V. *Integro-differential operators*

1. Fredholm-Stieltjes integro-differential operator

The most part of this chapter is devoted to the *Fredholm-Stieltjes integro-differential operators* of the form

$$\mathbf{x} \to \mathbf{x}'(t) - \int_0^1 \mathrm{d}_s[\mathbf{P}(t,s)]\,\mathbf{x}(s)\,.$$

The kernel $\mathbf{P}(t,s)$ *is assumed to be an* $n \times n$*-matrix valued function defined for a.e.* $t \in [0,1]$ *and any* $s \in [0,1]$ *and such that* $\mathbf{P}(.,s)$ *is measurable on* $[0,1]$ *for any* $s \in [0,1]$,

(1,1) $\qquad \varrho(t) = |\mathbf{P}(t,0)| + \mathrm{var}_0^1\,\mathbf{P}(t,.) = \|\mathbf{P}(t,.)\|_{BV} < \infty \qquad$ *a.e. on* $[0,1]$

and

(1,2) $$\|\varrho\|_{L^p} = \left(\int_0^1 (\varrho(t))^p\,\mathrm{d}t\right)^{1/p} < \infty\,,$$

where $1 \le p < \infty$.

Such kernels will be called $L^p[BV]$*-kernels.*

1.1. Remark. For $L^r \subset L^p$ if $p \le r$, any $L^r[BV]$-kernel is also an $L^p[BV]$-kernel for each p, $1 \le p \le r$. Furthermore

$$|\mathbf{P}(t,s)| \le |\mathbf{P}(t,0)| + |\mathbf{P}(t,s) - \mathbf{P}(t,0)| \le \varrho(t)$$

for all $s \in [0,1]$ and a.e. $t \in [0,1]$. Hence by (1,2)

$$\int_0^1 |\mathbf{P}(t,s)|^p\,\mathrm{d}t < \infty \qquad \text{for any} \quad s \in [0,1]\,.$$

1.2. Proposition. *If* $\mathbf{P}(t,s)$ *is an* $L^p[BV]$*-kernel, then the function*

$$\mathbf{Px}\colon\ t \in [0,1] \to \int_0^1 \mathrm{d}_s[\mathbf{P}(t,s)]\,\mathbf{x}(s) \in R_n$$

belongs to L^p_n *for any* $\mathbf{x} \in BV_n$ *and the operator*

$$(1,3) \qquad \mathbf{P}\colon \mathbf{x} \in BV_n \to \int_0^1 \mathrm{d}_s[\mathbf{P}(t,s)]\,\mathbf{x}(s) \in L^p_n$$

is linear and bounded.

Proof. By I.4.27 and I.4.37 $\mathbf{Px} \in L^p_n$ and

$$(1,4) \qquad |(\mathbf{Px})(t)| \le \varrho(t)\,(\sup_{s\in[0,1]} |\mathbf{x}(s)|) \qquad \text{a.e. on } [0,1]$$

for any $\mathbf{x} \in BV_n$. Since $\varrho \in L^p$ and $\sup_{s\in[0,1]} |\mathbf{x}(s)| \le \|\mathbf{x}\|_{BV}$, our assertion follows immediately.

1.3. Remark. Since (1,4) holds also for any $\mathbf{x} \in C_n$, the mapping $\mathbf{x} \to \mathbf{Px}$ is bounded as an operator $C_n \to L^p_n$, as well. Let us notice, furthermore, that if $\mathbf{x}_k, \mathbf{x} \in C_n$ $(k = 1, 2, \ldots)$ and $\lim_{k\to\infty} \|\mathbf{x}_k - \mathbf{x}\|_C = 0$, then in virtue of (1,4) $\lim_{k\to\infty} (\mathbf{Px}_k)(t) = (\mathbf{Px})(t)$ a.e. on $[0,1]$. In other words, $\mathbf{P}$ maps sequences converging uniformly on $[0,1]$ onto seuqences converging a.e. on $[0,1]$. It was shown in Kantorovič, Pinsker, Vulich [1] that

$$\mathbf{x} \in C_n \to \int_0^1 \mathrm{d}_s[\mathbf{P}(t,s)]\,\mathbf{x}(s) \in L^1_n,$$

with the $L^1[BV]$-kernel $\mathbf{P}(t,s)$, is a general form of operators $C_n \to L^1_n$ possessing this property.

1.4. Proposition. *If* $\mathbf{P}(t,s)$ *is an* $L^p[BV]$*-kernel, then the operator* $\mathbf{P}\colon BV_n \to L^p_n$ *given by* (1,3) *is compact.*

Proof. Let $\mathbf{x}_k \in BV_n$ and $\|\mathbf{x}\|_{BV} \le 1$ for each $k = 1, 2, \ldots$. By the Helly Choice Theorem the sequence $\{\mathbf{x}_k\}_{k=1}^\infty$ contains a subsequence $\{\mathbf{x}_{k_l}\}_{l=1}^\infty$ such that

$$\lim_{l\to\infty} \mathbf{x}_{k_l}(t) = \mathbf{x}(t) \qquad \text{on } [0,1]$$

for some $\mathbf{x} \in BV_n$. For $t, s \in [0,1]$ let us denote

$$p(t,s) = \mathrm{var}_0^s\, \mathbf{P}(t,.)$$

and

$$z_l(t) = \int_0^1 \mathrm{d}_s[p(t,s)]\,|\mathbf{x}_{k_l}(s) - \mathbf{x}(s)|.$$

Given $l = 1, 2, \ldots$ and $s \in [0,1]$,

$$|\mathbf{x}_{k_l}(s) - \mathbf{x}(s)| \le \|\mathbf{x}_{k_l} - \mathbf{x}\|_{BY} \le \|\mathbf{x}_{k_l}\|_{BV} + \|\mathbf{x}\|_{BV} \le 1 + \|\mathbf{x}\|_{BV} < \infty$$

and hence by I.4.27

$$|z_l(t)| \le (\mathrm{var}_0^1\, \mathbf{P}(t,.))\,(1 + \|\mathbf{x}\|_{BV}) \le (1 + \|\mathbf{x}\|_{BV})\,\varrho(t) \qquad \text{a.e. on } [0,1].$$

Moreover, according to I.4.24

$$\lim_{l\to\infty} z_l(t) = 0 \qquad \text{a.e. on } [0,1].$$

By the assumption $\varrho \in L^p$ and hence applying the classical Lebesgue Convergence Theorem we obtain

$$(1,5) \qquad \lim_{l\to\infty} \int_0^1 |z_l(t)|^p \, dt = 0.$$

Since for any $l = 1, 2, \ldots$

$$\int_0^1 \left| \int_0^1 d_s[\mathbf{P}(t,s)] \, (\mathbf{x}_{k_l}(s) - \mathbf{x}(s)) \right|^p dt \le \int_0^1 |z_l(t)|^p \, dt,$$

(1,5) implies

$$\lim_{l\to\infty} \|\mathbf{P}\mathbf{x}_{k_l} - \mathbf{P}\mathbf{x}\|_{L^p} = 0$$

and this completes the proof.

1.5. Notation. Throughout the chapter $\mathbf{P}$ denotes the operator defined by (1,3) or its restriction on W_n^p $(1 \le p < \infty)$, where W_n^p stands for the Sobolev space defined in I.5.10. Furthermore,

$$(1,6) \qquad \mathbf{D}\colon \mathbf{x} \in W_n^p \to \mathbf{x}' \in L_n^p$$

and

$$(1,7) \qquad \mathbf{L} = \mathbf{D} - \mathbf{P}\colon \mathbf{x} \in W_n^p \to \mathbf{x}' - \mathbf{P}\mathbf{x} \in L_n^p$$

for any $p \in R$, $p \ge 1$.

1.6. Remark. Clearly, $\mathbf{D}$ is linear and bounded for any $p \in R$, $p \ge 1$. Hence if $\mathbf{P}(t,s)$ is an $L^p[BV]$-kernel, then $\mathbf{L}$ is also linear and bounded. We shall show that it has a closed range and hence by I.3.14 it is normally solvable.

