{211}, noting that $E|X|^{\gamma}<\infty$, we have by Lemma 2.3 that

$$\begin{aligned}
I_{211} &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}(\log n)^{\tau}\sum_{i=1}^{n}\left[E|i^{\alpha}X_{ni}|^{\tau}I(|i^{\alpha}X_{ni}|\le n^{\theta q})+n^{\theta q\tau}P(|i^{\alpha}X_{ni}|>n^{\theta q})\right] \\
&\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}(\log n)^{\tau}\sum_{i=1}^{n}\left[E|i^{\alpha}X|^{\tau}I(|i^{\alpha}X|\le n^{\theta q})+n^{\theta q\tau}P(|i^{\alpha}X|>n^{\theta q})\right] \\
&\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-1-(1-q)\theta\tau}(\log n)^{\tau}<\infty.
\end{aligned}$$

For I_{212}, when $\gamma\ge 2$, we have $EX^2<\infty$. Since $\gamma(\beta+1)>1$ and $\beta>-\frac{1}{2}$, we have that $2(\beta+1)>1$. Thus, we have by Lemma 2.3 and (8) that

$$\begin{aligned}
I_{212} &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}g(n)(\log n)^{\tau} \\
&\quad \left(\sum_{i=1}^{n}E\left[|i^{\alpha}X_{ni}|^{2}I(|i^{\alpha}X_{ni}|\le n^{\theta q})+n^{2\theta q}I(|i^{\alpha}X_{ni}|>n^{\theta q})\right]\right)^{\frac{\tau}{2}} \\
&\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-[2(\beta+1)-1]\frac{\tau}{2}}g(n)(\log n)^{\tau}<\infty.
\end{aligned}$$

When $0<\gamma<2$, note that $\frac{2\theta-(\alpha\gamma+1)}{(2-\gamma)\theta}>1>q$ by $\gamma(\beta+1)-1>0$, and thus $\theta[2-(2-\gamma)q]-(\alpha\gamma+1)>0$. We have by Lemma 2.3 and (8) again that

$$\begin{aligned}
I_{212} &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}g(n)(\log n)^{\tau} \\
&\quad \left(\sum_{i=1}^{n}E\left[|i^{\alpha}X_{ni}|^{2}I(|i^{\alpha}X_{ni}|\le n^{\theta q})+n^{2\theta q}I(|i^{\alpha}X_{ni}|>n^{\theta q})\right]\right)^{\frac{\tau}{2}} \\
&\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-[\theta(2-(2-\gamma)q)-(\alpha\gamma+1)]\frac{\tau}{2}}g(n)(\log n)^{\tau}<\infty.
\end{aligned}$$

Therefore, $I_{212}<\infty$, and thus $I_{21}<\infty$.

For I_{22}, by C_r inequality and Lemma 2.4, we have that

$$I_{22} = C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}E\left|\sum_{i=1}^{n}\left(X_{ni}^{(2)}-EX_{ni}^{(2)}\right)\right|^{\tau}$$

$$
\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} g(n)\left(\sum_{i=1}^{n} E\left|X_{ni}^{(2)}\right|^2\right)^{\frac{\tau}{2}} + C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}\sum_{i=1}^{n} E\left|X_{ni}^{(2)}\right|^{\tau} \\
&:= I_{221}+I_{222}. \qquad (19)
\end{aligned}
$$

When $\gamma \ge 2$, we have $EX^2<\infty$ and $2(\beta+1)>1$. Thus, we have by Lemma 2.3 and (8) that

$$
\begin{aligned}
I_{221} &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} g(n)\left(\sum_{i=1}^{n} E\left[|i^{\alpha}X_{ni}|^2 I(|i^{\alpha}X_{ni}|\le 2n^{\theta})+n^{2\theta} I(|i^{\alpha}X_{ni}|>n^{\theta})\right]\right)^{\frac{\tau}{2}} \\
&\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-[2(\beta+1)-1]\frac{\tau}{2}} g(n)<\infty.
\end{aligned}
$$

When $0<\gamma<2$, note that $\gamma(\beta+1)-1>0$. We have by Lemma 2.3 and (8) again that

$$
\begin{aligned}
I_{221} &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} g(n)\left(\sum_{i=1}^{n} E|i^{\alpha}X_{ni}|^2 I(|i^{\alpha}X_{ni}|\le 2n^{\theta})+\sum_{i=1}^{n} n^{2\theta} P(|i^{\alpha}X_{ni}|>n^{\theta})\right)^{\frac{\tau}{2}} \\
&\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-[\gamma(\beta+1)-1]\frac{\tau}{2}} g(n)<\infty.
\end{aligned}
$$

Thus, $I_{221}<\infty$ holds.

For I_{222}, we have

$$
\begin{aligned}
I_{222} &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}\sum_{i=1}^{n}\left[E|i^{\alpha}X_{ni}|^{\tau} I(|i^{\alpha}X_{ni}|\le 2n^{\theta})+n^{\theta\tau}P(|i^{\alpha}X_{ni}|>n^{\theta})\right] \\
&= C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}\sum_{i=1}^{n} E|i^{\alpha}X|^{\tau} I(|i^{\alpha}X|\le 2n^{\theta})+C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2}\sum_{i=1}^{n} P(|i^{\alpha}X|>n^{\theta}) \\
&:= I_{222}^{'}+I_{222}^{''}. \qquad (20)
\end{aligned}
$$

For all $\tau>\gamma$, we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} E|n^{\alpha}X|^{\tau} I(|n^{\alpha}X|\le 2n^{\theta}) &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} E|n^{\alpha}X|^{\gamma} n^{\theta(\tau-\gamma)} \\
&\le CE|X|^{\gamma}. \qquad (21)
\end{aligned}
$$

