Nurten Gürses; Gülsüm Yeliz Şentürk; Salim Yüce

Investigating generalized quaternions with dual-generalized complex numbers

Mathematica Bohemica, Vol. 148 (2023), No. 3, 329–348

Persistent URL: http://dml.cz/dmlcz/151763

INVESTIGATING GENERALIZED QUATERNIONS WITH DUAL-GENERALIZED COMPLEX NUMBERS

Nurten Gürses, Gülsüm Yeliz Şentürk, Salim Yüce, Istanbul

Received June 30, 2021. Published online August 3, 2022.
Communicated by Clemens Fuchs

Abstract. We aim to introduce generalized quaternions with dual-generalized complex number coefficients for all real values α, β and $\mathfrak{p}$. Furthermore, the algebraic structures, properties and matrix forms are expressed as generalized quaternions and dual-generalized complex numbers. Finally, based on their matrix representations, the multiplication of these quaternions is restated and numerical examples are given.

Keywords: generalized quaternion; dual-generalized complex number; matrix representation

MSC 2020: 11R52, 15B33

1. Introduction

The discovery of the quaternions, which is an associative and non-commutative Clifford algebra over the real numbers, is one of the outstanding contributions of Hamilton (see [20], [21], [22]). As an extension of the quaternions, the octonions are a non-associative and non-commutative algebra. Later on, Cayley and Dickson discussed these algebras, sometimes called the Cayley-Dickson algebras. The extension originates from $\mathbb{R}$ (real numbers 1-D) to $\mathbb{C}$ (complex numbers 2-D) and continues as: from $\mathbb{C}$ to $\mathbb{H}$ (quaternions 4-D), from $\mathbb{H}$ to $\mathbb{O}$ (octonions 8-D), from $\mathbb{O}$ to $\mathbb{S}$ (sedenions 16-D) and from $\mathbb{S}$ to $\mathbb{T}$ (trigintaduonions 32-D) and has been generalized to algebras over fields and rings. This process is known as the Cayley-Dickson doubling process or the Cayley-Dickson process. Hence, one can see the following Cayley-Dickson doubling subalgebras chain:

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O} \subset \mathbb{S} \subset \mathbb{T} \subset \ldots$$

DOI: 10.21136/MB.2022.0096-21

Since the Cayley-Dickson process is inductive, it is possible to construct n-ions by applying this process in an arbitrarily repeated pattern.

A quaternion can be written as $q = a_0 + a_1e_1 + a_2e_2 + a_3e_3$, where $a_0, a_1, a_2, a_3 \in \mathbb{R}$ and e_1, e_2, e_3 are quaternionic units. Real quaternions are commonly used in theoretical and applied mathematics, computer animation and robotics. Cockle (see [10], [11]) discovered split quaternions (co-quaternions or para-quaternions). Moreover, the set of generalized quaternions, denoted by $Q_{\alpha\beta}$, is examined in [13], [18], [24], [27], [30], [33]. The algebra of generalized quaternions as a non-commutative system includes various well-known 4-dimensional algebras as special cases. For these quaternions, the conditions of the units are given by:

$$(1.1) \qquad e_1^2 = -\alpha, \quad e_2^2 = -\beta, \quad e_3^2 = -\alpha\beta,$$
$$e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = \beta e_1, \quad e_3e_1 = -e_1e_3 = \alpha e_2,$$

where $e_1, e_2, e_3 \notin \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$. For $\alpha = \beta = 1$ real quaternions, for $\alpha = 1$, $\beta = -1$ split quaternions, for $\alpha = 1$, $\beta = 0$ semi quaternions, for $\alpha = -1$, $\beta = 0$ split semi quaternions and for $\alpha = \beta = 0$ quasi quaternions are obtained.

When it comes to numbers and their relationship to one another, scholars have long been interested in the subject matter. One of the most significant contributions of number theory is the revelation of generalized complex numbers. The generalized complex numbers have the form:

$$\mathbb{C}_\mathfrak{p} := \{z = x_1 + x_2J\colon\ x_1, x_2 \in \mathbb{R},\ J^2 = \mathfrak{p} \in \mathbb{R},\ J \notin \mathbb{R}\}.$$

This is a commutative unitary ring and a vector space over $\mathbb{R}$, see [4], [5], [6], [23], [25], [37], [39]. Complex numbers $\mathbb{C}$ (ordinary numbers) in [38], hyperbolic numbers $\mathbb{P}$ (double, binary, split complex, perplex numbers) in [9], [16], [34] and dual numbers $\mathbb{D}$ in [29], [35] are obtained for $\mathfrak{p} = -1$, $\mathfrak{p} = 0$, and $\mathfrak{p} = 1$, respectively. Furthermore, the construction of the number systems by writing the coefficients as elements of the sets $\mathbb{C}$, $\mathbb{P}$ and $\mathbb{D}$ is another fascinating area for researchers. Hence it is no surprise that hyperbolic-complex numbers are examined in [2], [10], [25]. Furthermore, n-dimensional hyperbolic-complex and bicomplex numbers are investigated in [17], [31], [32], [36], respectively. Dual-complex numbers are examined in [7], [8], [26], [28]. Dual-hyperbolic numbers and their algebraic properties are discussed in [26]. Besides, the functions and various matrix representations of dual-hyperbolic numbers and complex-hyperbolic numbers are presented in [1]. Hyper-dual numbers are studied in [12], [14], [15]. Additionally, dual-generalized complex ($\mathcal{DGC}$) numbers have been constructed by doubling dual numbers over generalized complex numbers using the Cayley-Dickson process. This extension is examined in [19] and denoted by:

$$\mathbb{DC}_\mathfrak{p} := \{a = z_1 + z_2\varepsilon\colon\ z_1 = x_1 + x_2J,\ z_2 = x_3 + x_4J \in \mathbb{C}_\mathfrak{p}\},$$

where $J^2 = \mathfrak{p} \in \mathbb{R}$, $\varepsilon^2 = 0$, $\varepsilon \neq 0$, $J\varepsilon = \varepsilon J$, and $J, \varepsilon \notin \mathbb{R}$. $\mathbb{DC}_{\mathfrak{p}}$[1] generalizes with dual-complex numbers for $\mathfrak{p} = -1$ (see [7], [8], [28]), dual-hyperbolic numbers for $\mathfrak{p} = 1$ (see [1]), and hyper-dual numbers for $\mathfrak{p} = 0$ (see [12], [14], [15]).

The theoretical perspectives and literature review mentioned are the motivating factors of this study and as a result lead to the following:

Problem. Is it possible to combine the concepts of generalized quaternions and $\mathcal{DGC}$ numbers? If the answer is affirmative, what algebraic properties are satisfied?

In this regard, the present paper is organized as follows. In Section 2, generalized quaternions with $\mathcal{DGC}$ number coefficients are introduced for all real values α, β and $\mathfrak{p}$. Moreover, the algebraic notions are investigated as numbers and as quaternions. Then, several matrix representations are given. Finally, the multiplication of these quaternions is presented a using different method and numerical examples are given.

2. Generalized quaternions with dual-generalized complex number

The structure of this section is as follows: After a brief definition of new generalized quaternions, the algebraic properties and structures are discussed.

Definition 2.1. The set of generalized quaternions with $\mathcal{DGC}$ number coefficients is denoted by $\widetilde{\mathbb{Q}}_{\alpha\beta}$ and defined as:

$$\widetilde{\mathbb{Q}}_{\alpha\beta} := \{\tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \colon\ a_0, a_1, a_2, a_3 \in \mathbb{DC}_{\mathfrak{p}}\},$$

where e_1, e_2, e_3 are quaternionic units as given in equation (1.1) and $\alpha, \beta \in \mathbb{R}$.

It should be noted that the $\mathcal{DGC}$ units J, ε and $J\varepsilon$ commute with the three quaternionic units e_k; that is $e_k J = J e_k$, $e_k \varepsilon = \varepsilon e_k$ and $e_k J\varepsilon = J\varepsilon e_k$ for $1 \leqslant k \leqslant 3$. It is evident that e_1 is distinct from the usual complex unit for $\mathfrak{p} = -1$, $\alpha = 1$, distinct from the usual hyperbolic unit for $\mathfrak{p} = 1$, $\alpha = -1$, and distinct from the usual dual unit for $\mathfrak{p} = -1$, $\alpha = 0$. This condition also holds for the other quaternionic units.

[1] $\mathbb{DC}_{\mathfrak{p}}$ is a commutative unitary ring and a vector space over $\mathbb{R}$. For $a_1 = z_{11} + z_{12}\varepsilon$, $a_2 = z_{21} + z_{22}\varepsilon \in \mathbb{DC}_{\mathfrak{p}}$ and $\lambda \in \mathbb{R}$, the operations are given as follows (see [19]):

$$\begin{aligned}
&\text{equality:} && a_1 = a_2 \Leftrightarrow z_{11} + z_{12}\varepsilon = z_{21} + z_{22}\varepsilon \Leftrightarrow z_{11} = z_{21},\ z_{12} = z_{22},\\
&\text{addition:} && a_1 + a_2 = (z_{11} + z_{12}\varepsilon) + (z_{21} + z_{22}\varepsilon) = (z_{11} + z_{21}) + (z_{12} + z_{22})\varepsilon,\\
&\text{scalar multiplication:} && \lambda a_1 = \lambda(z_{11} + z_{12}\varepsilon) = (\lambda z_{11}) + (\lambda z_{12})\varepsilon,\\
&\text{multiplication:} && a_1 a_2 = (z_{11} + z_{12}\varepsilon)(z_{21} + z_{22}\varepsilon) = (z_{11} z_{21}) + (z_{11} z_{22} + z_{12} z_{21})\varepsilon.
\end{aligned}$$

Additionally, special cases of these quaternions are given by:

▷ If $\alpha = \beta = 1$, then $\widetilde{\mathbb{Q}}_{\alpha\beta}$ is the set of real quaternions.
▷ If $\alpha = 1$, $\beta = -1$, then $\widetilde{\mathbb{Q}}_{\alpha\beta}$ is the set of split/para/co quaternions.
▷ If $\alpha = 1$, $\beta = 0$, then $\widetilde{\mathbb{Q}}_{\alpha\beta}$ is the set of semi quaternions.
▷ If $\alpha = -1$, $\beta = 0$, then $\widetilde{\mathbb{Q}}_{\alpha\beta}$ is the set of split semi quaternions.
▷ If $\alpha = \beta = 0$, then $\widetilde{\mathbb{Q}}_{\alpha\beta}$ is the set $\frac{1}{4}$/quasi quaternions.