1.7. Proposition. *Let $\mathbf{P}\colon [0,1] \times [0,1] \to L(R_n)$ be an $L^p[BV]$-kernel $(1 \le p < \infty)$. Then the operator $\mathbf{L}\colon W_n^p \to L_n^p$ given by (1,7) has a closed range in L_n^p.*

Proof. Let $\mathbf{f} \in L_n^p$. Then $\mathbf{f} \in R(\mathbf{L})$ if and only if there exists $\mathbf{x} \in W_n^p$ such that

$$(1,8) \qquad \mathbf{x}(t) - \mathbf{x}(0) - \int_0^t \left(\int_0^1 d_s[\mathbf{P}(\tau,s)] \, \mathbf{x}(s) \right) d\tau = \int_0^t \mathbf{f}(\tau) \, d\tau.$$

Hence denoting

$$(1,9) \qquad \boldsymbol{\Psi}\colon \mathbf{h} \in L_n^p \to \int_0^t \mathbf{h}(\tau) \, d\tau \in W_n^p,$$

$$\boldsymbol{\Pi}\colon \mathbf{x} \in W_n^p \to \mathbf{z}(t) \equiv \mathbf{x}(0) \in W_n^p,$$

we have $\mathbf{f} \in R(\mathbf{L})$ if and only if $\boldsymbol{\Psi}\mathbf{f} \in R(\mathbf{I} - (\boldsymbol{\Pi} + \boldsymbol{\Psi}\mathbf{P}))$, where $\mathbf{I}$ stands for the identity operator on W_n^p.

The operators $\boldsymbol{\Pi}$ and $\boldsymbol{\Psi}$ are evidently linear and bounded. As $R(\boldsymbol{\Pi})$ is finite dimensional, $\boldsymbol{\Pi}$ is compact (cf. I.3.21). Since, given $\mathbf{x} \in W_n^p$, $\|\mathbf{x}\|_{BV} \leq \|\mathbf{x}\|_{W^p}$, it follows from 1.4 that also $\mathbf{P}$: $W_n^p \to L_n^p$ is compact. Hence the operator $\boldsymbol{\Theta} = \boldsymbol{\Pi} + \boldsymbol{\Psi}\mathbf{P}$: $W_n^p \to W_n^p$ is linear bounded and compact. Consequently $R(\mathbf{I} - \boldsymbol{\Theta})$ is closed (cf. I.3.20). Since $\boldsymbol{\Psi}(R(\mathbf{L})) = R(\mathbf{I} - \boldsymbol{\Theta})$, $R(\mathbf{L})$ is closed.

1.8. Proposition. *If $\mathbf{P}(t, s)$ is an $L^p[BV]$-kernel, then*

$$n \leq \dim N(\mathbf{L}) < \infty ,$$

while $\dim N(\mathbf{L}) = n$ *if and only if* $R(\mathbf{L}) = L_n^p$.

Proof. By the proof of 1.7 the equation $\mathbf{L}\mathbf{x} = \mathbf{f}$ is equivalent to the equation

$$\mathbf{x} - \boldsymbol{\Theta}\mathbf{x} = \boldsymbol{\Psi}\mathbf{f},$$

where $\boldsymbol{\Theta} = \boldsymbol{\Pi} + \boldsymbol{\Psi}\mathbf{P}$: $W_n^p \to W_n^p$ is defined by (1,9). Since $\boldsymbol{\Theta}$ is compact, by I.3.20 we have $\dim N(\mathbf{L}) = \dim N(\mathbf{I} - \boldsymbol{\Theta}) < \infty$ and

$$\dim N(\mathbf{L}) = \operatorname{codim} R(\mathbf{I} - \boldsymbol{\Theta}) = \dim W_n^p / R(\mathbf{I} - \boldsymbol{\Theta}) . \tag{1,10}$$

It follows from the definition of $\boldsymbol{\Theta}$ that

$$R(\mathbf{I} - \boldsymbol{\Theta}) \subset \{\mathbf{g} \in W_n^p;\ \mathbf{g}(0) = \mathbf{0}\} = V_n^p .$$

Consequently

$$\dim W_n^p / R(\mathbf{I} - \boldsymbol{\Theta}) \geq \dim W_n^p / V_n^p .$$

If $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis in R_n and $\xi_j(t) = \mathbf{e}_j$ on $[0, 1]$ $(j = 1, 2, \ldots, n)$, then the system of equivalence classes $\xi_j + V_n^p$ $(j = 1, 2, \ldots, n)$ forms a basis in W_n^p / V_n^p. Hence

$$\dim W_n^p / V_n^p = n$$

and by (1,10) $\dim N(\mathbf{L}) = n$ if and only if

$$\dim W_n^p / V_n^p = \dim W_n^p / R(\mathbf{I} - \boldsymbol{\Theta}) .$$

Since $R(\mathbf{I} - \boldsymbol{\Theta}) = V_n^p$ if and only if $R(\mathbf{L}) = L_n^p$, the proof will be completed by means of the following assertion.

1.9. Lemma. *Given a Banach space X and its closed linear subspaces M, N such that $M \subset N \subset X$, $\dim X/M = \dim X/N < \infty$ holds if and only if $M = N$.*

Proof. Let $\dim X/M = \dim X/N = k < \infty$ and let $\mathbf{x} \in N \setminus M$. Let $\Xi_j = \xi_j + N$ $(j = 1, 2, \ldots, k)$ be a basis in X/N and let

$$\alpha\mathbf{x} + \sum_{j=1}^{k} \lambda_j \xi_j \in M \subset N$$

for some real numbers α, λ_j $(j = 1, 2, \ldots, k)$. Since $\alpha \boldsymbol{x} \in N$, this may happen only if $\lambda_1 \xi_1 + \lambda_2 \xi_2 + \ldots + \lambda_k \xi_k \in N$, i.e. $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$. Thus $\alpha \boldsymbol{x} \in M$ and for $\boldsymbol{x} \notin M$, $\alpha = 0$. This means that the classes $\{\boldsymbol{x} + M,\ \xi_j + M;\ j = 1, 2, \ldots, k\}$ are linearly independent in X/M and $\dim X/M \geq k + 1 > \dim X/N$. This being contradictory to the assumption, we have $M = N$.

1.10. Remark. By 1.8 there exists an $n \times k$-matrix valued function $\boldsymbol{X}$ $(k = \dim N(\boldsymbol{L}))$ such that $\boldsymbol{x} \in W_n^p$ is a solution to $\boldsymbol{L}\boldsymbol{x} = \boldsymbol{0}$ if and only if $\boldsymbol{x}_0(t) \equiv \boldsymbol{X}(t)\,\boldsymbol{c}$ on $[0, 1]$ for some $\boldsymbol{c} \in R_k$. Unfortunately, even if $k = n$, it need not be $\det(\boldsymbol{X}(t)) \neq 0$ on $[0, 1]$. For example, the equation

$$\boldsymbol{x}'(t) - 4 \int_0^1 \boldsymbol{x}(\tau)\, \mathrm{d}\tau = \boldsymbol{f}(t) \qquad \text{a.e. on } [0, 1] \tag{1,11}$$

possesses for any $\boldsymbol{f} \in L_n^1$ and $\boldsymbol{c} \in R_n$ the unique solution

$$\boldsymbol{x}(t) = \boldsymbol{I}(1 - 4t)\,\boldsymbol{c} + 4t \int_0^1 \left(\int_0^s \boldsymbol{f}(\tau)\, \mathrm{d}\tau \right) \mathrm{d}s + \int_0^t \boldsymbol{f}(\tau)\, \mathrm{d}\tau \qquad \text{on } [0, 1]$$

such that $\boldsymbol{x}(0) = \boldsymbol{c}$. In particular, $\boldsymbol{x} \in AC_n$ is a solution of the corresponding homogeneous equation if and only if $\boldsymbol{x}(t) = \boldsymbol{I}(1 - 4t)\,\boldsymbol{c}$ for some $\boldsymbol{c} \in R_n$ and $\boldsymbol{X}(t) = \boldsymbol{I}(1 - 4t)$ is the fundamental matrix solution for (1,11). Let us notice that $\boldsymbol{X}(\frac{1}{4}) = \boldsymbol{0}$.