By (3.14), let $u=2x^{\theta}y^{-\alpha}$, $v=y$, we have $x=2^{-1/\theta}u^{1/\theta}v^{\alpha/\theta}$, $y=v$, $J=\frac{\partial(x,y)}{\partial(u,v)}=2^{-1/\theta}\frac{1}{\theta}u^{1/\theta-1}v^{\alpha/\theta}$, and thus

$$
\begin{aligned}
I_{222}^{'} \le\ & C\int_{1}^{\infty} x^{\gamma(\beta+1)-2-\theta\tau})\,\mathrm{d}x\int_{1}^{x} E|y^{\alpha}X|^{\tau} I(|X|\le 2x^{\theta}y^{-\alpha})\,\mathrm{d}y \\
&+\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} E|n^{\alpha}X|^{\tau} I(|n^{\alpha}X|\le n^{\theta})
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\int_1^\infty u^{\frac{\gamma(\beta+1)-1}{\theta}-1-\tau}du\int_1^{u^{\frac{1}{\beta+1}}} v^{\frac{\alpha[\gamma(\beta+1)-1]}{\theta}}E|X|^\tau I(|X|\leq u)dv+C\\
&\leq C\int_1^\infty u^{\gamma-\tau-1}E|X|^\tau I(|X|\leq u)du+C\\
&\leq C\int_1^\infty u^{\gamma-\tau-1}\left(\int_0^u x^{\tau-1}P(|X|>x))\,\mathrm{d}x\right)du+C\\
&\leq C\int_1^\infty x^{\tau-1}P(|X|>x))\,\mathrm{d}x\int_x^\infty u^{\gamma-\tau-1}du+C\\
&\leq C\int_1^\infty x^{\gamma-1}P(|X|>x))\,\mathrm{d}x+C\\
&\leq CE|X|^\gamma+C<\infty.
\end{aligned}
$$

Similar to the proof of Liang and Zhang [13], let $u=x^\theta y^{-\alpha}$, $v=y$, we have $x=u^{1/\theta}v^{\alpha/\theta}$, $y=v$, $J=\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\theta}u^{1/\theta-1}v^{\alpha/\theta}$, and thus

$$
\begin{aligned}
I_{222}^{''} &= C\sum_{n=1}^\infty n^{\gamma(\beta+1)-2}\sum_{i=1}^n P(|i^\alpha X|>n^\theta)\leq C\int_1^\infty x^{\gamma(\beta+1)-2}\,\mathrm{d}x\int_1^x P(|X|>y^{-\alpha}x^\theta)\,\mathrm{d}y\\
&\leq C\int_1^\infty u^{\frac{\gamma(\beta+1)-1}{\theta}-1}P(|X|>u)\,\mathrm{d}u\int_1^{u^{\frac{1}{1+\beta}}} v^{\frac{\alpha[\gamma(\beta+1)-1]}{\theta}}dv\\
&\leq C\left[\int_1^\infty u^{\gamma-1}P(|X|>u)\,\mathrm{d}u-\int_1^\infty u^{\frac{\gamma(\beta+1)-1}{\theta}-1}P(|X|>u)\,\mathrm{d}u\right]\\
&\leq CE|X|^\gamma<\infty.
\end{aligned}
$$

Thus, $I_{22}<\infty$ holds. Similar to the proof of I_{22}, we can derive that $I_{23}<\infty$.

When $0<\nu<1$ and $\eta>\gamma>\nu$, note that $E|X|^\eta<\infty$. For $I_{24}^{'}$, we have by C_r inequality and Lemma 2.3 that

$$
\begin{aligned}
I_{24}^{'} &= C\sum_{n=1}^\infty n^{\gamma(\beta+1)-2-\theta\nu}\sum_{i=1}^n E\left|X_{ni}^{(4)}\right|^\nu\\
&\leq C\sum_{n=1}^\infty n^{\gamma(\beta+1)-2-\theta\nu}\sum_{i=1}^n E\left|i^\alpha X_{ni}\right|^\nu I(|i^\alpha X_{ni}|>n^\theta)\\
&\leq C\sum_{n=1}^\infty n^{\gamma(\beta+1)-2-\theta\eta}\sum_{i=1}^n E\left|i^\alpha X\right|^\eta I(|X|\leq n^\theta)\\
&\quad +C\sum_{n=1}^\infty n^{\gamma(\beta+1)-2-\theta\nu}\sum_{i=1}^n E\left|i^\alpha X\right|^\nu I(|X|>n^\theta)\\
&\leq CE\left|X\right|^\eta+CE\left|X\right|^\gamma<\infty. \qquad (22)
\end{aligned}
$$

Thus $I_{24}^{'}<\infty$ holds. Similarly, we have $I_{25}^{'}<\infty$.

When $1 < \nu < 2$ and $\gamma < \eta$, for I_{24}, we have by C_r inequality and Lemma 2.4, Lemma 2.3 and (8) that

$$\begin{aligned} I_{24} &= C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} E\left|\sum_{i=1}^{n}\left(X_{ni}^{(4)} - EX_{ni}^{(4)}\right)\right|^{\nu} \\ &\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g(n)\sum_{i=1}^{n} E\left|\left(X_{ni}^{(4)} - EX_{ni}^{(4)}\right)\right|^{\nu} \\ &\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g(n)\sum_{i=1}^{n} E\left|X_{ni}^{(4)}\right|^{\nu} \\ &\leq CE|X|^{\eta} + CE|X|^{\gamma} < \infty. \end{aligned} \tag{23}$$

So $I_{24} < \infty$ holds. Similar to the proof of I_{24}, we can derive that $I_{25} < \infty$.