In all cases with $\mathcal{DGC}$ number coefficient.

It is also possible to study more specific quaternions with $\mathcal{DGC}$ number coefficients depending on the choice of the real values α and β.

Let $\tilde{q} = a_0 + a_1e_1 + a_2e_2 + a_3e_3$, $\tilde{p} = b_0 + b_1e_1 + b_2e_2 + b_3e_3 \in \widetilde{\mathbb{Q}}_{\alpha\beta}$. Algebraic structures are now defined on $\widetilde{\mathbb{Q}}_{\alpha\beta}$ considering a generalized quaternion form. In general, a quaternion $\tilde{q}$ has 2 parts, a scalar $S_{\tilde{q}} = a_0$, and a vector $V_{\tilde{q}} = a_1e_1 + a_2e_2 + a_3e_3$. So that $\tilde{q} = S_{\tilde{q}} + V_{\tilde{q}}$. Equality and addition (and hence subtraction) are component-wise defined as follows:

$$\tilde{p} = \tilde{q} \Leftrightarrow a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3 \Leftrightarrow S_{\tilde{p}} = S_{\tilde{q}},\ V_{\tilde{p}} = V_{\tilde{q}}$$

and

$$\begin{aligned}&\widetilde{\mathbb{Q}}_{\alpha\beta} \times \widetilde{\mathbb{Q}}_{\alpha\beta} \to \widetilde{\mathbb{Q}}_{\alpha\beta},\\ &\qquad (\tilde{q},\tilde{p}) \mapsto \tilde{q} + \tilde{p} = (a_0 + b_0) + (a_1 + b_1)e_1 + (a_2 + b_2)e_2 + (a_3 + b_3)e_3\\ &\qquad\qquad\qquad\qquad = (S_{\tilde{p}} + S_{\tilde{q}}) + (V_{\tilde{p}} + V_{\tilde{q}}).\end{aligned}$$

An element $\bar{\tilde{q}}$ called the conjugate of $\tilde{q}$ is defined by:

$$\begin{aligned}&\widetilde{\mathbb{Q}}_{\alpha\beta} \to \widetilde{\mathbb{Q}}_{\alpha\beta},\\ &\qquad \tilde{q} \mapsto \bar{\tilde{q}} = a_0 - a_1e_1 - a_2e_2 - a_3e_3 = S_{\tilde{q}} - V_{\tilde{q}}.\end{aligned} \tag{2.1}$$

The scalar multiplication refers to the product of $\tilde{q}$ by $c \in \mathbb{R}$ and is

$$\begin{aligned}&\mathbb{R} \times \widetilde{\mathbb{Q}}_{\alpha\beta} \to \widetilde{\mathbb{Q}}_{\alpha\beta},\\ &\qquad (c,\tilde{q}) \mapsto c\tilde{q} = ca_0 + ca_1e_1 + ca_2e_2 + ca_3e_3 = cS_{\tilde{q}} + cV_{\tilde{q}}.\end{aligned}$$

Moreover, multiplication of $\tilde{q}$ and $\tilde{p}$ is calculated as:

$$\begin{aligned}&\widetilde{\mathbb{Q}}_{\alpha\beta} \times \widetilde{\mathbb{Q}}_{\alpha\beta} \to \widetilde{\mathbb{Q}}_{\alpha\beta},\\ &\qquad (\tilde{q},\tilde{p}) \mapsto \tilde{q}\tilde{p} = (a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha\beta a_3b_3)\\ &\qquad\qquad\qquad\qquad + (a_0b_1 + a_1b_0 + \beta a_2b_3 - \beta a_3b_2)e_1\\ &\qquad\qquad\qquad\qquad + (a_0b_2 - \alpha a_1b_3 + a_2b_0 + \alpha a_3b_1)e_2\\ &\qquad\qquad\qquad\qquad + (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)e_3.\end{aligned} \tag{2.2}$$

The multiplication is non-commutative but associative and distributive over addition.

Corollary 2.1. *$\widetilde{\mathbb{Q}}_{\alpha\beta}$ is a 4-dimensional module over $\mathbb{DC}_{\mathfrak{p}}$ with base $\{1, e_1, e_2, e_3\}$ and an 8-dimensional module over $\mathbb{C}_{\mathfrak{p}}$ with base $\{1, \varepsilon, e_1, \varepsilon e_1, e_2, \varepsilon e_2, e_3, \varepsilon e_3\}$.*

P r o o f. Let us consider $\tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \widetilde{\mathbb{Q}}_{\alpha\beta}$, where $a_i = z_{i1} + z_{i2}\varepsilon \in \mathbb{DC}_{\mathfrak{p}}$ and $z_{i1}, z_{i2} \in \mathbb{C}_{\mathfrak{p}}$ for $i = 0, 1, 2, 3$. One can easily see that $\widetilde{\mathbb{Q}}_{\alpha\beta}$ is an abelian group with respect to addition. Considering the operation $\cdot\colon\ \mathbb{DC}_{\mathfrak{p}} \times \widetilde{\mathbb{Q}}_{\alpha\beta} \to \widetilde{\mathbb{Q}}_{\alpha\beta}$, the module properties are satisfied: for all $a, b \in \mathbb{DC}_{\mathfrak{p}}$ and for all $\tilde{p}, \tilde{q} \in \widetilde{\mathbb{Q}}_{\alpha\beta}$, $a \cdot (\tilde{q} + \tilde{p}) = a \cdot \tilde{q} + a \cdot \tilde{p}$, $(a + b) \cdot \tilde{q} = a \cdot \tilde{q} + b \cdot \tilde{q}$, $(ab) \cdot \tilde{q} = a \cdot (b \cdot \tilde{q})$, $1 \cdot \tilde{q} = \tilde{q} \cdot 1 = \tilde{q}$, where 1 is multiplicative identity of $\widetilde{\mathbb{Q}}_{\alpha\beta}$. Hence $\widetilde{\mathbb{Q}}_{\alpha\beta}$ is module over $\mathbb{DC}_{\mathfrak{p}}$ with base $\{1, e_1, e_2, e_3\}$ and dimension 4. Similarly, obtaining module properties considering the operation $\cdot\colon\ \mathbb{C}_{\mathfrak{p}} \times \widetilde{\mathbb{Q}}_{\alpha\beta} \to \widetilde{\mathbb{Q}}_{\alpha\beta}$ is a simple calculation. Thus the proof is completed. □

Definition 2.2. Let $\tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$, $\tilde{p} = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3 \in \widetilde{\mathbb{Q}}_{\alpha\beta}$. The scalar product and the vector product in $\widetilde{\mathbb{Q}}_{\alpha\beta}$ are defined as follows, respectively:

$$\begin{aligned}(2.3)\qquad & \widetilde{\mathbb{Q}}_{\alpha\beta} \times \widetilde{\mathbb{Q}}_{\alpha\beta} \to \mathbb{DC}_{\mathfrak{p}}, \\ & (\tilde{q}, \tilde{p}) \mapsto \langle \tilde{q}, \tilde{p} \rangle_g = S_{\tilde{q}} S_{\tilde{p}} + \langle V_{\tilde{q}}, V_{\tilde{p}} \rangle_g \\ & \qquad = a_0 b_0 + \alpha a_1 b_1 + \beta a_2 b_2 + \alpha\beta a_3 b_3 = S_{\tilde{q}\bar{\tilde{p}}},\end{aligned}$$

$$\begin{aligned}(2.4)\qquad & \widetilde{\mathbb{Q}}_{\alpha\beta} \times \widetilde{\mathbb{Q}}_{\alpha\beta} \to \widetilde{\mathbb{Q}}_{\alpha\beta}, \\ & (\tilde{q}, \tilde{p}) \mapsto \tilde{q} \times_g \tilde{p} = S_{\tilde{q}} V_{\bar{\tilde{p}}} + S_{\bar{\tilde{p}}} V_{\tilde{q}} - V_{\tilde{q}} \times_g V_{\tilde{p}},\end{aligned}$$

where $\langle\ ,\ \rangle_g$ is a generalized scalar product and $\times_g$ is a generalized vector product[2].

Lemma 2.1. *For all $\tilde{q}, \tilde{p} \in \widetilde{\mathbb{Q}}_{\alpha\beta}$, $\tilde{q}\bar{\tilde{p}} = \langle \tilde{q}, \tilde{p} \rangle_g + \tilde{q} \times_g \tilde{p}$.*

P r o o f. It is clear that

$$\begin{aligned}\tilde{q}\bar{\tilde{p}} &= (a_0 b_0 + \alpha a_1 b_1 + \beta a_2 b_2 + \alpha\beta a_3 b_3) + (-a_0 b_1 + a_1 b_0 - \beta a_2 b_3 + \beta a_3 b_2) e_1 \\ &\quad + (-a_0 b_2 + \alpha a_1 b_3 + a_2 b_0 - \alpha a_3 b_1) e_2 + (-a_0 b_3 - a_1 b_2 + a_2 b_1 + a_3 b_0) e_3 \\ &= \underbrace{S_{\tilde{q}} S_{\tilde{p}} + \langle V_{\tilde{q}}, V_{\tilde{p}} \rangle_g}_{\langle \tilde{q}, \tilde{p} \rangle_g} + \underbrace{S_{\tilde{q}} V_{\bar{\tilde{p}}} + S_{\bar{\tilde{p}}} V_{\tilde{q}} - V_{\tilde{q}} \times_g V_{\tilde{p}}}_{\tilde{q} \times_g \tilde{p}}.\end{aligned}$$

□

Definition 2.3. For any $\tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \widetilde{\mathbb{Q}}_{\alpha\beta}$, the norm operation of $\tilde{q}$ is defined by:

$$\begin{aligned}(2.5)\qquad & N\colon\ \widetilde{\mathbb{Q}}_{\alpha\beta} \times \widetilde{\mathbb{Q}}_{\alpha\beta} \to \mathbb{DC}_{\mathfrak{p}}, \\ & \tilde{q} \mapsto N_{\tilde{q}} = \tilde{q}\bar{\tilde{q}} = \bar{\tilde{q}}\tilde{q} = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2.\end{aligned}$$

[2] For a more general description of the generalized inner and cross product, see [24].