1.11. Remark. Putting $\boldsymbol{R}(t, s) = \boldsymbol{P}(t, s+) - \boldsymbol{P}(t, 1)$ for $s \in (0, 1)$, $\boldsymbol{R}(t, 0) = \boldsymbol{P}(t, 0) - \boldsymbol{P}(t, 1)$ and $\boldsymbol{R}(t, 1) = \boldsymbol{0}$, we would obtain

$$\begin{aligned} \boldsymbol{R}(t, s+) &= \boldsymbol{P}(t, s+) - \boldsymbol{P}(t, 1) \qquad \text{if } s \in [0, 1), \\ \boldsymbol{R}(t, s-) &= \boldsymbol{P}(t, s-) - \boldsymbol{P}(t, 1) \qquad \text{if } s \in (0, 1] \end{aligned}$$

and hence according to I.5.5

$$\int_0^1 \mathrm{d}_s[\boldsymbol{P}(t, s)]\, \boldsymbol{x}(s) = \int_0^1 \mathrm{d}_s[\boldsymbol{R}(t, s)]\, \boldsymbol{x}(s) \qquad \text{for each } \boldsymbol{x} \in AC_n.$$

Given a subdivision $\sigma = \{0 = s_0 < s_1 < \ldots < s_m = 1\}$ of $[0, 1]$ and $\delta > 0$ such that $0 = s_0 < s_0 + \delta < s_1 < s_1 + \delta < \ldots < s_{m-1} < s_{m-1} + \delta < s_m = 1$, we have

$$\begin{aligned} V_\delta(t) = {} & |\boldsymbol{P}(t, s_0 + \delta) - \boldsymbol{P}(t, 0)| + \sum_{j=1}^{m-1} |\boldsymbol{P}(t, s_j + \delta) - \boldsymbol{P}(t, s_{j-1} + \delta)| \\ & + |\boldsymbol{P}(t, 1) - \boldsymbol{P}(t, s_{m-1} + \delta)| \leq \varrho(t) \qquad \text{a.e. on } [0, 1]. \end{aligned}$$

Consequently

$$\sum_{j=1}^{m} |\boldsymbol{R}(t, s_j) - \boldsymbol{R}(t, s_{j-1})| = \lim_{\delta \to 0+} V_\delta(t) \leq \varrho(t)$$

and $\operatorname{var}_0^1 \boldsymbol{R}(t, .) \leq \varrho(t)$ a.e. on $[0, 1]$. Since $|\boldsymbol{R}(t, 0)| \leq 2\,\varrho(t)$ a.e. on $[0, 1]$ (cf. 1.1), it follows that $\boldsymbol{R}\colon [0, 1] \times [0, 1] \to L(R_n)$ is also an $L^p[BV]$-kernel.

This means that without any loos of generality we may assume that $\mathbf{P}(t, .)$ is right-continuous on $(0, 1)$ and $\mathbf{P}(t, 1) = \mathbf{0}$ for almost all $t \in [0, 1]$.

1.12. Remark. Let

$$\mathbf{P}(t, s) = \begin{cases} -\mathbf{A}(t) - \mathbf{C}(t) - \mathbf{D}(t) & \text{if} \quad s = 0, \\ -\mathbf{A}(t) - \mathbf{D}(t) & \text{if} \quad 0 < s < t, \\ -\mathbf{D}(t) & \text{if} \quad t \leq s < 1, \\ \mathbf{0} & \text{if} \quad s = 1, \end{cases}$$

where $\mathbf{A}$, $\mathbf{C}$, $\mathbf{D}$ are $n \times n$-matrix valued functions whose columns are elements of L^p_n. Then

$$\operatorname{var}_0^1 \mathbf{P}(t, .) = |\mathbf{A}(t)| + |\mathbf{C}(t)| + |\mathbf{D}(t)| \qquad \text{a.e. on } [0, 1]$$

and hence $\mathbf{P}(t, s)$ is an $L^p[BV]$-kernel. Furthermore, given $\mathbf{x} \in AC_n$,

$$\int_0^1 \mathrm{d}_s[\mathbf{P}(t, s)]\, \mathbf{x}(s) = \mathbf{A}(t)\, \mathbf{x}(t) + \mathbf{C}(t)\, \mathbf{x}(0) + \mathbf{D}(t)\, \mathbf{x}(1) \qquad \text{a.e. on } [0, 1]$$

and the integro-differential operator $\mathbf{L} = \mathbf{D} - \mathbf{P}$ reduces to the *differential-boundary operator*

$$\mathbf{x} \in W^p_n \to \mathbf{x}'(t) - \mathbf{A}(t)\, \mathbf{x}(t) - \mathbf{C}(t)\, \mathbf{x}(0) - \mathbf{D}(t)\, \mathbf{x}(1) \in L^p_n .$$

2. Duality theory

Our wish is now to establish the duality theory for BVP

$$\mathbf{x}'(t) - \int_0^1 \mathrm{d}_s[\mathbf{P}(t, s)]\, \mathbf{x}(s) = \mathbf{f}(t) \qquad \text{a.e. on } [0, 1], \tag{2,1}$$

$$\mathbf{S}\mathbf{x} \equiv \mathbf{M}\, \mathbf{x}(0) + \int_0^1 \mathbf{K}(t)\, \mathbf{x}'(t)\, \mathrm{d}t = \mathbf{r}. \tag{2,2}$$

In particular, we shall show the normal solvability and evaluate the index of this boundary value problem under the following assumptions.

2.1. Assumptions. $\mathbf{P}$: $[0, 1] \times [0, 1] \to L(R_n)$ *is an* L^p-$[BV]$-*kernel,* $1 \leq p < \infty$, $\mathbf{f} \in L^p_n$, $\mathbf{M} \in L(R_n, R_m)$, $\mathbf{K}$: $[0, 1] \to L(R_n, R_m)$, $\|\mathbf{K}\|_{L^q} < \infty$, $q = p/(p - 1)$ *if* $p > 1$, $q = \infty$ *if* $p = 1$ *and* $\mathbf{r} \in R_m$.

2.2. Definition. A function $\mathbf{x}$: $[0, 1] \to R_n$ is said to be a *solution of BVP* (2,1), (2,2) if $\mathbf{x} \in AC_n$ and (2,1), (2,2) hold for a.e. $t \in [0, 1]$.

2.3. Remark. According to 1.13 we may assume that for a.e. $t \in [0, 1]$ $\mathbf{P}(t, .)$ is right-continuous on $(0, 1)$ and $\mathbf{P}(t, 1) = \mathbf{0}$. Furthermore, let us mention, that if

$\boldsymbol{P}(t,s)$ is an $L^p[BV]$-kernel and $\boldsymbol{f} \in L_n^p$, then obviously $\boldsymbol{x}' \in L_n^p$ for any solution $\boldsymbol{x} \in AC_n$ of the integro-differential equation (2,1). Thus given a solution $\boldsymbol{x}$ of BVP (2,1), (2,2), $\boldsymbol{x} \in W_n^p$.

2.4. Notations. *The operators* $\boldsymbol{D} \in B(W_n^p, L_n^p)$ *and* $\boldsymbol{P} \in K(W_n^p, L_n^p)$ *are defined by* (1,3) *and* (1,6),

$$\boldsymbol{S}\colon \boldsymbol{x} \in W_n^p \to \boldsymbol{M}\,\boldsymbol{x}(0) + \int_0^1 \boldsymbol{K}(t)\,\boldsymbol{x}'(t)\,\mathrm{d}t \in R_m$$

and

$$\mathscr{L}\colon \boldsymbol{x} \in W_n^p \to \begin{bmatrix} \boldsymbol{D}\boldsymbol{x} - \boldsymbol{P}\boldsymbol{x} \\ \boldsymbol{S}\boldsymbol{x} \end{bmatrix} \in L_n^p \times R_m. \tag{2,3}$$

Making use of 2.4, we may reformulate BVP (2,1), (2,2) as the operator equation

$$\mathscr{L}\boldsymbol{x} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix}. \tag{2,4}$$

It appears to be convenient to handle instead of (2,4) the operator equation for $\xi = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{d} \end{pmatrix} \in W_n^p \times R_m$

$$\xi - \boldsymbol{T}\xi = \varphi, \tag{2,5}$$

where

$$\boldsymbol{\Psi}\colon \boldsymbol{u} \in L_n^p \to \int_0^1 \boldsymbol{u}(\tau)\,\mathrm{d}\tau \in W_n^p, \qquad \boldsymbol{\Phi}\colon \boldsymbol{x} \in W_n^p \to \boldsymbol{v}(t) \equiv \boldsymbol{x}(0) \in W_n^p, \tag{2,6}$$

$$\boldsymbol{T}\colon \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{d} \end{pmatrix} \in W_n^p \times R_m \to \begin{bmatrix} \boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{\Psi}\boldsymbol{P}\boldsymbol{x} \\ \boldsymbol{d} - \boldsymbol{S}\boldsymbol{x} \end{bmatrix} \in W_n^p \times R_m \quad \text{and} \quad \varphi = \begin{pmatrix} \boldsymbol{\Psi}\boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in W_n^p \times R_m.$$

Clearly, $\boldsymbol{x} \in W_n^p$ is a solution to BVP (2,1), (2,2) if and only if for an arbitrary $\boldsymbol{d} \in R_m$ the couple $\xi = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{d} \end{pmatrix}$ is a solution of (2,5). In particular,

$$\dim N(\boldsymbol{I} - \boldsymbol{T}) = \dim N(\mathscr{L}) + m. \tag{2,7}$$

Furthermore, $\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in L_n^p \times R_m$ belongs to $R(\mathscr{L})$ if and only if $\begin{pmatrix} \boldsymbol{\Psi}\boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in R(\boldsymbol{I} - \boldsymbol{T})$. As according to 1.4 and I.3.21 the linear operator $\boldsymbol{T}$ given by (2,6) is compact and the linear operator $\boldsymbol{W}\colon \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in L_n^p \times R_m \to \begin{pmatrix} \boldsymbol{\Psi}\boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in W_n^p \times R_m$ is obviously bounded, we have

2.5. Proposition. *Under the assumptions* 2.1 *the operator* $\mathscr{L}$ *given by* (2,3) *has a closed range in* $L_n^p \times R_m$.