When $\nu > 2$, for I_{24}, by C_r inequality and Lemma 2.4, we have that

$$\begin{aligned} I_{24} &= C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} E\left|\sum_{i=1}^{n}\left(X_{ni}^{(4)} - EX_{ni}^{(4)}\right)\right|^{\nu} \\ &\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu}\sum_{i=1}^{n} E\left|X_{ni}^{(4)}\right|^{\nu} + C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g(n)\left(\sum_{i=1}^{n} E\left|X_{ni}^{(4)}\right|^{2}\right)^{\frac{\nu}{2}} \\ &:= I_{241} + I_{242}. \end{aligned} \tag{24}$$

Similarly to the proof of (22), we get that $I_{241} < \infty$. Now for I_{242}, noting that $\gamma(\beta+1) - 2 - [(\beta+1)\eta - 1]\frac{\nu}{2} < -1$, by Lemma 2.3 and (8), we have that

$$\begin{aligned} I_{242} &= C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g(n)\left(\sum_{i=1}^{n} E\left|X_{ni}^{(4)}\right|^{2}\right)^{\frac{\nu}{2}} \\ &\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g(n)\left(\sum_{i=1}^{n} E\left|i^{\alpha}X_{ni}\right|^{2} I(\left|i^{\alpha}X_{ni}\right| > n^{\theta})\right)^{\frac{\nu}{2}} \\ &\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-[(\beta+1)\eta-1]\frac{\nu}{2}} g(n) < \infty. \end{aligned}$$

Therefore, $I_{24} < \infty$. In addition, we can obtain $I_{25} < \infty$ similarly. Therefore, $I_2 < \infty$ follows immediately from the statements above.

For I_1, we have that

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^{\alpha}X_{ni}\right| - \varepsilon n^{\theta}\right)_{+}^{\nu} \\ &\geq \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu}\int_{0}^{\infty} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^{\alpha}X_{ni}\right| - \varepsilon n^{\theta} > t^{1/\nu}\right) \mathrm{d}t \end{aligned}$$

$$\geq \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} \int_0^{(\varepsilon n^\theta)^\nu} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^k i^\alpha X_{ni}\right| - \varepsilon n^\theta > t^{1/\nu}\right) \mathrm{d}t$$

$$\geq \varepsilon^\nu \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^k i^\alpha X_{ni}\right| > 2\varepsilon n^\theta\right), \tag{25}$$

which together with the arbitrariness of $\varepsilon > 0$ yields that

$$I_1 \leq f(\delta)\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^k i^\alpha X_{ni}\right| > \varepsilon n^\theta\right) < \infty.$$

The proof is completed. □

The condition (9) holds by taking $f(t) = t^\zeta$, where $0 < \zeta < \nu < 1$. According to Theorem 3.1, we can obtain the following corollary.

Corollary 3.2. Let the condition (8) be fulfilled, $\alpha \geq 0$, and $\{X_{ni}, i \geq 1, n \geq 1\} \prec X$. Assume that $\gamma > 1$, $0 \geq \beta > \max\{-1/2, \gamma^{-1} - 1\}$, $\theta = \alpha + \beta + 1$, and $EX_{ni} = 0$. If $E|X|^\eta < \infty$ for some $\eta > \gamma$, then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^k i^\alpha X_{ni}\right| > \varepsilon n^\theta\right) < \infty. \tag{26}$$

Proof. Taking $f(t) = t^\delta$ for $0 < \delta < \nu < 1$, according to Theorem 3.1, we get

$$\infty > \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^k i^\alpha X_{ni}\right| n^{-\theta} - \varepsilon\right)_+^\delta$$

$$= \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\delta} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^k i^\alpha X_{ni}\right| - \varepsilon n^\theta\right)_+^\delta$$

$$= \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\delta} \int_0^{\infty} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^k i^\alpha X_{ni}\right| - \varepsilon n^\theta > t^{1/\delta}\right) \mathrm{d}t$$

$$\geq \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\delta} \int_0^{(\varepsilon n^\theta)^\delta} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^k i^\alpha X_{ni}\right| - \varepsilon n^\theta > t^{1/\delta}\right) \mathrm{d}t$$

$$\geq \varepsilon^\delta \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^k i^\alpha X_{ni}\right| > 2\varepsilon n^\theta\right). \tag{27}$$

According to the arbitrariness of $\varepsilon > 0$, we have completed the proof of the corollary. □

Theorem 3.1 is obtained under the condition (8). Below we consider the case without dominating coefficients.

Theorem 3.3. Let $\alpha \geq 0$, $\nu > 0$, $\{X_{ni},\ i \geq 1,\ n \geq 1\} \prec X$ and the condition (9) be fulfilled. Assume that $\gamma > 1$, $\gamma > \nu$ and $0 \geq \beta > \max\{-\frac{1}{2}, \gamma^{-1} - 1\}$ and $EX_{ni} = 0$. If $E|X|^{\eta} < \infty$ for some $\eta > \gamma$, then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} Ef(\{|S_n| - \varepsilon\}_+) < \infty, \tag{28}$$

and thus

$$\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^{\alpha} X_{ni}\right| > \varepsilon n^{\theta} g^{1/\nu}(n)\right) < \infty, \tag{29}$$

where $S_n = \max_{1\leq k\leq n} |\sum_{i=1}^{k} i^{\alpha} X_{ni}| g^{-1/\nu}(n) n^{-\theta}$.

P r o o f. Using the same segmentation and notations as those in Theorem 3.1, and similar to the proof of Theorem 3.1, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} Ef\left(\{|S_n| - \varepsilon\}_+\right) &= \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} \int_0^{\infty} P(|S_n| > \varepsilon + h(t))\,\mathrm{d}t \\
&= \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} \int_0^{f(\delta)} P(|S_n| > \varepsilon + h(t))\,\mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} \int_{f(\delta)}^{\infty} P(|S_n| > \varepsilon + h(t))\,\mathrm{d}t \\
&:= I_1 + I_2.
\end{aligned} \tag{30}$$

Thus, we need to prove $I_1 < \infty$ and $I_2 < \infty$.