Definition 2.4. For any $\tilde{q} = a_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \widetilde{\mathbb{Q}}_{\alpha\beta}$, the inverse of $\tilde{q}$ is defined by:

$$\widetilde{\mathbb{Q}}_{\alpha\beta} \to \widetilde{\mathbb{Q}}_{\alpha\beta},$$
$$\tilde{q} \mapsto (\tilde{q})^{-1} = \frac{\bar{\tilde{q}}}{N_{\tilde{q}}},$$

where $N_{\tilde{q}}$ is non-null[3] number.

For the elements of $\widetilde{\mathbb{Q}}_{\alpha\beta}$, let us present the following properties, which are properties analogous to the properties of alternative real Cayley-Dickson algebras:

Proposition 2.1. *Let $\tilde{q}, \tilde{p} \in \widetilde{\mathbb{Q}}_{\alpha\beta}$ and $c_1, c_2 \in \mathbb{R}$. Then the basic properties of conjugation and the norm can be given as follows:*

(i) $\bar{\bar{\tilde{q}}} = \tilde{q}$,
(ii) $\overline{c_1\tilde{p} + c_2\tilde{q}} = c_1\bar{\tilde{p}} + c_2\bar{\tilde{q}}$,
(iii) $\overline{\tilde{q}\tilde{p}} = \bar{\tilde{p}}\bar{\tilde{q}}$,
(iv) $N_{c_1\tilde{q}} = c_1^2 N_{\tilde{q}}$,
(v) $N_{\tilde{q}\tilde{p}} = N_{\tilde{q}}N_{\tilde{p}}$.

Proof. Let $\tilde{q} = a_0 + a_1e_1 + a_2e_2 + a_3e_3$, $\tilde{p} = b_0 + b_1e_1 + b_2e_2 + b_3e_3 \in \widetilde{\mathbb{Q}}_{\alpha\beta}$. Considering equation (2.1), items (i) and (ii) are obvious.

(iii) Taking the conjugate of equation (2.2), we get:

$$\begin{aligned}\overline{\tilde{q}\tilde{p}} &= (a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha\beta a_3b_3) - (a_0b_1 + a_1b_0 + \beta a_2b_3 - \beta a_3b_2)e_1 \\ &\quad - (a_0b_2 - \alpha a_1b_3 + a_2b_0 + \alpha a_3b_1)e_2 - (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)e_3.\end{aligned}$$

Thus, it is clear that $\overline{\tilde{q}\tilde{p}} = \bar{\tilde{p}}\bar{\tilde{q}}$.

(iv) Using item (ii) and equation (2.5), we have: $N_{c_1\tilde{q}} = (c_1\tilde{q})\overline{(c_1\tilde{q})} = c_1^2 N_{\tilde{q}}$.

(v) Having item (iii) and equation (2.5), we find:

$$N_{\tilde{q}\tilde{p}} = (\tilde{q}\tilde{p})\overline{(\tilde{q}\tilde{p})} = \tilde{q}\tilde{p}\bar{\tilde{p}}\bar{\tilde{q}} = N_{\tilde{q}}N_{\tilde{p}}.$$

□

Proposition 2.2. *For any $\tilde{q}$, $\tilde{p}$ and $\tilde{r} \in \widetilde{\mathbb{Q}}_{\alpha\beta}$, the inner product possesses the following properties:*

(i) $\langle \tilde{r}\tilde{q}, \tilde{r}\tilde{p}\rangle_g = N_{\tilde{r}}\langle \tilde{q}, \tilde{p}\rangle_g$,
(ii) $\langle \tilde{q}\tilde{r}, \tilde{p}\tilde{r}\rangle_g = N_{\tilde{r}}\langle \tilde{q}, \tilde{p}\rangle_g$,
(iii) $\langle \tilde{r}\tilde{q}, \tilde{p}\rangle_g = \langle \tilde{q}, \bar{\tilde{r}}\tilde{p}\rangle_g$,
(iv) $\langle \tilde{q}\tilde{r}, \tilde{p}\rangle_g = \langle \tilde{q}, \tilde{p}\bar{\tilde{r}}\rangle_g$.

[3] Null numbers are characterized by having zero norm in $\mathbb{DC}_{\mathfrak{p}}$.

Proof. Let $\tilde{q} = a_0 + a_1e_1 + a_2e_2 + a_3e_3$, $\tilde{p} = b_0 + b_1e_1 + b_2e_2 + b_3e_3$ and $\tilde{r} = c_0 + c_1e_1 + c_2e_2 + c_3e_3 \in \widetilde{\mathbb{Q}}_{\alpha\beta}$. Using equations (2.3) and (2.5), the following proofs can be given:

(i) $$\begin{aligned}\langle \tilde{r}\tilde{q}, \tilde{r}\tilde{p}\rangle_g &= S_{\tilde{r}\tilde{q}}S_{\tilde{r}\tilde{p}} + \langle V_{\tilde{r}\tilde{q}}, V_{\tilde{r}\tilde{p}}\rangle_g \\ &= (c_0a_0 - \alpha c_1a_1 - \beta c_2a_2 - \alpha\beta c_3a_3)(c_0b_0 - \alpha c_1b_1 - \beta c_2b_2 - \alpha\beta c_3b_3) \\ &\quad + \alpha(c_0a_1 + c_1a_0 + \beta c_2a_3 - \beta c_3a_2)(c_0b_1 + c_1b_0 + \beta c_2b_3 - \beta c_3b_2) \\ &\quad + \beta(c_0a_2 - \alpha c_1a_3 + c_2a_0 + \alpha c_3a_1)(c_0b_2 - \alpha c_1b_3 + c_2b_0 + \alpha c_3b_1) \\ &\quad + \alpha\beta(c_0a_3 + c_1a_2 - c_2a_1 + c_3a_0)(c_0b_3 + c_1b_2 - c_2b_1 + c_3b_0) \\ &= (c_0^2 + \alpha c_1^2 + \beta c_2^2 + \alpha\beta c_3^2)(a_0b_0 + \alpha a_1b_1 + \beta a_2b_2 + \alpha\beta a_3b_3) \\ &= N_{\tilde{r}}\langle \tilde{q}, \tilde{p}\rangle_g.\end{aligned}$$

(iv) $$\begin{aligned}\langle \tilde{q}\tilde{r}, \tilde{p}\rangle_g = S_{\tilde{q}\tilde{r}}S_{\tilde{p}} + \langle V_{\tilde{q}\tilde{r}}, V_{\tilde{p}}\rangle_g &= (a_0c_0 - \alpha a_1c_1 - \beta a_2c_2 - \alpha\beta a_3c_3)b_0 \\ &\quad + \alpha(a_0c_1 + a_1c_0 + \beta a_2c_3 - \beta a_3c_2)b_1 \\ &\quad + \beta(a_0c_2 - \alpha a_1c_3 + a_2c_0 + \alpha a_3c_1)b_2 \\ &\quad + \alpha\beta(a_0c_3 + a_1c_2 - a_2c_1 + a_3c_0)b_3\end{aligned}$$

and

$$\begin{aligned}\langle \tilde{q}, \tilde{p}\bar{\tilde{r}}\rangle_g = S_{\tilde{q}}S_{\tilde{p}\bar{\tilde{r}}} + \langle V_{\tilde{q}}, V_{\tilde{p}\bar{\tilde{r}}}\rangle_g &= a_0(b_0c_0 + \alpha b_1c_1 + \beta b_2c_2 + \alpha\beta b_3c_3) \\ &\quad + \alpha(-b_0c_1 + b_1c_0 - \beta b_2c_3 + \beta b_3c_2)a_1 \\ &\quad + \beta(-b_0c_2 + \alpha b_1c_3 + b_2c_0 - \alpha b_3c_1)a_2 \\ &\quad + \alpha\beta(-b_0c_3 - b_1c_2 + b_2c_1 + b_3c_0)a_3.\end{aligned}$$

Hence we have $\langle \tilde{q}\tilde{r}, \tilde{p}\rangle_g = \langle \tilde{q}, \tilde{p}\bar{\tilde{r}}\rangle_g$. The other items can be proved similarly. □

Example 2.1. Let us consider the following elements of $\widetilde{\mathbb{Q}}_{21}$ as generalized quaternions with $\mathcal{DGC}$ number coefficients:

(2.6) $$\begin{aligned}\tilde{q} &= (1 - J + 2\varepsilon - J\varepsilon) + (-1 + 2\varepsilon + J\varepsilon)e_1 + (J - \varepsilon - J\varepsilon)e_2 + (2\varepsilon - J\varepsilon)e_3, \\ \tilde{p} &= (1 + J + \varepsilon + J\varepsilon) + (-1 + 2J + 3\varepsilon - J\varepsilon)e_1 + (1 - J\varepsilon)e_2 + (1 - J + \varepsilon)e_3, \\ \tilde{r} &= 1 + (1 + J\varepsilon)e_2.\end{aligned}$$

For $\mathfrak{p} = 1$, we have generalized quaternions with dual-hyperbolic number coefficients. Using Definition 2.2, we obtain:

$$\langle \tilde{q}, \tilde{p}\rangle_g = 2 - 3J - \varepsilon + 2J\varepsilon,$$

and

$$\tilde{q} \times_g \tilde{p} = (3 - 5J + 4\varepsilon - J\varepsilon)e_1 + (-2 + 4J + 4\varepsilon - 11J\varepsilon)e_2 + (1 + J - 7\varepsilon + 5J\varepsilon)e_3.$$

Besides, using Proposition 2.2 item (ii), we verify:

$$\langle \tilde{q}\tilde{r}, \tilde{p}\tilde{r}\rangle_g = 4 - 6J - 8\varepsilon + 8J\varepsilon = N_{\tilde{r}}\langle \tilde{q}, \tilde{p}\rangle_g,$$

where $N_{\tilde{r}} = 2 + 2J\varepsilon$. Moreover, similar calculations can be conducted for the above generalized quaternions with hyper-dual number coefficients ($\mathfrak{p} = 0$) and dual-complex number coefficients ($\mathfrak{p} = -1$) as a particular case.