Since by I.5.13 the dual space $(W_n^p)^*$ to W_n^p is isometrically isomorphic with $L_n^q \times R_n^*$ and $(L_n^p \times R_m)^*$ is isometrically isomorphic with $L_n^q \times R_m^*$ (cf. I.3.9 and

I.3.10), the adjoint operator to $\mathscr{L}$ may be represented analytically by the linear bounded operator

$$(2,8)\qquad \mathscr{L}^*\colon (\mathbf{y}^*, \lambda^*) \in L_n^q \times R_m^* \to (\mathbf{L}_1^*(\mathbf{y}^*, \lambda^*), \mathbf{L}_2^*(\mathbf{y}^*, \lambda^*)) \in L_n^q \times R_n^*$$

which is defined by the relation

$$(2,9)\quad \int_0^1 \mathbf{y}^*(t)\,[\mathbf{D}\mathbf{x} - \mathbf{P}\mathbf{x}]\,(t)\,\mathrm{d}t + \lambda^*[\mathbf{S}\mathbf{x}] = \int_0^1 \mathbf{L}_1^*(\mathbf{y}^*, \lambda^*)\,(t)\,\mathbf{x}'(t)\,\mathrm{d}t + \mathbf{L}_2^*(\mathbf{y}^*, \lambda^*)\,\mathbf{x}(0)$$
$$\text{for all}\quad \mathbf{x} \in W_n^p,\quad \mathbf{y}^* \in L_n^q \quad\text{and}\quad \lambda^* \in R_m^*.$$

Analogously, the operator

$$(2,10)\qquad \mathbf{T}^*\colon (\mathbf{y}^*, \varkappa^*, \lambda^*) \in L_n^q \times R_n^* \times R_m^*$$
$$\to (\mathbf{T}_1^*(\mathbf{y}^*, \varkappa^*, \lambda^*), \mathbf{T}_2^*(\mathbf{y}^*, \varkappa^*, \lambda^*), \mathbf{T}_3^*(\mathbf{y}^*, \varkappa^*, \lambda^*)) \in L_n^q \times R_n^* \times R_m^*$$

defined by

$$(2,11)\qquad \int_0^1 \mathbf{y}^*(t)\,(\mathbf{P}\mathbf{x})\,(t)\,\mathrm{d}t + \varkappa^*\,\mathbf{x}(0) + \lambda^*(\mathbf{d} - \mathbf{S}\mathbf{x})$$
$$= \int_0^1 \mathbf{T}_1^*(\mathbf{y}^*, \varkappa^*, \lambda^*)\,\mathbf{x}'(t)\,\mathrm{d}t + \mathbf{T}_2^*(\mathbf{y}^*, \varkappa^*, \lambda^*)\,\mathbf{x}(0) + \mathbf{T}_3^*(\mathbf{y}^*, \varkappa^*, \lambda^*)\,\mathbf{d}$$
$$\text{for all}\quad \mathbf{x} \in W_n^p,\quad \mathbf{d} \in R_m,\quad \mathbf{y}^* \in L_n^q,\quad \varkappa^* \in R_n^*,\quad \lambda^* \in R_m^*$$

represents analytically the adjoint operator to the operator $\mathbf{T}$.

2.6. Theorem. *If* 2,1 *holds and* $\mathbf{P}(t,1) = \mathbf{0}$ *a.e. on* $[0,1]$, *then the operator* $\mathscr{L}^*\colon L_n^q \times R_m^* \to L_n^q \times R_n^*$ *given by* (2,8) *verifies* (2,9) *if and only if*

$$(2,12)\quad \mathbf{L}_1^*(\mathbf{y}^*, \lambda^*)\,(t) = \mathbf{y}^*(t) + \int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,t)\,\mathrm{d}s + \lambda^*\,\mathbf{K}(t) \qquad \textit{a.e. on } [0,1],$$

$$(2,13)\qquad \mathbf{L}_2^*(\mathbf{y}^*, \lambda^*) = \lambda^*\mathbf{M} + \int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,0)\,\mathrm{d}s.$$

Proof. Let $\mathbf{x} \in W_n^p$, $\mathbf{y}^* \in L_n^q$ and $\lambda^* \in R_m^*$. By I.4.38

$$\int_0^1 \mathbf{y}^*(t)\,(\mathbf{P}\mathbf{x})\,(t)\,\mathrm{d}t = \int_0^1 \mathrm{d}_t\left[\int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,t)\,\mathrm{d}s\right]\mathbf{x}(t).$$

Furthermore, integrating by parts (I.4.33) and taking into account the assumption $\mathbf{P}(t,1) = \mathbf{0}$ a.e. on $[0,1]$, we obtain

$$\int_0^1 \mathbf{y}^*(t)\,(\mathbf{P}\mathbf{x})\,(t)\,\mathrm{d}t = -\left(\int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,0)\,\mathrm{d}s\right)\mathbf{x}(0) - \int_0^1\left(\int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,t)\,\mathrm{d}s\right)\mathbf{x}'(t)\,\mathrm{d}t.$$

Hence

$$\int_0^1 \mathbf{y}^*(t)\,[\mathbf{D}\mathbf{x} - \mathbf{P}\mathbf{x}]\,(t)\,\mathrm{d}t + \lambda^*(\mathbf{S}\mathbf{x})$$

$$= \left[\lambda^* \mathbf{M} + \int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,0)\,\mathrm{d}s\right]\mathbf{x}(0) + \int_0^1 \left[\mathbf{y}^*(t) + \int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,t)\,\mathrm{d}s + \lambda^*\,\mathbf{K}(t)\right]\mathbf{x}'(t)\,\mathrm{d}t$$

for all $\mathbf{x} \in W_n^p$, $\mathbf{y}^* \in L_n^q$ and $\lambda^* \in R_m^*$.

In virtue of (2,9) this yields that

$$\int_0^1 \left[\mathbf{L}_1^*(\mathbf{y}^*, \lambda^*)\,(t) - \mathbf{y}^*(t) - \int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,t)\,\mathrm{d}s - \lambda^*\,\mathbf{K}(t)\right]\mathbf{x}'(t)\,\mathrm{d}t$$

$$+ \left[\mathbf{L}_2^*(\mathbf{y}^*, \lambda^*) - \lambda^* \mathbf{M} - \int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,0)\,\mathrm{d}s\right]\mathbf{x}(0) = 0$$

holds for all $\mathbf{x} \in W_n^p$, $\mathbf{y}^* \in L_n^q$ and $\lambda^* \in R_m^*$.

The proof will be completed by making use of I.5.15.

Similarly

2.7. Proposition. *If* 2.1 *holds and* $\mathbf{P}(t, 1) = \mathbf{0}$ *a.e. on* $[0, 1]$, *then the operator* $\mathbf{T}^*$: $L_n^q \times R_n^* \times R_m^* \to L_n^q \times R_n^* \times R_m^*$ *given by* (2,10) *verifies* (2,11) *if and only if*

$$\mathbf{T}_1^*(\mathbf{y}^*, \varkappa^*, \lambda^*)\,(t) = -\mathbf{L}_1^*(\mathbf{y}^*, \lambda^*)\,(t) + \mathbf{y}^*(t) \qquad \textit{a.e. on } [0,1]\,,$$

$$\mathbf{T}_2^*(\mathbf{y}^*, \varkappa^*, \lambda^*) = -\mathbf{L}_2^*(\mathbf{y}^*, \lambda^*) + \varkappa^*\,, \qquad \mathbf{T}_3^*(\mathbf{y}^*, \varkappa^*, \lambda^*) = \lambda^*$$

for all $\mathbf{y}^* \in L_n^q$, $\varkappa^* \in R_n^*$ *and* $\lambda^* \in R_m^*$.

2.8. Corollary. $\dim N(\mathscr{L}^*) = \dim N(\mathbf{I} - \mathbf{T}^*) - n < \infty$.

Proof follows readily from 2.6, 2.7 and I.3.20.