First, we will deal with I_2. By Markov's inequality and (9), it can be included that

$$\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} \int_{f(\delta)}^{\infty} P(|S_n| > \varepsilon + h(t))\,\mathrm{d}t \\
&\leq \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g^{-1}(n) E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^{\alpha} X_{ni}\right| - \varepsilon n^{\theta} g^{1/\nu}(n)\right)_+^{\nu}.
\end{aligned} \tag{31}$$

Noting that $g(n) \geq 1$, when $\nu > 1$, we have

$$\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g^{-1}(n) E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^{\alpha} X_{ni}\right| - \varepsilon n^{\theta} g^{1/\nu}(n)\right)_+^{\nu} \\
&\leq \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g^{-1}(n) E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^{\alpha} X_{ni}\right| - \varepsilon n^{\theta}\right)_+^{\nu}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} g^{-1}(n) E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}(X_{ni}^{(1)}-EX_{ni}^{(1)})\right|^{\tau}\right)\\
&\quad +C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} g^{-1}(n) E\left|\sum_{i=1}^{n}\left(X_{ni}^{(2)}-EX_{ni}^{(2)}\right)\right|^{\tau}\\
&\quad +C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} g^{-1}(n) E\left|\sum_{i=1}^{n}\left(X_{ni}^{(3)}-EX_{ni}^{(3)}\right)\right|^{\tau}\\
&\quad +C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g^{-1}(n) E\left|\sum_{i=1}^{n}\left(X_{ni}^{(4)}-EX_{ni}^{(4)}\right)\right|^{\nu}\\
&\quad +C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g^{-1}(n) E\left|\sum_{i=1}^{n}\left(X_{ni}^{(5)}-EX_{ni}^{(5)}\right)\right|^{\nu}\\
&:= I_{21}+I_{22}+I_{23}+I_{24}+I_{25}. \qquad (32)
\end{aligned}
$$

When $0<\nu<1$, we have that

$$
\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g^{-1}(n) E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^{\alpha}X_{ni}\right|-\varepsilon n^{\theta} g^{1/\nu}(n)\right)_{+}^{\nu}\\
&\leq \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g^{-1}(n) E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^{\alpha}X_{ni}\right|-\varepsilon n^{\theta}\right)_{+}^{\nu}\\
&\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} g^{-1}(n) E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}(X_{ni}^{(1)}-EX_{ni}^{(1)})\right|^{\tau}\right)\\
&\quad +C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} g^{-1}(n) E\left|\sum_{i=1}^{n}\left(X_{ni}^{(2)}-EX_{ni}^{(2)}\right)\right|^{\tau}\\
&\quad +C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} g^{-1}(n) E\left|\sum_{i=1}^{n}\left(X_{ni}^{(3)}-EX_{ni}^{(3)}\right)\right|^{\tau}\\
&\quad +C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g^{-1}(n) E\left|\sum_{i=1}^{n} X_{ni}^{(4)}\right|^{\nu}\\
&\quad +C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g^{-1}(n) E\left|\sum_{i=1}^{n} X_{ni}^{(5)}\right|^{\nu}\\
&:= I_{21}+I_{22}+I_{23}+I_{24}^{'}+I_{25}^{'}. \qquad (33)
\end{aligned}
$$

For I_{21}, taking $\tau>\max\{2,\gamma,\frac{2\gamma(\beta+1)}{2\beta+1},\frac{2\gamma(\beta+1)}{\theta[2-(2-\gamma)q]-(\alpha\gamma+1)},\frac{\gamma(\beta+1)}{(1-q)\theta},\frac{2\gamma(\beta+1)}{\gamma(\beta+1)-1}\}$, we obtain by Lemma 2.5 and C_r inequality, that

$$
I_{21} = C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau} g^{-1}(n) E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}(X_{ni}^{(1)}-EX_{ni}^{(1)})\right|^{\tau}\right)
$$

$$\begin{aligned} &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}(\log n)^{\tau}\sum_{i=1}^{n} E|X_{ni}^{(1)}|^{\tau} \\ &\quad + C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}(\log n)^{\tau}\left(\sum_{i=1}^{n} E|X_{ni}^{(1)}|^{2}\right)^{\frac{\tau}{2}} \\ &:= I_{211}+I_{212}. \end{aligned} \tag{34}$$

For I_{211}, we have by Lemma 2.3 and $E|X|^{\gamma}<\infty$ that

$$\begin{aligned} I_{211} &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}(\log n)^{\tau}\sum_{i=1}^{n}\left[E|i^{\alpha}X_{ni}|^{\tau}I(|i^{\alpha}X_{ni}|\le n^{\theta q})+n^{\theta q\tau}P(|i^{\alpha}X_{ni}|>n^{\theta q})\right] \\ &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-1-(1-q)\theta\tau}(\log n)^{\tau}<\infty. \end{aligned}$$

For I_{212}, when $\gamma\ge 2$, we have $EX^2<\infty$. Since $\gamma(\beta+1)>1$ and $\beta>-\frac{1}{2}$, we have that $2(\beta+1)>1$. Thus, we have by Lemma 2.3 that

$$\begin{aligned} I_{212} &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}(\log n)^{\tau} \\ &\quad\cdot\left(\sum_{i=1}^{n} E\left[|i^{\alpha}X_{ni}|^{2}I(|i^{\alpha}X_{ni}|\le n^{\theta q})+n^{2\theta q}I(|i^{\alpha}X_{ni}|>n^{\theta q})\right]\right)^{\frac{\tau}{2}} \\ &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-[2(\beta+1)-1]\frac{\tau}{2}}(\log n)^{\tau}<\infty. \end{aligned}$$

When $0<\gamma<2$, note that $\frac{2\theta-(\alpha\gamma+1)}{(2-\gamma)\theta}>1>q$ by $\gamma(\beta+1)-1>0$, and thus $\theta[2-(2-\gamma)q]-(\alpha\gamma+1)>0$. We have by Lemma 2.3 again that

$$\begin{aligned} I_{212} &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}(\log n)^{\tau} \\ &\quad\cdot\left(\sum_{i=1}^{n} E\left[|i^{\alpha}X_{ni}|^{2}I(|i^{\alpha}X_{ni}|\le n^{\theta q})+n^{2\theta q}I(|i^{\alpha}X_{ni}|>n^{\theta q})\right]\right)^{\frac{\tau}{2}} \\ &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-[\theta(2-(2-\gamma)q)-(\alpha\gamma+1)]\frac{\tau}{2}}(\log n)^{\tau}<\infty. \end{aligned}$$

Therefore, $I_{212}<\infty$, and thus $I_{21}<\infty$.