The same results can be obtained if these quaternions are rewritten as $\mathcal{DGC}$ numbers with generalized quaternion coefficients. One can see the details easily through the following remark:

Remark 2.1. For $\tilde{q} \in \widetilde{\mathbb{Q}}_{\alpha\beta}$, the following equation can be written:

$$\tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 = q_0 + q_1 J + q_2 \varepsilon + q_3 J\varepsilon,$$

where $a_j = x_{j1} + x_{j2}J + x_{j3}\varepsilon + x_{j4}J\varepsilon \in \mathbb{DC}_{\mathfrak{p}}$ and $q_{i-1} = x_{0i} + x_{1i}e_1 + x_{2i}e_2 + x_{3i}e_3 \in Q_{\alpha\beta}$, $0 \leqslant j \leqslant 3$, $1 \leqslant i \leqslant 4$. Thus, it should be noted that there is no difference between a generalized quaternion with $\mathcal{DGC}$ number coefficients and a $\mathcal{DGC}$ number with generalized quaternion coefficients.

For $\tilde{q} = q_0 + q_1 J + q_2\varepsilon + q_3 J\varepsilon$ and $\tilde{p} = p_0 + p_1 J + p_2\varepsilon + p_3 J\varepsilon$, the algebraic operations considering $\mathcal{DGC}$ number form are given as follows, respectively:

$$\begin{aligned}
\text{equality:} \quad & \tilde{p} = \tilde{q} \Leftrightarrow p_0 = q_0,\ p_1 = q_1,\ p_2 = q_2,\ p_3 = q_3,\\
\text{addition (and hence subtraction):} \quad & \tilde{p} + \tilde{q} = (p_0 + q_0) + (p_1 + q_1)J\\
& \quad + (p_2 + q_2)\varepsilon + (p_3 + q_3)J\varepsilon,\\
\text{scalar multiplication:} \quad & c\tilde{q} = cq_0 + cq_1 J + cq_2\varepsilon + cq_3 J\varepsilon,\ c \in \mathbb{R},\\
\text{multiplication:} \quad & \tilde{p}\tilde{q} = (p_0q_0 + \mathfrak{p}p_1q_1) + (p_0q_1 + p_1q_0)J\\
& \quad + (p_0q_2 + \mathfrak{p}p_1q_3 + p_2q_0 + \mathfrak{p}p_3q_1)\varepsilon\\
& \quad + (p_0q_3 + p_1q_2 + p_2q_1 + p_3q_0)J\varepsilon,\\
\text{generalized complex conjugate:} \quad & \tilde{q}^{\dagger_1} = q_0 - q_1 J + q_2\varepsilon - q_3 J\varepsilon,\\
\text{dual conjugate:} \quad & \tilde{q}^{\dagger_2} = q_0 + q_1 J - q_2\varepsilon - q_3 J\varepsilon,\\
\text{coupled conjugate:} \quad & \tilde{q}^{\dagger_3} = q_0 - q_1 J - q_2\varepsilon + q_3 J\varepsilon.
\end{aligned}$$

Hence, $\widetilde{\mathbb{Q}}_{\alpha\beta}$ is a 4-dimensional module over $\mathbb{Q}_{\alpha\beta}$ with base $\{1, J, \varepsilon, J\varepsilon\}$ and thus a 16-dimensional vector space over $\mathbb{R}$ with base

$$\{1, J, \varepsilon, J\varepsilon, e_1, Je_1, \varepsilon e_1, J\varepsilon e_1, e_2, Je_2, \varepsilon e_2, J\varepsilon e_2, e_3, Je_3, \varepsilon e_3, J\varepsilon e_3\}.$$

For $1 \leqslant i \leqslant 3$, the following norm operations for $\tilde{q}$ are defined:

generalized complex module: $N_{\tilde{q}}^{\dagger_1} = \tilde{q}\tilde{q}^{\dagger_1}$,

dual module: $N_{\tilde{q}}^{\dagger_2} = \tilde{q}\tilde{q}^{\dagger_2}$,

coupled module: $N_{\tilde{q}}^{\dagger_3} = \tilde{q}\tilde{q}^{\dagger_3}$.

Additionally, the inverse of a non-null $\tilde{q}$ is defined by:

$$(\tilde{q})^{-1}_{\dagger_i} = \frac{\tilde{q}^{\dagger_i}}{N_{\tilde{q}}^{\dagger_i}}.$$

For the elements of $\widetilde{\mathbb{Q}}_{\alpha\beta}$, the following properties, which are properties analogues to the properties of alternative real Cayley-Dickson algebras, are given as:

Proposition 2.3. *Let $\tilde{q}, \tilde{p} \in \widetilde{\mathbb{Q}}_{\alpha\beta}$ and $c_1, c_2 \in \mathbb{R}$. Then, for $1 \leqslant i \leqslant 3$, the properties of conjugation and the norm can be given as follows:*

(i) $(\tilde{q}^{\dagger_i})^{\dagger_i} = \tilde{q}$,

(ii) $(c_1\tilde{q} \pm c_2\tilde{p})^{\dagger_i} = c_1\tilde{q}^{\dagger_i} \pm c_2\tilde{p}^{\dagger_i}$,

(iii) $(\tilde{q}\tilde{p})^{\dagger_i} \neq \tilde{p}^{\dagger_i}\tilde{q}^{\dagger_i}$ *in general*,

(iv) $\tilde{q} + \tilde{q}^{\dagger_1} = 2(q_0 + q_2\varepsilon)$,

(v) $\tilde{q} + \tilde{q}^{\dagger_2} = 2(q_0 + q_1 J)$,

(vi) $\tilde{q} + \tilde{q}^{\dagger_3} = 2(q_0 + q_3 J\varepsilon)$,

(vii) $\tilde{q} - \varepsilon\tilde{q}^{\dagger_4} = q_0 + q_1 J$,

(viii) $\varepsilon\tilde{q} + \tilde{q}^{\dagger_4} = q_2 + q_3 J$,

(ix) $N_{c_1\tilde{q}}^{\dagger_i} = {c_1}^2 N_{\tilde{q}}^{\dagger_i}$,

(x) $N_{\tilde{q}\tilde{p}}^{\dagger_i} \neq N_{\tilde{q}}^{\dagger_i} N_{\tilde{p}}^{\dagger_i}$ *in general.*

Proof. Let us consider $\tilde{q} = (1 + e_1) + J$ and $\tilde{p} = (1 + e_2) + J\varepsilon$ for items (iii) and (x).

(iii) One can easily see the following:

$$\tilde{q}\tilde{p} = (1 + e_1 + e_2 + e_3) + (1 + e_2)J + \mathfrak{p}\varepsilon + (1 + e_1)J\varepsilon, \tag{2.7}$$

$$(\tilde{q}\tilde{p})^{\dagger_2} = (1 + e_1 + e_2 + e_3) + (1 + e_2)J - \mathfrak{p}\varepsilon - (1 + e_1)J\varepsilon, \tag{2.8}$$

and

$$\tilde{p}^{\dagger_2}\tilde{q}^{\dagger_2} = (1 + e_1 + e_2 - e_3) + (1 + e_2)J - \mathfrak{p}\varepsilon - (1 + e_1)J\varepsilon.$$

So $(\tilde{q}\tilde{p})^{\dagger_2} \neq \tilde{p}^{\dagger_2}\tilde{q}^{\dagger_2}$. Since the generalized quaternions are non-commutative, we also get $(\tilde{q}\tilde{p})^{\dagger_i} \neq \tilde{p}^{\dagger_i}\tilde{q}^{\dagger_i}$ for $i = 1, 3$.

(x) By substituting equations (2.7) and (2.8) into $N_{\tilde{q}\tilde{p}}^{\dagger_2} = (\tilde{q}\tilde{p})(\tilde{q}\tilde{p})^{\dagger_2}$, we have:

$$\begin{aligned} N_{\tilde{q}\tilde{p}}^{\dagger_2} &= (1 + e_1 + e_2 + e_3)^2 + \mathfrak{p}(1 + e_2)^2 + 2(1 - \beta + e_1 + 2e_2 + e_3)J \\ &\quad + 2\mathfrak{p}\varepsilon + 2(e_3 - \alpha e_2)J\varepsilon \end{aligned}$$

and

$$N_{\tilde{q}}^{\dagger_2} N_{\tilde{p}}^{\dagger_2} = ((1+e_1)+J)((1+e_1)+J)((1+e_2)+J\varepsilon)((1+e_2)-J\varepsilon)$$
$$= ((1+e_1)^2+\mathfrak{p})(1+e_2)^2 + 2(1+e_1)(1+e_2)^2 J.$$

Hence $N_{\tilde{q}\tilde{p}}^{\dagger_2} \neq N_{\tilde{q}}^{\dagger_2} N_{\tilde{p}}^{\dagger_2}$. Since the generalized quaternions are non-commutative, $N_{\tilde{q}\tilde{p}}^{\dagger_i} \neq N_{\tilde{q}}^{\dagger_i} N_{\tilde{p}}^{\dagger_i}$ for $i = 1, 3$.