2.9. Theorem. *If* 2.1 *holds and* $\mathbf{P}(t, 1) = \mathbf{0}$ *a.e. on* $[0, 1]$, *then*

$$\operatorname{ind}(\mathscr{L}) = \dim N(\mathscr{L}^*) - \dim N(\mathscr{L}) = m - n\,.$$

Proof. By 2.5 and I.3.15 $\operatorname{codim} R(\mathscr{L}) = \dim N(\mathscr{L}^*)$. Hence by (2,7) and 2.8 and I.3.20

$$\operatorname{ind}(\mathscr{L}) = \dim N(\mathbf{I} - \mathbf{T}^*) - n - \dim N(\mathbf{I} - \mathbf{T}) + m = m - n\,.$$

2.10. Remark. The relation (2,9), where $\mathbf{L}_1^*(\mathbf{y}^*, \lambda^*)$ and $\mathbf{L}_2^*(\mathbf{y}^*, \lambda^*)$ are given by (2,12) and (2,13) is the *Green formula* for BVP (2,1), (2,2).

2.11. Remark. Let $\mathbf{A}, \mathbf{C}, \mathbf{D}$: $[0, 1] \to L(R_n)$ be Lebesgue integrable on $[0, 1]$, let $\mathbf{P}$: $[0, 1] \times [0, 1] \to L(R_n)$ be an $L^1[BV]$-kernel and let $\mathbf{K}$: $[0, 1] \to L(R_n, R_m)$ be of bounded variation on $[0, 1]$ and $\mathbf{M}, \mathbf{N} \in L(R_n, R_m)$. Let us consider the problem of determining $\mathbf{x} \in AC_n$ which verifies the system

$$\mathbf{x}'(t) - \mathbf{A}(t)\,\mathbf{x}(t) - [\mathbf{C}(t)\,\mathbf{x}(0) + \mathbf{D}(t)\,\mathbf{x}(1)] - \int_0^1 \mathrm{d}_s[\mathbf{P}(t,s)]\,\mathbf{x}(s) = \mathbf{f}(t) \quad \text{a.e. on } [0,1] \tag{2,14}$$

and

$$\mathbf{M}\,\mathbf{x}(0) + \mathbf{N}\,\mathbf{x}(1) + \int_0^1 \mathrm{d}[\mathbf{K}(t)]\,\mathbf{x}(t) = \mathbf{r}\,, \tag{2,15}$$

where $\mathbf{f} \in L_n^1$ and $\mathbf{r} \in R_m$. Again we may assume that $\mathbf{P}(t,\,.)$ is for almost all $t \in [0,1]$ right-continuous on $(0,1)$. Moreover, if we put

$$\mathbf{P}_0(t,s) = \begin{cases} \mathbf{P}(t,0+) - \mathbf{P}(t,1-) & \text{if} \quad s = 0\,, \\ \mathbf{P}(t,s) \quad\; - \mathbf{P}(t,1-) & \text{if} \quad 0 < s < 1\,, \\ \mathbf{0} & \text{if} \quad s = 1 \end{cases}$$

and $\mathbf{C}_0(t) = \mathbf{C}(t) - [\mathbf{P}(t,0+) - \mathbf{P}(t,0)]$, $\mathbf{D}_0(t) = \mathbf{D}(t) - [\mathbf{P}(t,1) - \mathbf{P}(t,1-)]$, for any $\mathbf{x} \in AC_n$ we should obtain

$$\begin{aligned} &\mathbf{C}(t)\,\mathbf{x}(0) + \mathbf{D}(t)\,\mathbf{x}(1) + \int_0^1 \mathrm{d}_s[\mathbf{P}(t,s)]\,\mathbf{x}(s) \\ &= \mathbf{C}_0(t)\,\mathbf{x}(0) + \mathbf{D}_0(t)\,\mathbf{x}(1) + \int_0^1 \mathrm{d}_s[\mathbf{P}_0(t,s)]\,\mathbf{x}(s)\,. \end{aligned}$$

Hence, without any loss of generality we may assume that for almost all $t \in [0,1]$ $\mathbf{P}(t,\,.)$ is right-continuous on $[0,1)$, left-continuous at 1 and $\mathbf{P}(t,1) = \mathbf{0}$. Analogously, $\mathbf{K}$ may be assumed right-continuous on $[0,1)$, left-continuous at 1 and $\mathbf{K}(1) = \mathbf{0}$.

According to 1.12 we may rewrite the equation (2,14) in the form

$$\mathbf{x}'(t) - \int_0^1 \mathrm{d}_s[\mathbf{R}(t,s)]\,\mathbf{x}(s) = \mathbf{f}(t) \qquad \text{a.e. on } [0,1]\,, \tag{2,16}$$

where

$$\mathbf{R}(t,s) = \mathbf{P}(t,s) + \begin{cases} -\mathbf{A}(t) - \mathbf{C}(t) - \mathbf{D}(t) & \text{if} \quad s = 0\,, \\ -\mathbf{A}(t) - \mathbf{D}(t) & \text{if} \quad 0 < s < t\,, \\ -\mathbf{D}(t) & \text{if} \quad t \le s < 1\,, \\ \mathbf{0} & \text{if} \quad s = 1 \end{cases}$$

is again an $L^1[BV]$-kernel. Furthermore, applying the integration-by-parts formula and taking into account that $\mathbf{K}(1) = \mathbf{0}$ and

$$\mathbf{x}(1) = \mathbf{x}(0) + \int_0^1 \mathbf{x}'(\tau)\,\mathrm{d}\tau \qquad \text{for any} \quad \mathbf{x} \in AC_n\,,$$

we transfer the side condition (2,15) into

$$\mathbf{H}\,\mathbf{x}(0) + \int_0^1 \mathbf{F}(t)\,\mathbf{x}'(t)\,\mathrm{d}t = \mathbf{r}\,, \tag{2,17}$$

where

$$\mathbf{H} = \mathbf{M} + \mathbf{N} - \mathbf{K}(0)\,, \qquad \mathbf{F}(t) = \mathbf{N} - \mathbf{K}(t)\,.$$

The system (2,16), (2,17) may be written as the operator equation

$$\mathscr{R}\mathbf{x} = \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix}$$

with $\mathscr{R}\colon AC_n \to L_n^1 \times R_m$ defined in an obvious way. Now, proceeding analogously as in the close of the proof of IV.3.13 we may deduce from 2.6 that $(\mathbf{y}^*, \boldsymbol{\lambda}^*) \in N(\mathscr{R}^*)$ if and only if there exists $\mathbf{z} \in BV_n$ such that $\mathbf{z}(t) = \mathbf{y}(t)$ a.e. on $[0, 1]$, $\mathbf{z}(0+) = \mathbf{z}(0)$, $\mathbf{z}(1-) = \mathbf{z}(1)$ and

$$\mathbf{z}^*(t) + \int_0^1 \mathbf{z}^*(s)\,\mathbf{R}(s, t)\,\mathrm{d}s + \boldsymbol{\lambda}^*\,\mathbf{F}(t) = \mathbf{0} \qquad \text{on } (0, 1)\,, \tag{2,18}$$

$$\boldsymbol{\lambda}^*\mathbf{H} + \int_0^1 \mathbf{z}^*(s)\,\mathbf{R}(s, 0)\,\mathrm{d}s = \mathbf{0}\,. \tag{2,19}$$

As $\mathbf{F}(1-) = \mathbf{F}(1) = \mathbf{N}$ and $\mathbf{R}(t, 1-) = -\mathbf{D}(t)$ for almost all $t \in [0, 1]$, we have by (2,18)

$$\mathbf{z}^*(1) = \int_0^1 \mathbf{z}^*(s)\,\mathbf{D}(s)\,\mathrm{d}s - \boldsymbol{\lambda}^*\mathbf{N}\,. \tag{2,20}$$

Since $\mathbf{F}(0+) = \mathbf{F}(0) = \mathbf{N} - \mathbf{K}(0)$ and $\mathbf{R}(t, 0+) = \mathbf{P}(t, 0) - \mathbf{A}(t) - \mathbf{D}(t)$ for almost all $t \in [0, 1]$, the relations (2,18) and (2,19) imply

$$\begin{aligned}
\mathbf{z}^*(0) &= -\int_0^1 \mathbf{z}^*(s)\,\mathbf{P}(s, 0)\,\mathrm{d}s + \int_0^1 \mathbf{z}^*(s)\,\mathbf{D}(s)\,\mathrm{d}s + \int_0^1 \mathbf{z}^*(s)\,\mathbf{A}(s)\,\mathrm{d}s - \boldsymbol{\lambda}^*\mathbf{N} + \boldsymbol{\lambda}^*\,\mathbf{K}(0) \\
&= -\left[\boldsymbol{\lambda}^*\mathbf{H} + \int_0^1 \mathbf{z}^*(s)\,\mathbf{R}(s, 0)\,\mathrm{d}s\right] - \int_0^1 \mathbf{z}^*(s)\,\mathbf{C}(s)\,\mathrm{d}s + \boldsymbol{\lambda}^*\mathbf{M} \\
&= -\int_0^1 \mathbf{z}^*(s)\,\mathbf{C}(s)\,\mathrm{d}s + \boldsymbol{\lambda}^*\mathbf{M}\,.
\end{aligned}$$