For I_{22}, we have by Lemma 2.4 and C_r inequality that

$$\begin{aligned} I_{22} &= C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}g^{-1}(n)E\left|\sum_{i=1}^{n}(X_{ni}^{(2)}-EX_{ni}^{(2)})\right|^{\tau} \\ &\le C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}\left(\sum_{i=1}^{n}E(X_{ni}^{(2)})^{2}\right)^{\frac{\tau}{2}}+C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}\sum_{i=1}^{n}E|X_{ni}^{(2)}|^{\tau} \end{aligned}$$

$$:= \quad I_{221} + I_{222}. \tag{35}$$

When $\gamma \geq 2$, we have $EX^2 < \infty$ and $2(\beta+1) > 1$. Thus, we have by Lemma 2.3 that

$$\begin{aligned} I_{221} &\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}\left(\sum_{i=1}^{n} E\left[|i^{\alpha}X_{ni}|^2 I(|i^{\alpha}X_{ni}| \leq 2n^{\theta}) + n^{2\theta} I(|i^{\alpha}X_{ni}| > n^{\theta})\right]\right)^{\frac{\tau}{2}} \\ &\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-[2(\beta+1)-1]\frac{\tau}{2}} < \infty. \end{aligned}$$

When $0 < \gamma < 2$, note that $\gamma(\beta+1) - 1 > 0$. We have by Lemma 2.3 again that

$$\begin{aligned} I_{221} &\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\tau}\left(\sum_{i=1}^{n} E|i^{\alpha}X_{ni}|^2 I(|i^{\alpha}X_{ni}| \leq 2n^{\theta}) + \sum_{i=1}^{n} n^{2\theta} P(|i^{\alpha}X_{ni}| > n^{\theta})\right)^{\frac{\tau}{2}} \\ &\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-[\gamma(\beta+1)-1]\frac{\tau}{2}} < \infty. \end{aligned}$$

Thus, $I_{221} < \infty$ holds. The proof of I_{222} is similar to that of (20), then we get $I_{222} < \infty$. Thus, $I_{22} < \infty$ holds. The proof of I_{23} is similar to that of I_{22}, we can get that $I_{23} < \infty$.

For $I_{24}^{'}$, when $0 < \nu \leq 1$, noting that $g(n) \geq 1$, similar to the proof of (22), we have that

$$\begin{aligned} I_{24}^{'} &= C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g^{-1}(n) \sum_{i=1}^{n} E\left|X_{ni}^{(4)}\right|^{\nu} \\ &\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} \sum_{i=1}^{n} E\left|X_{ni}^{(4)}\right|^{\nu} \\ &\leq CE|X|^{\eta} + CE|X|^{\gamma} < \infty. \end{aligned} \tag{36}$$

Thus $I_{24}^{'} < \infty$. Similarly, we have $I_{25}^{'} < \infty$.

When $1 < \nu \leq 2$, and $\nu < \gamma$, similar to the proof of (22), we have that

$$\begin{aligned} I_{24} &= C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g^{-1}(n) E\left|\sum_{i=1}^{n}\left(X_{ni}^{(4)} - EX_{ni}^{(4)}\right)\right|^{\nu} \\ &\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g^{-1}(n) \sum_{i=1}^{n} E\left|X_{ni}^{(4)} - EX_{ni}^{(4)}\right|^{\nu} \\ &\quad + C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} \sum_{i=1}^{n} E\left|X_{ni}^{(4)} - EX_{ni}^{(4)}\right|^{\nu} \\ &\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} \sum_{i=1}^{n} E\left|X_{ni}^{(4)}\right|^{\nu} \\ &\leq CE|X|^{\eta} + CE|X|^{\gamma} < \infty. \end{aligned} \tag{37}$$

When $\nu > 2$, and $\nu < \gamma$, we have that

$$I_{24} = C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} g^{-1}(n) E\left|\sum_{i=1}^{n}\left(X_{ni}^{(4)} - EX_{ni}^{(4)}\right)\right|^{\nu}$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu}\sum_{i=1}^{n} E|X_{ni}^{(4)}|^{\nu} + C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu}\left(\sum_{i=1}^{n} E|X_{ni}^{(4)}|^{2}\right)^{\frac{\nu}{2}} \\
&:= I_{241} + I_{242}. \qquad (38)
\end{aligned}
$$

Similar to the proof of (22), we can see that $I_{241} < \infty$ holds.

Now for I_{242}, noting that $\gamma(\beta+1) - 2 - [(\beta+1)\eta - 1]\frac{\nu}{2} < -1$, by Lemma 2.3, we have that

$$
\begin{aligned}
I_{242} &= C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu}\left(\sum_{i=1}^{n} E\left|X_{ni}^{(4)}\right|^{2}\right)^{\frac{\nu}{2}} \\
&\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu}\left(\sum_{i=1}^{n} E\left|i^{\alpha}X_{ni}\right|^{2} I(|i^{\alpha}X_{ni}| > n^{\theta})\right)^{\frac{\nu}{2}} \\
&\leq C\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-[(\beta+1)\eta-1]\frac{\nu}{2}} < \infty.
\end{aligned}
$$

In summary, $I_{24} < \infty$ holds. The proof of $I_{25} < \infty$ is similar to that of $I_{24} < \infty$.