The proof of the other items is a simple calculation so we can omit it. □

Through the analogy between a generalized quaternion with $\mathcal{DGC}$ numbers and a $\mathcal{DGC}$ number with generalized quaternions, the following remark can be given:

Remark 2.2. Let $\tilde{q} = q_0+q_1J+q_2\varepsilon+q_3J\varepsilon$ and $\tilde{p} = p_0+p_1J+p_2\varepsilon+p_3J\varepsilon \in \widetilde{\mathbb{Q}}_{\alpha\beta}$. The analogue of the scalar product on $\widetilde{\mathbb{Q}}_{\alpha\beta}$ is defined as follows:

$$\langle \tilde{q}, \tilde{p}\rangle_g = S_{q_0\bar{p}_0} + \mathfrak{p}S_{q_1\bar{p}_1} + (S_{q_0\bar{p}_1} + S_{q_1\bar{p}_0})J + (S_{q_0\bar{p}_2} + S_{q_2\bar{p}_0} + \mathfrak{p}(S_{q_1\bar{p}_3} + S_{q_3\bar{p}_1}))\varepsilon$$
$$+ (S_{q_0\bar{p}_3} + S_{q_1\bar{p}_2} + S_{q_2\bar{p}_1} + S_{q_3\bar{p}_0})J\varepsilon.$$

3. Matrix representations in view of generalized quaternions and $\mathcal{DGC}$ numbers

In this section, we formulate the key concepts, including matrix correspondences. The following matrix approaches provide an alternative formulation of multiplication.

Theorem 3.1. *Every generalized quaternion with $\mathcal{DGC}$ number coefficients can be represented by a 4×4 $\mathcal{DGC}$ matrix.*

Proof. Let us define the bijective linear map $f_{\tilde{q}}\colon \widetilde{\mathbb{Q}}_{\alpha\beta} \to \widetilde{\mathbb{Q}}_{\alpha\beta}$ by $f_{\tilde{q}}(\tilde{p}) = \tilde{q}\tilde{p}$ for every $\tilde{p} \in \widetilde{\mathbb{Q}}_{\alpha\beta}$. By using the equations:

$$f_{\tilde{q}}(1) = \tilde{q} = a_0 + a_1e_1 + a_2e_2 + a_3e_3,$$
$$f_{\tilde{q}}(e_1) = \tilde{q}e_1 = -\alpha a_1 + a_0e_1 + \alpha a_3e_2 - a_2e_3,$$
$$f_{\tilde{q}}(e_2) = \tilde{q}e_2 = -\beta a_2 - \beta a_3e_1 + a_0e_2 + a_1e_3,$$
$$f_{\tilde{q}}(e_3) = \tilde{q}e_3 = -\alpha\beta a_3 + \beta a_2e_1 - \alpha a_1e_2 + a_0e_3,$$

a 4×4 left $\mathcal{DGC}$ matrix representation of $\tilde{q} = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ concerning the standard basis $\{1, e_1, e_2, e_3\}$ is given by

$$\mathcal{A}_{\tilde{q}}^{l} = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}. \tag{3.1}$$

Denote by M the following subset of $\mathbb{M}_4(\mathbb{DC}_\mathfrak{p})$ as:

$$\mathsf{M} := \left\{ \mathcal{A}^l_{\tilde q} \in \mathbb{M}_4(\mathbb{DC}_\mathfrak{p}) \colon \ \mathcal{A}^l_{\tilde q} = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \right\}.$$

Hence it can be concluded that there exists a correspondence between $\widetilde{\mathbb{Q}}_{\alpha\beta}$ and M via the bijective map $\mathtt{M}\colon \widetilde{\mathbb{Q}}_{\alpha\beta} \to \mathsf{M}$, $\tilde q \mapsto \mathcal{A}^l_{\tilde q}$.

Similarly, by the bijective linear map $f_{\tilde q}(\tilde p) = \tilde p \tilde q$, 4×4 right $\mathcal{DGC}$ matrix representation of $\tilde q$ is also computed as below:

$$\mathcal{A}^r_{\tilde q} = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ a_2 & -\alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}.$$

Thus, there exists a correspondence between $\widetilde{\mathbb{Q}}_{\alpha\beta}$ and N via the bijective map $\mathtt{N}\colon \widetilde{\mathbb{Q}}_{\alpha\beta} \to \mathsf{N}$, $\tilde q \mapsto \mathcal{A}^r_{\tilde q}$ where

$$\mathsf{N} := \left\{ \mathcal{A}^r_{\tilde q} \in \mathbb{M}_4(\mathbb{DC}_\mathfrak{p}) \colon \ \mathcal{A}^r_{\tilde q} = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ a_2 & -\alpha a_3 & a_0 & \alpha a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix} \right\}.$$

The proof is completed. □

Theorem 3.2. *Every $\mathcal{DGC}$ number with generalized quaternion coefficients can be represented by a 4×4 generalized quaternion matrix.*

Proof. By considering the bijective linear map $F_{\tilde q}\colon \widetilde{\mathbb{Q}}_{\alpha\beta} \to \widetilde{\mathbb{Q}}_{\alpha\beta}$ by $F_{\tilde q}(\tilde p) = \tilde q \tilde p$ for every $\tilde p \in \widetilde{\mathbb{Q}}_{\alpha\beta}$, we have:

$$\begin{aligned} F_{\tilde q}(1) &= \tilde q = q_0 + q_1 J + q_2 \varepsilon + q_3 J\varepsilon, \\ F_{\tilde q}(J) &= \tilde q J = \mathfrak{p} q_1 + q_0 J + \mathfrak{p} q_3 \varepsilon + q_2 J\varepsilon, \\ F_{\tilde q}(\varepsilon) &= \tilde q \varepsilon = q_0 \varepsilon + q_1 J\varepsilon, \\ F_{\tilde q}(J\varepsilon) &= \tilde q J\varepsilon = \mathfrak{p} q_1 \varepsilon + q_0 J\varepsilon. \end{aligned}$$

Namely, a 4×4 generalized quaternion matrix representation of $\tilde q = q_0 + q_1 J + q_2\varepsilon + q_3 J\varepsilon$ for the standard basis $\{1, J, \varepsilon, J\varepsilon\}$ is

$$\mathcal{B}_{\tilde q} = \begin{bmatrix} q_0 & \mathfrak{p} q_1 & 0 & 0 \\ q_1 & q_0 & 0 & 0 \\ q_2 & \mathfrak{p} q_3 & q_0 & \mathfrak{p} q_1 \\ q_3 & q_2 & q_1 & q_0 \end{bmatrix}. \tag{3.2}$$

Hence there exists a correspondence between $\widetilde{\mathbb{Q}}_{\alpha\beta}$ and the set K via the bijective map $\mathtt{K}\colon\ \widetilde{\mathbb{Q}}_{\alpha\beta} \to \mathsf{K}$, $\tilde{q} \mapsto \mathcal{B}_{\tilde{q}}$ where

$$\mathsf{K} := \left\{ \mathcal{B}_{\tilde{q}} \in \mathbb{M}_4(Q_{\alpha\beta})\colon\ \mathcal{B}_{\tilde{q}} = \begin{bmatrix} q_0 & \mathfrak{p}q_1 & 0 & 0 \\ q_1 & q_0 & 0 & 0 \\ q_2 & \mathfrak{p}q_3 & q_0 & \mathfrak{p}q_1 \\ q_3 & q_2 & q_1 & q_0 \end{bmatrix} \right\}.$$

□

Corollary 3.1. *Let $\tilde{q} \in \widetilde{\mathbb{Q}}_{\alpha\beta}$. Then, the following statements can be given:*

(i) *The left $\mathcal{DGC}$ matrix representation of $\tilde{q} = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ can be determined in the following form:*

$$\mathcal{A}^l_{\tilde{q}} = a_0I_4 + a_1E^l_1 + a_2E^l_2 + a_3E^l_3,$$

where

$$e_1 \mapsto E^l_1 = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad e_2 \mapsto E^l_2 = \begin{bmatrix} 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$e_3 \mapsto E^l_3 = \begin{bmatrix} 0 & 0 & 0 & -\alpha\beta \\ 0 & 0 & -\beta & 0 \\ 0 & \alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The right $\mathcal{DGC}$ matrix representation of $\tilde{q} = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ can be determined in the following form:

$$\mathcal{A}^r_{\tilde{q}} = a_0I_4 + a_1E^r_1 + a_2E^r_2 + a_3E^r_3,$$

where

$$e_1 \mapsto E^r_1 = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad e_2 \mapsto E^r_2 = \begin{bmatrix} 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\beta \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$e_3 \mapsto E^r_3 = \begin{bmatrix} 0 & 0 & 0 & -\alpha\beta \\ 0 & 0 & \beta & 0 \\ 0 & -\alpha & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(ii) *The generalized quaternion matrix representation of* $\tilde{q} = q_0 + q_1 J + q_2\varepsilon + q_3 J\varepsilon$, *is also in the form*

$$\mathcal{B}_{\tilde{q}} = q_0 I_4 + q_1\mathcal{J} + q_2\mathcal{E} + q_3\mathcal{J}\mathcal{E},$$

where

$$J \mapsto \mathcal{J} = \begin{bmatrix} 0 & \mathfrak{p} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathfrak{p} \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \varepsilon \mapsto \mathcal{E} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$J\varepsilon \mapsto \mathcal{J}\mathcal{E} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathfrak{p} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

P r o o f. Taking into account the bijective linear maps f and F, it is obvious that $\mathcal{A}_1^l = \mathcal{A}_1^r = \mathcal{B}_1 = I_4$, $\mathcal{A}_{e_i}^l = E_i^l$, $\mathcal{A}_{e_i}^r = E_i^r$ and $\mathcal{B}_J = \mathcal{J}$, $\mathcal{B}_\varepsilon = \mathcal{E}$, $\mathcal{B}_{J\varepsilon} = \mathcal{J}\mathcal{E}$ where $i = 1, 2, 3$. □

After this part, the representation $\mathcal{A}_{\tilde{q}}^l$ will be considered and similar computations can be given for $\mathcal{A}_{\tilde{q}}^r$.