By the definition of $\mathbf{R}$ and $\mathbf{F}$ we have for any $\mathbf{z} \in BV_n$ and $\boldsymbol{\lambda} \in R_m$ fulfilling (2,20)

$$\begin{aligned}
&\int_0^1 \mathbf{z}^*(s)\,\mathbf{R}(s, t)\,\mathrm{d}s + \boldsymbol{\lambda}^*\,\mathbf{F}(t) \\
&= \left(\boldsymbol{\lambda}^*\mathbf{N} - \int_0^1 \mathbf{z}^*(s)\,\mathbf{D}(s)\,\mathrm{d}s\right) - \int_t^1 \mathbf{z}^*(s)\,\mathbf{A}(s)\,\mathrm{d}s + \int_0^1 \mathbf{z}^*(s)\,\mathbf{P}(s, t)\,\mathrm{d}s - \boldsymbol{\lambda}^*\,\mathbf{K}(t) \\
&= -\mathbf{z}^*(1) - \int_t^1 \mathbf{z}^*(s)\,\mathbf{A}(s)\,\mathrm{d}s + \int_0^1 \mathbf{z}^*(s)\,\mathbf{P}(s, t)\,\mathrm{d}s - \boldsymbol{\lambda}^*\,\mathbf{K}(t) \qquad \text{on } [0, 1]\,.
\end{aligned}$$

Thus, the adjoint problem to BVP (2,14), (2,15) is equivalent to the problem of determining $\mathbf{z} \in BV_n$ and $\boldsymbol{\lambda}^* \in R_m^*$ such that

$$(2,21)\quad \mathbf{z}^*(t) = \mathbf{z}^*(1) + \int_t^1 \mathbf{z}^*(s)\,\mathbf{A}(s)\,\mathrm{d}s - \int_0^1 \mathbf{z}^*(s)\,\mathbf{P}(s,t)\,\mathrm{d}s - \boldsymbol{\lambda}^*\,\mathbf{K}(t) \qquad \text{on } [0,1]$$

$$(2,22)\qquad \mathbf{z}^*(0) + \boldsymbol{\lambda}^*\mathbf{M} + \int_0^1 \mathbf{z}^*(s)\,\mathbf{C}(s)\,\mathrm{d}s = \mathbf{0}\,,$$

$$\mathbf{z}^*(1) - \boldsymbol{\lambda}^*\mathbf{N} - \int_0^1 \mathbf{z}^*(s)\,\mathbf{D}(s)\,\mathrm{d}s = \mathbf{0}\,.$$

2.12. Theorem. *Let us assume* 2.1 *and* $\mathbf{P}(t,1) = \mathbf{0}$ *a.e. on* $[0,1]$. *Then for given* $\mathbf{f} \in L_n^p$ *and* $\mathbf{r} \in R_m$ *BVP* (2,1), (2,2) *possesses a solution if and only if*

$$\int_0^1 \mathbf{y}^*(t)\,\mathbf{f}(t)\,\mathrm{d}t + \boldsymbol{\lambda}^*\mathbf{r} = 0$$

for any couple $(\mathbf{y}^*, \boldsymbol{\lambda}^*) \in L_n^q \times R_m^*$ *which verifies the adjoint system*

$$(2,23)\qquad \mathbf{y}^*(t) + \int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,t)\,\mathrm{d}s + \boldsymbol{\lambda}^*\,\mathbf{K}(t) = \mathbf{0} \qquad \text{a.e. on } [0,1]\,,$$

$$(2,24)\qquad \boldsymbol{\lambda}^*\mathbf{M} + \int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,0)\,\mathrm{d}s = \mathbf{0}\,.$$

Proof follows from 2.5, 2.6 and I.3.14 (cf. I.3.23).

2.13. Theorem. *Let us assume* 2.1 *and* $\mathbf{P}(t,1) = \mathbf{0}$ *a.e. on* $[0,1]$. *Then for given* $\mathbf{g}^* \in L_n^q$ *and* $\mathbf{q}^* \in R_n^*$, *the system*

$$\mathbf{y}^*(t) + \int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,t)\,\mathrm{d}s + \boldsymbol{\lambda}^*\,\mathbf{K}(t) = \mathbf{g}^*(t) \qquad \text{a.e. on } [0,1]\,,$$

$$\boldsymbol{\lambda}^*\mathbf{M} + \int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,0)\,\mathrm{d}s = \mathbf{q}^*$$

possesses a solution $(\mathbf{y}^*, \boldsymbol{\lambda}^*) \in L_n^q \times R_m^*$ *if and only if*

$$\int_0^1 \mathbf{g}^*(t)\,\mathbf{x}'(t)\,\mathrm{d}t + \mathbf{q}^*\,\mathbf{x}(0) = 0$$

holds for any solution $\mathbf{x} \in W_n^p$ *of the homogeneous problem* $\mathscr{L}\mathbf{x} = \mathbf{0}$.

Proof follows again from 2.5, 2.6 and I.3.14.

2.14. Remark. Let us notice that the side condition (2,2) is linearly dependent if there exists $\mathbf{q} \in R_m$ such that $\mathbf{q}^*\mathbf{M} = \mathbf{q}^*\,\mathbf{K}(t) = \mathbf{0}$ a.e. on $[0,1]$ $(\mathbf{q}^*(\mathbf{S}\mathbf{x}) = 0$ for all

$\boldsymbol{x} \in W_n^p$ implies that

$$\boldsymbol{x} \in W_n^p \to \boldsymbol{q}^*(\boldsymbol{S}\boldsymbol{x}) = (\boldsymbol{q}^*\boldsymbol{M})\,\boldsymbol{x}(0) + \int_0^1 (\boldsymbol{q}^*\,\boldsymbol{K}(t))\,\boldsymbol{x}'(t)\,\mathrm{d}t \in R$$

is the zero functional on W_n^p).

Analogously as in the case of Stieltjes-integral side conditions (cf. IV.1.14, where no use of the special form of side conditions was made), we can also show that to any nonzero linear operator $\boldsymbol{S}_0\colon W_n^p \to R_k$ and $\boldsymbol{r}_0 \in R_k$ such that $\boldsymbol{q}^*(\boldsymbol{S}_0\boldsymbol{x}) = \boldsymbol{0}$ for any $\boldsymbol{x} \in AC_n$ implies $\boldsymbol{q}^*\boldsymbol{r}_0 = \boldsymbol{0}$, there exist $m \leq k$, $\boldsymbol{S}\colon W_n^p \to R_m$ and $\boldsymbol{r} \in R_m$ such that the condition $\boldsymbol{S}\boldsymbol{x} = \boldsymbol{r}$ is linearly independent and equivalent to $\boldsymbol{S}_0\boldsymbol{x} = \boldsymbol{r}_0$.

2.15. Remark. It follows from the proof of IV.1.15 that if (2,2) is reasonable and linearly independent, then there exists a regular $m \times m$-matrix $\boldsymbol{\Theta}$ such that

$$\boldsymbol{\Theta}[\boldsymbol{M}, \boldsymbol{K}(t)] = \begin{bmatrix} \boldsymbol{M}_0, & \boldsymbol{0} \\ \boldsymbol{M}_1, & \boldsymbol{K}_1(t) \\ \boldsymbol{0}, & \boldsymbol{K}_2(t) \end{bmatrix} \qquad \text{a.e. on } [0,1],$$

where $\boldsymbol{M}_0 \in L(R_n, R_{m_0})$, $\boldsymbol{M}_1$ and $\boldsymbol{K}_1(t) \in L(R_n, R_{m_1})$ and $\boldsymbol{K}_2(t) \in L(R_n, R_{m_2})$ are such that $m_0 + m_1 + m_2 = m$, $\operatorname{rank}\begin{bmatrix}\boldsymbol{M}_0\\ \boldsymbol{M}_1\end{bmatrix} = m_0 + m_1$ and the rows of $\begin{bmatrix}\boldsymbol{K}_1(t)\\ \boldsymbol{K}_2(t)\end{bmatrix}$ are linearly independent in L_n^q, i.e.

$$\boldsymbol{q}^*\begin{bmatrix}\boldsymbol{K}_1(t)\\ \boldsymbol{K}_2(t)\end{bmatrix} = \boldsymbol{0} \qquad \text{a.e. on } [0,1]$$

implies $\boldsymbol{q}^* = \boldsymbol{0}$. The system

$$\begin{aligned} \boldsymbol{M}_0\,\boldsymbol{x}(0) &= \boldsymbol{r}_0\,, \\ \boldsymbol{M}_1\,\boldsymbol{x}(0) + \int_0^1 \boldsymbol{K}_1(t)\,\boldsymbol{x}'(t)\,\mathrm{d}t &= \boldsymbol{r}_1 \qquad \left(\begin{pmatrix}\boldsymbol{r}_0\\ \boldsymbol{r}_1\\ \boldsymbol{r}_2\end{pmatrix} = \boldsymbol{\Theta}\boldsymbol{r}\right), \\ \int_0^1 \boldsymbol{K}_2(t)\,\boldsymbol{x}'(t)\,\mathrm{d}t &= \boldsymbol{r}_2 \end{aligned}$$

is the canonical form of the side condition (2,2).