For I_1, we have that

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^{\alpha}X_{ni}\right| - \varepsilon n^{\theta} g^{1/\nu}(n)\right)_{+}^{\nu} \\
&\geq \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu}\int_{0}^{\infty} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^{\alpha}X_{ni}\right| - \varepsilon n^{\theta} g^{1/\nu}(n) > t^{1/\nu}\right) \mathrm{d}t \\
&\geq \sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2-\theta\nu}\int_{0}^{(\varepsilon n^{\theta} g^{1/\nu}(n))^{\nu}} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^{\alpha}X_{ni}\right| - \varepsilon n^{\theta} g^{1/\nu}(n) > t^{1/\nu}\right) \mathrm{d}t \\
&\geq \varepsilon^{\nu} g(n)\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^{\alpha}X_{ni}\right| > 2\varepsilon n^{\theta} g^{1/\nu}(n)\right) \\
&\geq \varepsilon^{\nu}\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^{\alpha}X_{ni}\right| > 2\varepsilon n^{\theta} g^{1/\nu}(n)\right), \qquad (39)
\end{aligned}
$$

which together with the arbitrariness of $\varepsilon > 0$ yields that

$$
I_1 \leq f(\delta)\sum_{n=1}^{\infty} n^{\gamma(\beta+1)-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} i^{\alpha}X_{ni}\right| > \varepsilon n^{\theta} g^{1/\nu}(n)\right) < \infty.
$$

The proof is completed. □

Remark 3.4. The complete f-moment convergence for weighted sums of WOD arrays is obtained in Theorem 3.1, which generalizes the result of Liang and Zhang [13] for the complete convergence. If $g(n) = M \geq 1$, the result is still valid for the END arrays.

4. AN APPLICATION TO NONPARAMETRIC REGRESSION MODELS BASED ON WOD ERRORS

In this section, we will give an application to nonparametric regression models based on WOD errors by using the complete convergence obtained in Corollary 3.2.

Let us consider the following nonparametric regression model:

$$Y_{ni} = f(x_{ni}) + \epsilon_{ni}, \quad i = 1, 2, \ldots, n, \ n \geq 1, \tag{40}$$

where x_{ni} are known fixed design points from A, $A \subset \mathbb{R}^d$ is a given compact set for some $d \geq 1$. $f(\cdot)$ is an unknown regression function defined on A, and ϵ_{ni} are random errors. As an estimator of $f(\cdot)$, we will consider the following weighted linear regression estimator:

$$f_n(x) = \sum_{i=1}^{n} W_{ni}(x) Y_{ni}, \quad x \in A \subset \mathbb{R}^d, \tag{41}$$

where $W_{ni}(x) = W_{ni}(x; x_{n1}, x_{n2}, \ldots, x_{nn})$, $i = 1, 2, \ldots, n$ are the weight functions.

This above estimator (41) was first introduced by Stone [24]. After that, many authors have studied the asymptotic properties of the estimator (41), including strong consistency, complete consistency, and asymptotic normality based on independent errors or dependent errors. To learn more about these properties of the estimator, we recommend readers these articles such as Georgiev [5], Georgiev and Greblicki [6], Roussas et al. [17], Tran et al. [25], Liang and Jing [12], Zhou et al. [36], Wang et al. [37], Shen [19], Yang et al. [34], Shen et al. [21], Wu et al. [33], Zhang et al. [35] and so on.

4.1. Theoretical results

In this subsection, let $c(f)$ denote the set of continuity points of the function f on A. The symbol $||x||$ denotes the Euclidean norm. For any point $x \in A$, the weight function $W_{ni}(x)$ satisfies the following assumptions:

1. (A_1) $\sum_{i=1}^{n} W_{ni}(x) \to 1$ as $n \to \infty$;
2. (A_2) $\sum_{i=1}^{n} |W_{ni}(x)| \leq C < \infty$ for all n;
3. (A_3) $\sum_{i=1}^{n} |W_{ni}(x)| \cdot |f(x_{ni}) - f(x)| I(||x_{ni} - x|| > a) \to 0$ *as* $n \to \infty$ for all $a > 0$.

Based on these assumptions, we will use Corollary 3.2 to further study the complete consistency for the nonparametric regression estimator $f_n(x)$.

Theorem 4.1. Let $\{\varepsilon_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of rowwise zero mean WOD random variables, which is stochastically dominated by a random variable X with $E|X|^\eta < \infty$ for some $\eta > 2$. Suppose that the conditions $(A_1) - (A_3)$ and (8) are satisfied, and $\max_{1 \leq i \leq n} |W_{ni}(x)| = O(n^{-\theta})$ for some $\max\{1/2, 2/\eta\} < \theta \leq 1$. Then for any $x \in c(f)$,

$$f_n(x) \to f(x) \ \text{ completely}. \tag{42}$$

Proof. Note that

$$|f_n(x) - f(x)| \leq |f_n(x) - Ef_n(x)| + |Ef_n(x) - f(x)|. \tag{43}$$

For $a > 0$ and $x \in c(f)$, we obtain from (40) and (41) that

$$\begin{aligned} |Ef_n(x) - f(x)| &\leq \sum_{i=1}^{n} |W_{ni}(x)| \cdot |f(x_{ni}) - f(x)| I(\| x_{ni} - x \| \leq a) \\ &+ \sum_{i=1}^{n} |W_{ni}(x)| \cdot |f(x_{ni}) - f(x)| I(\| x_{ni} - x \| \geq a) \\ &+ |f(x)| \cdot \left| \sum_{i=1}^{n} W_{ni}(x) - 1 \right|. \end{aligned} \tag{44}$$

It follows from $x \in c(f)$ that for all $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for all $x^{'}$ satisfying $\| x^{'} - x \| < \delta$, $|f(x^{'}) - f(x)| < \varepsilon$. Setting $0 < a < \delta$ in (44), we have

$$\begin{aligned} |Ef_n(x) - f(x)| &\leq \varepsilon \sum_{i=1}^{n} |W_{ni}(x)| \\ &+ \sum_{i=1}^{n} |W_{ni}(x)| \cdot |f(x_{ni}) - f(x)| I(\| x_{ni} - x \| \geq a) \\ &+ |f(x)| \cdot \left| \sum_{i=1}^{n} W_{ni}(x) - 1 \right|. \end{aligned} \tag{45}$$

Therefore by assumptions $(A_1) - (A_3)$ and the arbitrariness of $\varepsilon > 0$, we have that for all $x \in c(f)$,

$$\lim_{n \to \infty} Ef_n(x) = f(x). \tag{46}$$

In view of (46), to prove (42), it suffices to show

$$f_n(x) - Ef_n(x) = \sum_{i=1}^{n} W_{ni}(x) \epsilon_{ni} \to \ 0 \text{ completely.}$$