Corollary 3.2. *The column matrix representation of* $\tilde{p} = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3 \in \widetilde{\mathbb{Q}}_{\alpha\beta}$ *with respect to the basis* $\{1, e_1, e_2, e_3\}$ *is given by:*

$$\tilde{p} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix}^\top.$$

Using the matrix in equation (3.1), *the multiplication of* $\tilde{q}, \tilde{p} \in \widetilde{\mathbb{Q}}_{\alpha\beta}$ *can also be expressed by:*

$$\tilde{q}\tilde{p} = \mathcal{A}_{\tilde{q}}^l \tilde{p} = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Moreover, using equation (3.2) *and* $\tilde{p} = \begin{bmatrix} p_0 & p_1 & p_2 & p_3 \end{bmatrix}^\top$, *we have:*

$$\tilde{q}\tilde{p} = \begin{bmatrix} q_0 & \mathfrak{p}q_1 & 0 & 0 \\ q_1 & q_0 & 0 & 0 \\ q_2 & \mathfrak{p}q_3 & q_0 & \mathfrak{p}q_1 \\ q_3 & q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

By writing $\tilde{q} \in \widetilde{\mathbb{Q}}_{\alpha\beta}$ as in the form $\tilde{q} = (a_0 + a_1 e_1) + (a_2 + a_3 e_1)e_2$, we can state the following:

Proposition 3.1. *Let $\tilde{q} = (a_0 + a_1 e_1) + (a_2 + a_3 e_1)e_2 \in \widetilde{\mathbb{Q}}_{\alpha\beta}$. Then, we have*

$$\sigma \mathcal{A}^l_{\tilde{q}} \sigma = \mathcal{A}^l_{\tilde{q}^*},$$

where $\sigma = \operatorname{diag}(1, 1, -1, -1)$ and $\tilde{q}^* = (a_0 + a_1 e_1) - (a_2 + a_3 e_1)e_2 \in \widetilde{\mathbb{Q}}_{\alpha\beta}$.

Proof. It is clear that

$$\begin{aligned}
\sigma \mathcal{A}^l_{\tilde{q}} \sigma &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} a_0 & -\alpha a_1 & \beta a_2 & \alpha\beta a_3 \\ a_1 & a_0 & \beta a_3 & -\beta a_2 \\ -a_2 & -\alpha a_3 & a_0 & -\alpha a_1 \\ -a_3 & a_2 & a_1 & a_0 \end{bmatrix}.
\end{aligned}$$

Hence, the last matrix corresponds to $\mathcal{A}^l_{\tilde{q}^*}$. □

Standard elementary matrix operations establish the following theorems.

Theorem 3.3. *For any $\tilde{q}, \tilde{p} \in \widetilde{\mathbb{Q}}_{\alpha\beta}$ and $\lambda \in \mathbb{R}$, the following properties are satisfied:*

(i) $\tilde{q} = \tilde{p} \Leftrightarrow \mathcal{A}^l_{\tilde{q}} = \mathcal{A}^l_{\tilde{p}}$,
(ii) $\mathcal{A}^l_{\lambda\tilde{q}} = \lambda(\mathcal{A}^l_{\tilde{q}})$,
(iii) $\mathcal{A}^l_{\tilde{q}\tilde{p}} = \mathcal{A}^l_{\tilde{q}}\mathcal{A}^l_{\tilde{p}}$,
(iv) $\tilde{q} = \tilde{p} \Leftrightarrow \mathcal{B}_{\tilde{q}} = \mathcal{B}_{\tilde{p}}$,
(v) $\mathcal{B}_{\lambda\tilde{q}} = \lambda(\mathcal{B}_{\tilde{q}})$,
(vi) $\mathcal{B}_{\tilde{q}\tilde{p}} = \mathcal{B}_{\tilde{q}}\mathcal{B}_{\tilde{p}}$.

Theorem 3.4. *Let $\bar{\tilde{q}}$ be the conjugate and $\tilde{q}^{-1}$ be the inverse of non-null $\tilde{q} \in \widetilde{\mathbb{Q}}_{\alpha\beta}$. Then,*

$$\mathcal{A}^l_{\tilde{q}^{-1}} = \frac{1}{\sqrt{\det(\mathcal{A}^l_{\tilde{q}})}} \mathcal{A}^l_{\bar{\tilde{q}}},$$

where $\det(\mathcal{A}^l_{\tilde{q}}) = (a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2)^2 = N^2_{\tilde{q}}$.

Proof. By considering Definition 2.4 and Theorem 3.3 item (ii), the proof is clear. □

Let us define a valuable construction of the vector representation of $\tilde{q}$ and give its properties.

Definition 3.1. Let $\tilde{q} = q_0 + q_1 J + q_2 \varepsilon + q_3 J\varepsilon \in \widetilde{\mathbb{Q}}_{\alpha\beta}$. The vector representation of $\tilde{q}$ is defined as:

$$\vec{\tilde{q}} = [\,\vec{q}_0^{\top} \quad \vec{q}_1^{\top} \quad \vec{q}_2^{\top} \quad \vec{q}_3^{\top}\,]^{\top} = \begin{bmatrix} \vec{q}_0 \\ \vec{q}_1 \\ \vec{q}_2 \\ \vec{q}_3 \end{bmatrix} \in \mathbb{M}_{16\times 1}(\mathbb{R}),$$

where $q_{i-1} = x_{0i} + x_{1i}e_1 + x_{2i}e_2 + x_{3i}e_3 \in Q_{\alpha\beta}$ and $\vec{q}_{i-1} = (x_{0i}, x_{1i}, x_{2i}, x_{3i})^{\top}$ are vectors for $1 \leqslant i \leqslant 4$.

Theorem 3.5. *Let* $\tilde{q} = q_0 + q_1 J + q_2 \varepsilon + q_3 J\varepsilon \in \widetilde{\mathbb{Q}}_{\alpha\beta}$. *Then,*

(i) $\mathcal{X}\vec{\tilde{q}} = \vec{\tilde{q}^{\dagger_1}}$,
(ii) $\mathcal{Y}\vec{\tilde{q}} = \vec{\tilde{q}^{\dagger_2}}$,
(iii) $\mathcal{Z}\vec{\tilde{q}} = \vec{\tilde{q}^{\dagger_3}}$,

where

$$\begin{aligned} \mathcal{X} &= \operatorname{diag}(1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1) \in \mathbb{M}_{16}(\mathbb{R}), \\ \mathcal{Y} &= \operatorname{diag}(1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1) \in \mathbb{M}_{16}(\mathbb{R}), \\ \mathcal{Z} &= \operatorname{diag}(1,1,1,1,-1,-1,-1-1,-1,-1,-1,-1,1,1,1,1) \in \mathbb{M}_{16}(\mathbb{R}). \end{aligned}$$

P r o o f. For $1 \leqslant i \leqslant 4$, let us consider $\tilde{q} = q_0 + q_1 J + q_2 \varepsilon + q_3 J\varepsilon \in \widetilde{\mathbb{Q}}_{\alpha\beta}$, where $q_{i-1} = x_{0i} + x_{1i}e_1 + x_{2i}e_2 + x_{3i}e_3 \in Q_{\alpha\beta}$ and $\vec{q}_{i-1} = (x_{0i}, x_{1i}, x_{2i}, x_{3i})^{\top}$. Taking $\mathcal{Y} = \operatorname{diag}(1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1) \in \mathbb{M}_{16}(\mathbb{R})$, we have:

$$\mathcal{Y}\vec{\tilde{q}} = \begin{bmatrix} I_4 & 0 & 0 & 0 \\ 0 & I_4 & 0 & 0 \\ 0 & 0 & -I_4 & 0 \\ 0 & 0 & 0 & -I_4 \end{bmatrix} \begin{bmatrix} [\,x_{01} \quad x_{11} \quad x_{21} \quad x_{31}\,]^{\top} \\ [\,x_{02} \quad x_{12} \quad x_{22} \quad x_{32}\,]^{\top} \\ [\,x_{03} \quad x_{13} \quad x_{23} \quad x_{33}\,]^{\top} \\ [\,x_{04} \quad x_{14} \quad x_{24} \quad x_{34}\,]^{\top} \end{bmatrix}.$$

It is clear that multiplication gives $\vec{\tilde{q}^{\dagger_2}}$. The other items can be proved similarly. □

Theorem 3.6. *Every $\mathcal{DGC}$ number with generalized quaternion coefficients can be represented by an 8×8 generalized complex number matrix.*

P r o o f. Let us consider $\tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \widetilde{\mathbb{Q}}_{\alpha\beta}$, where $a_i = z_{i1} + z_{i2}\varepsilon \in \mathbb{DC}_{\mathfrak{p}}$ and $z_{i1}, z_{i2} \in \mathbb{C}_{\mathfrak{p}}$ for $i = 0, 1, 2, 3$. Applying the bijective map

$$\gamma(a_i) = \begin{bmatrix} z_{i1} & 0 \\ z_{i2} & z_{i1} \end{bmatrix},$$

which is from $\mathcal{DGC}$ numbers to the subset of 2×2 generalized complex number matrices, into equation (3.1), we can write:

$$\begin{bmatrix} \gamma(a_0) & \gamma(-\alpha a_1) & \gamma(-\beta a_2) & \gamma(-\alpha\beta a_3) \\ \gamma(a_1) & \gamma(a_0) & \gamma(-\beta a_3) & \gamma(\beta a_2) \\ \gamma(a_2) & \gamma(\alpha a_3) & \gamma(a_0) & \gamma(-\alpha a_1) \\ \gamma(a_3) & \gamma(-a_2) & \gamma(a_1) & \gamma(a_0) \end{bmatrix}. \tag{3.3}$$

It follows that

$$\begin{bmatrix} z_{01} & 0 & -\alpha z_{11} & 0 & -\beta z_{21} & 0 & -\alpha\beta z_{31} & 0 \\ z_{02} & z_{01} & -\alpha z_{12} & -\alpha z_{11} & -\beta z_{22} & -\beta z_{21} & -\alpha\beta z_{32} & -\alpha\beta z_{31} \\ z_{11} & 0 & z_{01} & 0 & \beta z_{31} & 0 & \beta z_{21} & 0 \\ z_{12} & z_{11} & z_{02} & z_{01} & -\beta z_{32} & \beta z_{31} & \beta z_{22} & \beta z_{21} \\ z_{21} & 0 & \alpha z_{31} & 0 & z_{01} & 0 & -\alpha z_{11} & 0 \\ z_{22} & z_{21} & \alpha z_{32} & \alpha z_{31} & z_{02} & z_{01} & -\alpha z_{12} & -\alpha z_{11} \\ z_{31} & 0 & -z_{21} & 0 & z_{11} & 0 & z_{01} & 0 \\ z_{32} & z_{31} & -z_{22} & -z_{21} & z_{12} & z_{11} & z_{02} & z_{01} \end{bmatrix}. \tag{3.4}$$