2.16. Remark. Another possible functional analytic way of attacking BVP (2,1), (2,2) with $\boldsymbol{r} \in R_m$ fixed consists in considering the linear operator $\mathscr{L}_r$ defined on $D(\mathscr{L}_r) = \{\boldsymbol{x} \in W_n^p;\ \boldsymbol{S}\boldsymbol{x} = \boldsymbol{r}\} \subset W_n^p$ by

$$\mathscr{L}_r\colon\ \boldsymbol{x} \in D(\mathscr{L}_r) \to \boldsymbol{D}\boldsymbol{x} - \boldsymbol{P}\boldsymbol{x} \in L_n^p\,.$$

BVP (2,1), (2,2) may be rewritten as the operator equation

$$\mathscr{L}_r\boldsymbol{x} = \boldsymbol{f}\,.$$

As $R(\mathscr{L}_r)$ is the set of all $\mathbf{f} \in L_n^p$ for which $\begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} \in R(\mathscr{L})$ and $R(\mathscr{L})$ is closed by 2.5, $R(\mathscr{L}_r)$ is also closed. By 2.12 $R(\mathscr{L}_r)$ is the set of all $\mathbf{f} \in L_n^p$ which fulfil the relation

$$\int_0^1 \mathbf{y}^*(t)\,\mathbf{f}(t)\,\mathrm{d}t + \boldsymbol{\lambda}^*\mathbf{r} = 0$$

for all couples $(\mathbf{y}^*, \boldsymbol{\lambda}^*) \in N(\mathscr{L}^*) \subset L_n^q \times R_m^*$. In particular, if N_0^* denotes the set of all $\mathbf{y}^* \in L_n^q$ for which there exists $\boldsymbol{\lambda}^* \in R_m^*$ such that $(\mathbf{y}^*, \boldsymbol{\lambda}^*) \in N(\mathscr{L}^*)$, then

$$R(\mathscr{L}_0) = {}^{\perp}(N_0^*)$$

(the set of all $\mathbf{f} \in L_n^p$ for which $\langle \mathbf{f}, \mathbf{y}^* \rangle_L = 0$ for any $\mathbf{y}^* \in N_0^*$).

2.17. Proposition. $R(\mathscr{L}_0)^{\perp} = N_0^*$, *where* $R(\mathscr{L}_0)^{\perp}$ *denotes the set of all* $\mathbf{y}^* \in L_n^q$ *such that*

$$\int_0^1 \mathbf{y}^*(t)\,\mathbf{f}(t)\,\mathrm{d}t = 0 \qquad \text{for any} \quad \mathbf{f} \in R(\mathscr{L}_0)$$

and N_0^* *is the set of all* $\mathbf{y}^* \in L_n^q$ *for which there exists* $\boldsymbol{\lambda}^* \in R_m^*$ *such that* $(\mathbf{y}^*, \boldsymbol{\lambda}^*) \in N(\mathscr{L}^*)$ *(i.e.* (2,23), (2,24) *hold).*

Proof. Let $\mathbf{y}^* \in L_n^q$. Then $\mathbf{y}^* \in R(\mathscr{L}_0)^{\perp}$ if and only if

$$0 = \int_0^1 \mathbf{y}^*(t)\,[\mathbf{D}\mathbf{x} - \mathbf{P}\mathbf{x}](t)\,\mathrm{d}t$$

$$= \int_0^1 \left[\mathbf{y}^*(t) + \int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,t)\,\mathrm{d}s\right]\mathbf{x}'(t)\,\mathrm{d}t + \left[\int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,0)\,\mathrm{d}s\right]\mathbf{x}(0)$$

holds for every $\mathbf{x} \in D(\mathscr{L}_0) = N(\mathbf{S})$.

This is true if and only if $(\mathbf{u}^*, \mathbf{v}^*) \in N(\mathbf{S})^{\perp}$, where

$$\mathbf{u}^*(t) = \mathbf{y}^*(t) + \int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,t)\,\mathrm{d}s \qquad \text{on } [0,1], \tag{2,25}$$

$$\mathbf{v}^* = \int_0^1 \mathbf{y}^*(s)\,\mathbf{P}(s,0)\,\mathrm{d}s\,.$$

Since $R(\mathbf{S})$ is a linear subspace in R_m, it is certainly closed and thus according to I.3.14 $N(\mathbf{S})^{\perp} = R(\mathbf{S}^*)$, where

$$\mathbf{S}^*\colon \boldsymbol{\lambda}^* \in R_m^* \to (\mathbf{S}_1^*\boldsymbol{\lambda}^*, \mathbf{S}_2^*\boldsymbol{\lambda}^*) \in L_n^q \times R_n^*$$

is the adjoint of $\mathbf{S}$ defined by the relation

$$\boldsymbol{\lambda}^*(\mathbf{S}\mathbf{x}) = \int_0^1 (\mathbf{S}_1^*\boldsymbol{\lambda}^*)(t)\,\mathbf{x}'(t)\,\mathrm{d}t + (\mathbf{S}_2^*\boldsymbol{\lambda}^*)\,\mathbf{x}(0) \qquad \text{for all} \quad \mathbf{x} \in W_n^p \quad \text{and} \quad \boldsymbol{\lambda}^* \in R_m^*.$$

Obviously, $(\mathbf{S}_1^*\boldsymbol{\lambda}^*)(t) = \boldsymbol{\lambda}^* \mathbf{K}(t)$ a.e. on $[0,1]$ and $\mathbf{S}_2^*\boldsymbol{\lambda}^* = \boldsymbol{\lambda}^*\mathbf{M}$. This means that $(\mathbf{u}^*, \mathbf{v}^*) \in N(\mathbf{S})^\perp$ if and only if there exists $\boldsymbol{\lambda}^* \in R_m^*$ such that

$$\mathbf{u}^*(t) = \boldsymbol{\lambda}^* \mathbf{K}(t) \qquad \text{a.e. on } [0,1], \qquad \mathbf{v}^* = \boldsymbol{\lambda}^*\mathbf{M},$$

wherefrom $R(\mathscr{L}_0)^\perp = N_0^*$ follows immediately by (2,25).

2.18. Remark. Since by 2.8 $\dim N_0^* < \infty$, Proposition 2.17 is a consequence of the following general assertion due to J. Dieudonné (cf. Goldberg II.3.6).

If Y is a linear normed space, $N \subset Y^*$, $\dim N < \infty$, then $({}^\perp N)^\perp = N$.

3. Green's function

Let us continue the investigation of the operator

$$\mathscr{L}\colon \mathbf{x} \in W_n^p \to \begin{bmatrix} \mathbf{Dx} - \mathbf{Px} \\ \mathbf{Sx} \end{bmatrix} \in L_n^p \times R_m,$$

given by (2,15). (cf. also (1,6), (1,3) and (2,2).) We assume again that 2.1 holds. Moreover, we assume that $\mathbf{P}(t,1) = \mathbf{0}$ a.e. on $[0,1]$ (cf. 1.15 and 2.2).

Of particular interest is the case when the operator equation

$$\mathscr{L}\mathbf{x} = \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} \tag{3,1}$$

(or BVP (2,1), (2,2)) has a unique solution for any $\mathbf{f} \in L_n^p$ and $\mathbf{r} \in R_m$.

3.1. Notation. Throughout the section $l = \dim N(\mathbf{D} - \mathbf{P})$, $\mathbf{X}(t)$ is an arbitrary $n \times k$-matrix valued function whose columns form a basis in $N(\mathbf{D} - \mathbf{P})$ and $(\mathbf{SX})$ is the $m \times l$-matrix

$$(\mathbf{SX}) = \mathbf{M}\,\mathbf{X}(0) + \int_0^1 \mathbf{K}(t)\,\mathbf{X}'(t)\,\mathrm{d}t. \tag{3,2}$$

(According to 1.8 $n \le l < \infty$.)