In other words, for any $\varepsilon > 0$, we need to verify that

$$\sum_{n=1}^{\infty} P\left(\left| \sum_{i=1}^{n} W_{ni}(x) \epsilon_{ni} \right| > \varepsilon \right) < \infty. \tag{47}$$

Applying Corollary 3.2 with $X_{ni} = \epsilon_{nk_i}$ and $\alpha = 0$, by taking $\beta = \theta - 1$ and $\gamma = 2/\theta$, we can know that $\eta > \gamma \geq 2$ and $\gamma(\beta + 1) = 2$, and thus

$$\sum_{n=1}^{\infty} P\left(\left| \sum_{i=1}^{n} W_{ni}(x) \epsilon_{ni} \right| > \varepsilon \right) \leq \sum_{n=1}^{\infty} P\left(\max_{1 \leq i \leq n} |W_{ni}(x)| \cdot \max_{1 \leq m \leq n} \left| \sum_{i=1}^{m} \epsilon_{nk_i} \right| > \varepsilon \right)$$

$$\leq \sum_{n=1}^{\infty} P\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m}\epsilon_{nk_i}\right| > Cn^{\theta}\right) < \infty, \qquad (48)$$

where $(k_1, \ldots, k_n)$ is a permutation of $(1, 2, \ldots, n)$. Hence (47) follows immediately from (48). The proof is completed. □

As an application of Theorem 4.1, we give the complete consistency for the nearest estimator neighbor of $f(x)$. Without loss of generality, let $A = [0, 1]$ and let $x_{ni} = i/n, i = 1, 2, \ldots, n$. For any $x \in A$, we rewrite

$$|x_{n1} - x|, |x_{n2} - x|, \ldots, |x_{nn} - x|$$

as follows:

$$|x_{n,R_1(x)} - x|, |x_{n,R_2(x)} - x|, \ldots, |x_{n,R_n(x)} - x|,$$

if $|x_{ni} - x| = |x_{nj} - x|$, then $|x_{ni} - x|$ is permuted before $|x_{nj} - x|$ when $x_{ni} < x_{nj}$. Let $1 \leq k_n \leq n$, the nearest neighbor weight function is defined as follows:

$$\widetilde{f}_n(x) = \sum_{i=1}^{n} \widetilde{W}_{ni}(x) Y_{ni}, \qquad (49)$$

where

$$\widetilde{W}_{ni}(x) = \begin{cases} 1/k_n, & \text{if } |x_{ni} - x| \leq |x_{n,R_{k_n}(x)} - x|, \\ 0, & \text{otherwise.} \end{cases}$$

4.2. Numerical simulation

The data are generated from model (40). For any fixed $n \geq 3$, let $(\epsilon_{n1}, \epsilon_{n2}, \ldots, \epsilon_{nn}) \sim N_n(\mathbf{0}, \mathbf{\Sigma})$, where $\mathbf{0}$ represents zero vector and

$$\mathbf{\Sigma} = \begin{pmatrix} 1+\theta^2 & -\theta & 0 & \cdots & 0 & 0 & 0 \\ -\theta & 1+\theta^2 & -\theta & \cdots & 0 & 0 & 0 \\ 0 & -\theta & 1+\theta^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\theta^2 & -\theta & 0 \\ 0 & 0 & 0 & \cdots & -\theta & 1+\theta^2 & -\theta \\ 0 & 0 & 0 & \cdots & 0 & -\theta & 1+\theta^2 \end{pmatrix}_{n\times n}, \quad 0 < \theta < 1.$$

By Joag-Dev and Proschan [9], we can see that $(\epsilon_{n1}, \epsilon_{n2}, \ldots, \epsilon_{nn})$ is a NA vector for each $n \geq 3$ with finite moment of any order, and thus is a WOD vector. We choose $\theta = 0.5$ and $\theta = 0.8$, respectively and take $k_n = \lfloor n^{0.6} \rfloor$. Taking the points $x = 0.25, 0.5, 0.75$ and the sample sizes n as $n = 50, 100, 200, 500$, respectively, we use R software to compute $\widetilde{f_n}(x) - f(x)$ with $f(x) = \sin x - x$ and $f(x) = x^2 - x$ for 1000 times and obtain the boxplots of $\widetilde{f_n}(x) - f(x)$ in Figures 1-4. We show the Mean Squared Error (MSE, for short) of $\widetilde{f_n}(x)$ in Tables 1 and 2.

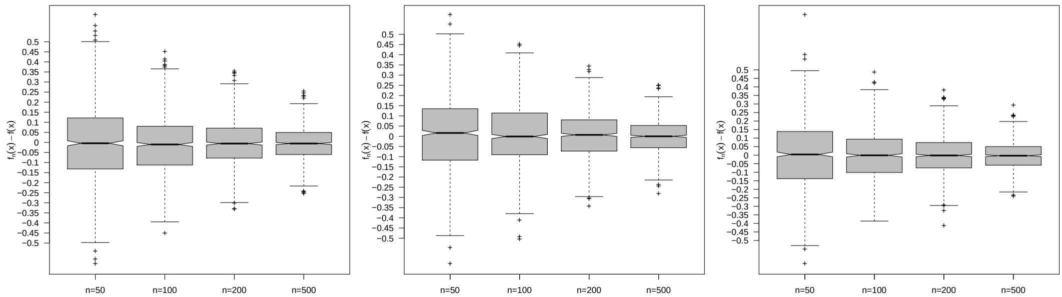

Fig. 1. Boxplots of $\widetilde{f_n}(x) - f(x)$ with $\theta = 0.5$, $x = 0.25, 0.5, 0.75$ and $f(x) = \sin x - x$.