This is a representation of $\tilde{q}$ with respect to the base $\{1, \varepsilon, e_1, \varepsilon e_1, e_2, \varepsilon e_2, e_3, \varepsilon e_3\}$. It is called the left generalized complex matrix representation of $\tilde{q}$ and denoted by $\mathcal{C}^l_{\tilde{q}}$. □

Theorem 3.7. *Every $\mathcal{DGC}$ number with generalized quaternion coefficients can be represented by a 16×16 real matrix.*

P r o o f. Let us consider $\tilde{q} = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \widetilde{\mathbb{Q}}_{\alpha\beta}$, where $a_i = z_{i1} + z_{i2}\varepsilon \in \mathbb{DC}_{\mathfrak{p}}$ and $z_{i1} = x_{i1} + x_{i2}J \in \mathbb{C}_{\mathfrak{p}}$ for $i = 0, 1, 2, 3$. We have the following 16×16 real matrix:

(3.5)

$$\begin{bmatrix} \Gamma(z_{01}) & \Gamma(0) & \Gamma(-\alpha z_{11}) & \Gamma(0) & \Gamma(-\beta z_{21}) & \Gamma(0) & \Gamma(-\alpha\beta z_{31}) & \Gamma(0) \\ \Gamma(z_{02}) & \Gamma(z_{01}) & \Gamma(-\alpha z_{12}) & \Gamma(-\alpha z_{11}) & \Gamma(-\beta z_{22}) & \Gamma(-\beta z_{21}) & \Gamma(-\alpha\beta z_{32}) & \Gamma(-\alpha\beta z_{31}) \\ \Gamma(z_{11}) & \Gamma(0) & \Gamma(z_{01}) & \Gamma(0) & \Gamma(\beta z_{31}) & \Gamma(0) & \Gamma(\beta z_{21}) & \Gamma(0) \\ \Gamma(z_{12}) & \Gamma(z_{11}) & \Gamma(z_{02}) & \Gamma(z_{01}) & \Gamma(-\beta z_{32}) & \Gamma(\beta z_{31}) & \Gamma(\beta z_{22}) & \Gamma(\beta z_{21}) \\ \Gamma(z_{21}) & \Gamma(0) & \Gamma(\alpha z_{31}) & \Gamma(0) & \Gamma(z_{01}) & \Gamma(0) & \Gamma(-\alpha z_{11}) & \Gamma(0) \\ \Gamma(z_{22}) & \Gamma(z_{21}) & \Gamma(\alpha z_{32}) & \Gamma(\alpha z_{31}) & \Gamma(z_{02}) & \Gamma(z_{01}) & \Gamma(-\alpha z_{12}) & \Gamma(-\alpha z_{11}) \\ \Gamma(z_{31}) & \Gamma(0) & \Gamma(-z_{21}) & \Gamma(0) & \Gamma(z_{11}) & \Gamma(0) & \Gamma(z_{01}) & \Gamma(0) \\ \Gamma(z_{32}) & \Gamma(z_{31}) & \Gamma(-z_{22}) & \Gamma(-z_{21}) & \Gamma(z_{12}) & \Gamma(z_{11}) & \Gamma(z_{02}) & \Gamma(z_{01}) \end{bmatrix},$$

which is computed by applying the bijective map $\Gamma(z_{i1}) = \begin{bmatrix} x_{i1} & \mathfrak{p} x_{i2} \\ x_{i2} & x_{i1} \end{bmatrix}$ into equation (3.4). Here Γ is defined from generalized complex numbers to the subset of 2×2 real matrices. Equation (3.5) is representation of $\tilde{q}$ with respect to the base

$$\{1, J, \varepsilon, J\varepsilon, e_1, Je_1, \varepsilon e_1, J\varepsilon e_1, e_2, Je_2, \varepsilon e_2, J\varepsilon e_2, e_3, Je_3, \varepsilon e_3, J\varepsilon e_3\}.$$

It is called the left real matrix representation of $\tilde{q}$ and denoted by $\mathcal{D}^l_{\tilde{q}}$. □

Example 3.1. Let

$$\tilde{q} = (18 + 6J + 8\varepsilon) + (-2 + 9\varepsilon)e_1 + (-7 + 3\varepsilon + 8J\varepsilon)e_2 + (19 + J - \varepsilon + 3J\varepsilon)e_3$$

be a generalized quaternion in $\widetilde{\mathbb{Q}}_{32}$ and $\mathfrak{p} = \frac{1}{2}$. Then,

$$\mathcal{A}^l_{\tilde{q}} = \begin{bmatrix} 18+6J+8\varepsilon & -3(-2+9\varepsilon) & -2(-7+3\varepsilon+8J\varepsilon) & -6(19+J-\varepsilon+3J\varepsilon) \\ -2+9\varepsilon & 18+6J+8\varepsilon & -2(19+J-\varepsilon+3J\varepsilon) & 2(-7+3\varepsilon+8J\varepsilon) \\ -7+3\varepsilon+8J\varepsilon & 3(19+J-\varepsilon+3J\varepsilon) & 18+6J+8\varepsilon & -3(-2+9\varepsilon) \\ 19+J-\varepsilon+3J\varepsilon & -(-7+3\varepsilon+8J\varepsilon) & -2+9\varepsilon & 18+6J+8\varepsilon \end{bmatrix},$$

$$\mathcal{B}_{\tilde{q}} = \begin{bmatrix} 18-2e_1-7e_2+19e_3 & \frac{1}{2}(6+e_3) & 0 & 0 \\ 6+e_3 & 18-2e_1-7e_2+19e_3 & 0 & 0 \\ 8+9e_1+3e_2-e_3 & \frac{1}{2}(8e_2+3e_3) & 18-2e_1-7e_2+19e_3 & \frac{1}{2}(6+e_3) \\ 8e_2+3e_3 & 8+9e_1+3e_2-e_3 & 6+e_3 & 18-2e_1-7e_2+19e_3 \end{bmatrix},$$

$$\mathcal{C}^l_{\tilde{q}} = \begin{bmatrix} 18+6J & 0 & 6 & 0 & 14 & 0 & -114-6J & 0 \\ 8 & 18+6J & -27 & 6 & -6-16J & 14 & 6-18J & -114-6J \\ -2 & 0 & 18+6J & 0 & -38-2J & 0 & -14 & 0 \\ 9 & -2 & 8 & 18+6J & 2-6J & -38-2J & 6+16J & -14 \\ -7 & 0 & 57+3J & 0 & 18+6J & 0 & 6 & 0 \\ 3+8J & -7 & -3+9J & 57+3J & 8 & 18+6J & -27 & 6 \\ 19+J & 0 & 7 & 0 & -2 & 0 & 18+6J & 0 \\ -1+3J & 19+J & -3-8J & 7 & 9 & -2 & 8 & 18+6J \end{bmatrix},$$

$$\mathcal{D}^l_{\tilde{q}} = \begin{bmatrix} 18 & 3 & 0 & 0 & 6 & 0 & 0 & 0 & 14 & 0 & 0 & 0 & -114 & -6 & 0 & 0 \\ 6 & 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & -114 & 0 & 0 \\ 8 & 0 & 18 & 3 & -27 & 0 & 6 & 0 & -6 & -16 & 14 & 0 & 6 & -9 & -114 & -3 \\ 0 & 0 & 6 & 18 & 0 & 0 & 0 & 0 & -16 & -6 & 0 & 0 & -18 & 6 & -6 & -114 \\ -2 & 0 & 0 & 0 & 18 & 3 & 0 & 0 & -38 & -1 & 0 & 0 & -14 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 18 & 0 & 0 & -2 & -38 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & -2 & 0 & 8 & 0 & 18 & 3 & 2 & -3 & -38 & -1 & 6 & 8 & -14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 18 & -6 & 2 & -2 & -38 & 16 & 6 & 0 & 0 \\ -7 & 0 & 0 & 0 & 57 & \frac{3}{2} & 0 & 0 & 18 & 3 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 57 & 0 & 0 & 6 & 18 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & -7 & 0 & -3 & \frac{9}{2} & 57 & 1 & 8 & 0 & 18 & 3 & -27 & 0 & 6 & 0 \\ 8 & 3 & 0 & 0 & 9 & -3 & 3 & 57 & 0 & 0 & 6 & 18 & 0 & 0 & 0 & 0 \\ 19 & \frac{1}{2} & 0 & 0 & 7 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 18 & 3 & 0 & 0 \\ 1 & 19 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 18 & 0 & 0 \\ -1 & \frac{3}{2} & 19 & \frac{1}{2} & -3 & -4 & 7 & 0 & 9 & 0 & -2 & 0 & 8 & 0 & 18 & 3 \\ 3 & -1 & 1 & 19 & -8 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 18 \end{bmatrix}.$$

Moreover, the matrix $\mathcal{A}^l_{\tilde{q}^{-1}}$ is as follows:

$$\frac{1}{\sqrt{\det(\mathcal{A}^l_{\tilde{q}})}}\begin{bmatrix} 18+6J+8\varepsilon & 3(-2+9\varepsilon) & 2(-7+3\varepsilon+8J\varepsilon) & 6(19+J-\varepsilon+3J\varepsilon) \\ 2-9\varepsilon & 18+6J+8\varepsilon & 2(19+J-\varepsilon+3J\varepsilon) & -2(-7+3\varepsilon+8J\varepsilon) \\ 7-3\varepsilon-8J\varepsilon & -3(19+J-\varepsilon+3J\varepsilon) & -18-6J-8\varepsilon & 3(-2+9\varepsilon) \\ -19-J+\varepsilon-3J\varepsilon & (-7+3\varepsilon+8J\varepsilon) & 18+6J+8\varepsilon & 18+6J+8\varepsilon \end{bmatrix},$$

where

$$\det(\mathcal{A}^l_{\tilde{q}}) = (2621+444J-114\varepsilon+544J\varepsilon)^2.$$

Also, the vector representation of $\tilde{q}^{\dagger_2}$ is computed by:

$$\begin{aligned} \vec{\tilde{q}^{\dagger_2}} = \mathcal{Y}\vec{\tilde{q}} &= \begin{bmatrix} I_4 & 0 & 0 & 0 \\ 0 & I_4 & 0 & 0 \\ 0 & 0 & -I_4 & 0 \\ 0 & 0 & 0 & -I_4 \end{bmatrix} \begin{bmatrix} [18 \quad -2 \quad -7 \quad 19]^\top \\ [6 \quad 0 \quad 0 \quad 1]^\top \\ [8 \quad 9 \quad 3 \quad -1]^\top \\ [0 \quad 0 \quad 8 \quad 3]^\top \end{bmatrix} \\ &= [18 \quad -2 \quad -7 \quad 19 \quad 6 \quad 0 \quad 0 \quad 1 \quad 8 \quad 9 \quad 3 \quad -1 \quad 0 \quad 0 \quad 8 \quad 3]^\top. \end{aligned}$$

4. Conclusion

This study develops the theory of generalized quaternions with $\mathcal{DGC}$ number coefficients for any real number $\mathfrak{p}$. With this purpose, the algebraic structures and properties are investigated by considering them as a generalized quaternion and as a $\mathcal{DGC}$ number. In addition, different matrix representations are investigated and examples are presented. The crucial part of this paper is that one can find the different types of generalized quaternions included in the following Table 1:

	Type of components	Ref.
$\mathfrak{p}=-1$	dual-complex	
$\mathfrak{p}=0$	hyper-dual	see [3] (for $\alpha=1$, $\beta=-1$)
$\mathfrak{p}=1$	dual-hyperbolic	

Table 1. Classification of generalized quaternions regarding components.

Moreover, it is worth pointing out that real, split, semi, split semi, and quasi quaternions are also obtained in this study by taking special values for α and β.

Acknowledgments. The authors are grateful to the editor and the referee for the critical comments and helpful suggestions.

[1] *M. Akar, S. Yüce, S. Şahin*: On the dual hyperbolic numbers and the complex hyperbolic numbers. J. Computer Sci. Comput. Math. *8* (2018), 1–6. doi

[2] *D. Alfsmann*: On families of 2^N-dimensional hypercomplex algebras suitable for digital signal processing. 14th European Signal Processing Conference. IEEE, Piscataway, 2006, pp. 1–4.

[3] *S. Aslan, M. Bekar, Y. Yaylı*: Hyper-dual split quaternions and rigid body motion. J. Geom. Phys. *158* (2020), Article ID 103876, 12 pages. zbl MR doi

[4] *F. Catoni, D. Boccaletti, R. Cannata, V. Catoni, E. Nichelatti, P. Zampetti*: The Mathematics of Minkowski Space-Time: With an Introduction to Commutative Hypercomplex Numbers. Frontiers in Mathematics. Birkhäuser, Basel, 2008. zbl MR doi

[5] *F. Catoni, R. Cannata, V. Catoni, P. Zampetti*: Two-dimensional hypercomplex numbers and related trigonometries and geometries. Adv. Appl. Clifford Algebr. *14* (2004), 47–68. zbl MR doi

[6] *F. Catoni, R. Cannata, V. Catoni, P. Zampetti*: N-dimensional geometries generated by hypercomplex numbers. Adv. Appl. Clifford Algebr. *15* (2005), 1–25. zbl MR doi

[7] *H. H. Cheng, S. Thompson*: Dual polynomials and complex dual numbers for analysis of spatial mechanisms. Design Engineering Technical Conferences and Computers in Engineering. Conference ASME 1996. ASME, Irvine, 1996, pp. 19–22. doi

[8] *H. H. Cheng, S. Thompson*: Singularity analysis of spatial mechanisms using dual polynomials and complex dual numbers. J. Mech. Des. *121* (1999), 200–205. doi

[9] *W. K. Clifford*: Mathematical Papers. Chelsea Publishing, New York, 1968. MR

[10] *J. Cockle*: On a new imaginary in algebra. Phil. Mag. (3) *34* (1849), 37–47. doi

[11] *J. Cockle*: On systems of algebra involving more than one imaginary; and on equations of the fifth degree. Phil. Mag. (3) *35* (1849), 434–437. doi

[12] *A. Cohen, M. Shoham*: Principle of transference: An extension to hyper-dual numbers. Mech. Mach. Theory *125* (2018), 101–110. doi

[13] *L. E. Dickson*: On the theory of numbers and generalized quaternions. Am. J. Math. *46* (1924), 1–16. zbl MR doi

[14] *J. A. Fike, J. J. Alonso*: The development of hyper-dual numbers for exact second-derivative calculations. 49th AIAA Aerospace Sciences Meeting including the New Horizons Forum and Aerospace Exposition. American Institute of Aeronautics and Astronautics, New York, 2011, pp. 1–17. doi

[15] *J. A. Fike, J. J. Alonso*: Automatic differentiation through the use of hyper-dual numbers for second derivatives. Recent Advances in Algorithmic Differentiation. Lecture Notes in Computational Science and Engineering 87. Springer, Berlin, 2012, pp. 163–173. zbl MR doi

[16] *P. Fjelstad*: Extending special relativity via the perplex numbers. Am. J. Phys. *54* (1986), 416–422. MR doi

[17] *P. Fjelstad, G. S. Gal*: n-dimensional hyperbolic complex numbers. Adv. Appl. Clifford Algebr. *8* (1998), 47–68. zbl MR doi

[18] *L. W. Griffiths*: Generalized quaternion algebras and the theory of numbers. Am. J. Math. *50* (1928), 303–314. zbl MR doi

[19] *N. Gürses, G. Y. Şentürk, S. Yüce*: A study on dual-generalized complex and hyperbolic-generalized complex numbers. Gazi Univ. J. Sci. *34* (2021), 180–194. doi

[20] *W. R. Hamilton*: On quaternions; or on a new system of imaginaries in algebra. Phil. Mag. (3) *25* (1844), 10–13. doi

[21] *W. R. Hamilton*: Lectures on Quaternions. Hodges and Smith, Dublin, 1853.

[22] *W. R. Hamilton*: Elements of Quaternions. Chelsea Publishing, New York, 1969. MR doi

[23] *A. A. Harkin, J. B. Harkin*: Geometry of generalized complex numbers. Math. Mag. *77* (2004), 118–129. zbl MR doi

[24] *M. Jafari, Y. Yayli*: Generalized quaternions and their algebraic properties. Commun. Fac. Sci. Univ. Ank., Ser. A1 *64* (2015), 15–27. doi
[25] *I. L. Kantor, A. S. Solodovnikov*: Hypercomplex Numbers: An Elementary Introduction to Algebras. Springer, New York, 1989. zbl MR
[26] *V. Majerník*: Multicomponent number systems. Acta Phys. Polon. A *90* (1996), 491–498. MR doi
[27] *A. B. Mamagani, M. Jafari*: On properties of generalized quaternion algebra. J. Novel Appl. Sci. *2* (2013), 683–689.
[28] *F. Messelmi*: Dual-complex numbers and their holomorphic functions. Available at `https://hal.archives-ouvertes.fr/hal-01114178` (2015), 11 pages.
[29] *E. Pennestrì, R. Stefanelli*: Linear algebra and numerical algorithms using dual numbers. Multibody Syst. Dyn. *18* (2007), 323–344. zbl MR doi
[30] *H. Pottman, J. Wallner*: Computational Line Geometry. Mathematics and Visualization. Springer, Berlin, 2001. zbl MR doi
[31] *G. B. Price*: An Introduction to Multicomplex Spaces and Functions. Pure and Applied Mathematics 140. Marcel Dekker, New York, 1991. zbl MR doi
[32] *D. Rochon, M. Shapiro*: On algebraic properties of bicomplex and hyperbolic numbers. An. Univ. Oradea, Fasc. Mat. *11* (2004), 71–110. zbl MR
[33] *D. Savin, C. Flaut, C. Ciobanu*: Some properties of the symbol algebras. Carpathian J. Math. *25* (2009), 239–245. zbl MR
[34] *G. Sobczyk*: The hyperbolic number plane. Coll. Math. J. *26* (1995), 268–280. doi
[35] *E. Study*: Geometrie der Dynamen: Die Zusammensetzung von Kräften und verwandte Gegenstände der Geometrie. B. G. Teubner, Leipzig, 1903. (In German.) zbl
[36] *H. Toyoshima*: Computationally efficient bicomplex multipliers for digital signal processing. IEICE Trans. Inform. Syst. *E81-D* (1998), 236–238.
[37] *S. Veldsman*: Generalized complex numbers over near-fields. Quaest. Math. *42* (2019), 181–200. zbl MR doi
[38] *I. M. Yaglom*: Complex Numbers in Geometry. Academic Press, New York, 1968. zbl MR
[39] *I. M. Yaglom*: A Simple Non-Euclidean Geometry and Its Physical Basis: An Elementary Account of Galilean Geometry and the Galilean Principle of Relativity. Springer, New York, 1979. zbl MR doi

Authors' addresses: *Nurten Gürses* (corresponding author), Department of Mathematics, Faculty of Arts and Sciences, Yildiz Technical University, 34220, Istanbul, Turkey, e-mail: `nbayrak@yildiz.edu.tr`; *Gülsüm Yeliz Şentürk*, Department of Computer Engineering, Faculty of Engineering and Architecture, Istanbul Gelisim University, 34310, Istanbul, Turkey, e-mail: `gysenturk@gelisim.edu.tr`; *Salim Yüce*, Department of Mathematics, Faculty of Arts and Sciences, Yildiz Technical University, 34220, Istanbul, Turkey, e-mail: `sayuce@yildiz.edu.tr`.