When $\theta = 0.5$, Figure 1 is about the boxplots of $\widetilde{f_n}(x) - f(x)$ for $f(x) = \sin x - x$ and Figure 2 is about the boxplots of $\widetilde{f_n}(x) - f(x)$ for $f(x) = x^2 - x$ with $x = 0.25, 0.5, 0.75$, respectively. We can see that no matter $f(x) = x^2 - x$ or $f(x) = \sin x - x$, for $x = 0.25, 0.5, 0.75$, the differences $\widetilde{f_n}(x) - f(x)$ fluctuate around zero, and the range decreases significantly as the increasing of sample size n. Table 1 accurately reflects that the MSE of $\widetilde{f_n}(x)$ decreases significantly with the increasing of n. These simulation results are consistent with theoretical results.

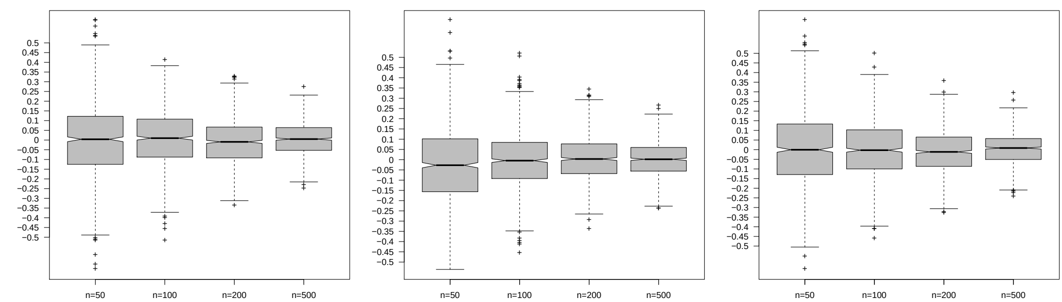

Fig. 2. Boxplots of $\widetilde{f_n}(x) - f(x)$ with $\theta = 0.5$, $x = 0.25, 0.5, 0.75$ and $f(x) = x^2 - x$.

When $\theta = 0.8$, Figure 3 is about the boxplots of $\widetilde{f_n}(x) - f(x)$ for $f(x) = \sin x - x$ and Figure 4 is about the boxplots of $\widetilde{f_n}(x) - f(x)$ for $f(x) = x^2 - x$ with $x = 0.25, 0.5, 0.75$, respectively. We can obtain the same conclusions as those when $\theta = 0.5$. Table 2 also reflects precisely that the MSE of $\widetilde{f_n}(x)$ decreases markedly as n increases. These simulation results also agree with the theoretical results again.

$f(x)$	x	n=50	n=100	n=200	n=500
	0.25	0.03326531	0.02130204	0.01244919	0.006727868
$\sin x - x$	0.5	0.04076935	0.02238789	0.01209324	0.006697934
	0.75	0.03552043	0.02072919	0.01202523	0.006288669
	0.25	0.03745238	0.02211743	0.01218106	0.006506393
$x^2 - x$	0.5	0.03999099	0.01973842	0.01258507	0.006043364
	0.75	0.03077941	0.02259765	0.01304094	0.006813427

Tab. 1. MSE of the estimator $\widetilde{f_n}(x)$ when $\theta = 0.5$.

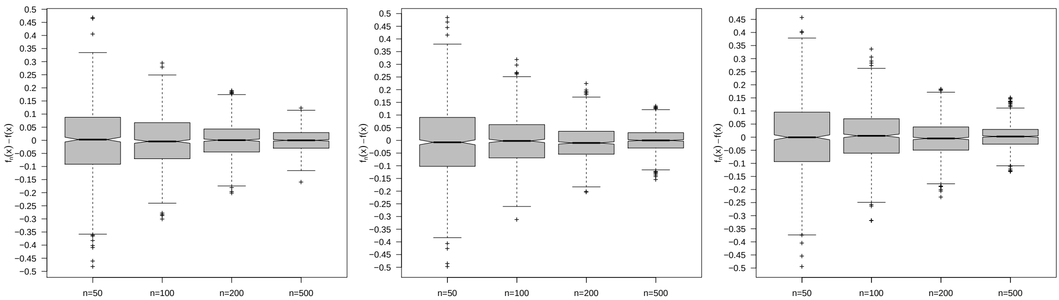

Fig. 3. Boxplots of $\widetilde{f_n}(x) - f(x)$ with $\theta = 0.8$, $x = 0.25, 0.5, 0.75$ and $f(x) = \sin x - x$.

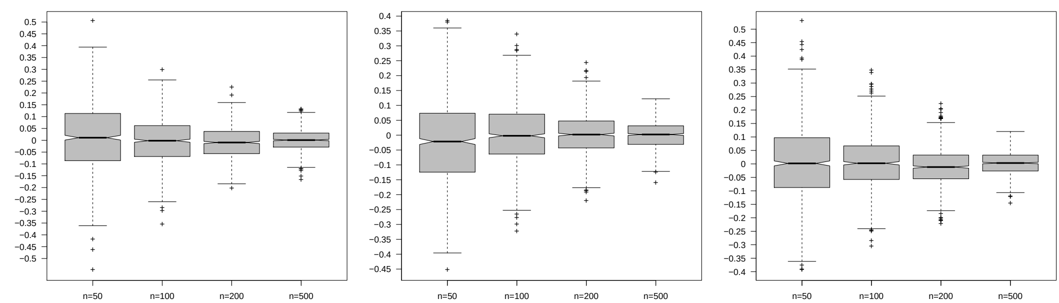

Fig. 4. Boxplots of $\widetilde{f_n}(x) - f(x)$ with $\theta = 0.8$, $x = 0.25, 0.5, 0.75$ and $f(x) = \sin x - x$.

$f(x)$	x	n=50	n=100	n=200	n=500
	0.25	0.01941749	0.009855784	0.004323809	0.002056822
$\sin x - x$	0.5	0.02051351	0.009541484	0.004377528	0.001906454
	0.75	0.02016247	0.009742266	0.004503567	0.002023224
	0.25	0.02131077	0.01001992	0.004521118	0.001939339
$x^2 - x$	0.5	0.02111859	0.01011867	0.004202791	0.001768192
	0.75	0.01976998	0.01026442	0.00456359	0.00189953

Tab. 2. MSE of the estimator $\widetilde{f_n}(x)$ when $\theta = 0.8$.

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