3.2. Lemma. $\dim N(\mathscr{L}) = l - \operatorname{rank}(\mathbf{SX})$.

Proof. By the definition of $\mathbf{X}(t)$ we have $\mathbf{x} \in N(\mathscr{L})$ if and only if $\mathbf{x}(t) \equiv \mathbf{X}(t)\,\mathbf{c}$ on $[0,1]$, where $\mathbf{c} \in R_l$ is such that

$$(\mathbf{SX})\,\mathbf{c} = \mathbf{0}. \tag{3,3}$$

Obviously, the functions $\mathbf{X}(t)\,\mathbf{c}_j$ with $\mathbf{c}_j \in R_l$ $(j = 1, 2, \ldots, v)$ are linearly dependent in W_n^p if and only if the vectors $\mathbf{c}_j$ $(j = 1, 2, \ldots, v)$ are linearly dependent. The assertion of the lemma follows immediately.

3.3. Remark. Since rank $(\mathbf{SX}) \leq m$ and $l \geq n$, 3.2 implies

$$\dim N(\mathscr{L}) \geq n - m\,.$$

3.4. Lemma. *$R(\mathscr{L}) = L_n^p \times R_m$ if and only if $\dim N(\mathscr{L}) = n - m$.*

Proof. Since by 2.5 $R(\mathscr{L})$ is closed in $L_n^p \times R_m$, $R(\mathscr{L}) = L_n^p \times R_m$ if and only if

$$0 = \operatorname{codim} R(\mathscr{L}) = \dim\left((L_n^p \times R_m)/R(\mathscr{L})\right) = \dim N(\mathscr{L}^*) \tag{3,4}$$

(cf. I.3.11). According to 2.9

$$\dim N(\mathscr{L}^*) = \dim N(\mathscr{L}) + m - n$$

wherefrom by (3,4) the assertion of the lemma follows.

3.5. Corollary. *BVP (2,1), (2,2) possesses a unique solution for any $\mathbf{f} \in L_n^p$ and $\mathbf{r} \in R_m$ if and only if*

$$m = n \quad \textit{and} \quad \dim N(\mathscr{L}) = 0\,. \tag{3,5}$$

Proof follows from 3.4 taking into account that (3,1) has a unique solution for any $\begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} \in R(\mathscr{L})$ if and only if $\dim N(\mathscr{L}) = 0$.

Analogously as in the case of ordinary differential equations we want to represent solutions to (3,1) in the form

$$\mathbf{x}(t) = \int_0^1 \mathbf{G}(t, s)\, \mathbf{f}(s)\, \mathrm{d}s + \mathbf{H}(t)\, \mathbf{r} \qquad \text{on } [0, 1]\,. \tag{3,6}$$

3.6. Definition. A couple of functions $\mathbf{G}\colon [0,1] \times [0,1] \to L(R_n)$ and $\mathbf{H}\colon [0,1] \to L(R_n)$ is said to be a *Green couple of BVP* (2,1), (2,2) if for any $t \in [0, 1]$ the rows of $\mathbf{G}(t, .)$ are elements of L_n^q and the function (3,6) is for any $\mathbf{f} \in L_n^p$ and $\mathbf{r} \in R_n$ the unique solution of BVP (2,1), (2,2).

Clearly, (3,6) verifies (3,1) for any $\mathbf{f} \in L_n^p$ and $\mathbf{r} \in R_n$ if and only if

$$\begin{aligned} \mathbf{x}(t) = {} & \int_0^1 \mathbf{G}(t, s) \left[\mathbf{x}'(s) - \int_0^1 \mathrm{d}_\sigma[\mathbf{P}(s, \sigma)]\, \mathbf{x}(\sigma) \right] \mathrm{d}s \\ & + \mathbf{H}(t) \left[\mathbf{M}\, \mathbf{x}(0) + \int_0^1 \mathbf{K}(s)\, \mathbf{x}'(s)\, \mathrm{d}s \right] \qquad \text{on } [0, 1] \end{aligned} \tag{3,7}$$

holds for any $\mathbf{x} \in W_n^p$. If for any $t \in [0, 1]$ the rows of $\mathbf{G}(t, .)$ are elements of L_n^q, then by I.4.33 and I.4.38

$$\begin{aligned} & \int_0^1 \mathbf{G}(t, \sigma) \left(\int_0^1 \mathrm{d}_s[\mathbf{P}(\sigma, s)]\, \mathbf{x}(s)\, \mathrm{d}\sigma \right) = \int_0^1 \mathrm{d}_s \left[\int_0^1 \mathbf{G}(t, \sigma)\, \mathbf{P}(\sigma, s)\, \mathrm{d}\sigma \right] \mathbf{x}(s) \\ & = - \left(\int_0^1 \mathbf{G}(t, \sigma)\, \mathbf{P}(\sigma, 0)\, \mathrm{d}\sigma \right) \mathbf{x}(0) - \int_0^1 \left(\int_0^1 \mathbf{G}(t, \sigma)\, \mathbf{P}(\sigma, s)\, \mathrm{d}\sigma \right) \mathbf{x}'(s)\, \mathrm{d}s \end{aligned}$$

for any $t \in [0, 1]$ and any $\boldsymbol{x} \in W_n^p$. (We assume $\boldsymbol{P}(t, 1) = \boldsymbol{0}$.) Consequently the right-hand side of (3,7) becomes

$$\int_0^1 \left[\boldsymbol{G}(t, s) + \int_0^1 \boldsymbol{G}(t, \sigma)\, \boldsymbol{P}(\sigma, s)\, d\sigma + \boldsymbol{H}(t)\, \boldsymbol{K}(s)\right] \boldsymbol{x}'(s)\, ds$$
$$+ \left[\boldsymbol{H}(t)\, \boldsymbol{M} + \int_0^1 \boldsymbol{G}(t, \sigma)\, \boldsymbol{P}(\sigma, 0)\, d\sigma\right] \boldsymbol{x}(0).$$

Thus, since for any $\boldsymbol{x} \in W_n^p$

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) + \int_0^t \boldsymbol{x}'(\tau)\, d\tau = \boldsymbol{x}(0) + \int_0^1 \Delta(t, s)\, \boldsymbol{x}'(s)\, ds \qquad \text{on } [0, 1],$$

where

$$(3,8) \qquad \Delta(t, s) = \begin{cases} \boldsymbol{0} & \text{if } t < s, \\ \boldsymbol{I} & \text{if } t \geq s, \end{cases}$$

the relation (3,7) may be rewritten as follows

$$(3,9) \qquad \int_0^1 \left[\boldsymbol{G}(t, s) + \int_0^1 \boldsymbol{G}(t, \sigma)\, \boldsymbol{P}(\sigma, s)\, d\sigma + \boldsymbol{H}(t)\, \boldsymbol{K}(s) - \Delta(t, s)\right] \boldsymbol{x}'(s)\, ds$$
$$+ \left[\boldsymbol{H}(t)\, \boldsymbol{M} + \int_0^1 \boldsymbol{G}(t, \sigma)\, \boldsymbol{P}(\sigma, 0)\, d\sigma - \boldsymbol{I}\right] \boldsymbol{x}(0) = \boldsymbol{0} \qquad \text{for any} \quad \boldsymbol{x} \in W_n^p.$$

Applying I.5.15 we complete the proof of the following

3.7. Proposition. *Let us assume* 2.1 *and* $\boldsymbol{P}(t, 1) = \boldsymbol{0}$ *a.e. on* $[0, 1]$. *Let* $\boldsymbol{G}$: $[0, 1] \times [0, 1] \to L(R_n)$ *and* $\boldsymbol{H}$: $[0, 1] \to L(R_n)$ *and let* $\boldsymbol{G}(t, .)$ *be* L^q*-intergrable on* $[0, 1]$ *for any* $t \in [0, 1]$. *Then* $\boldsymbol{G}(t, s)$, $\boldsymbol{H}(t)$ *is a Green couple of BVP* (2,1), (2,2) *if and only if* (3,5) *holds and for any* $t \in [0, 1]$

$$(3,10) \quad \boldsymbol{G}(t, s) + \int_0^1 \boldsymbol{G}(t, \sigma)\, \boldsymbol{P}(\sigma, s)\, d\sigma + \boldsymbol{H}(t)\, \boldsymbol{K}(s) = \Delta(t, s) \qquad \textit{for a.e.} \quad s \in [0, 1],$$
$$\boldsymbol{H}(t)\, \boldsymbol{M} + \int_0^1 \boldsymbol{G}(t, \sigma)\, \boldsymbol{P}(\sigma, 0)\, d\sigma = \boldsymbol{I},$$

where $\Delta(t, s)$ *is given by* (3,8).

Moreover, we have

3.8. Proposition. *Let the assumptions of* 3.7 *be satisfied. If* $m = n$ *and for any* $t \in [0, 1]$ $\boldsymbol{G}(t, s)$ *and* $\boldsymbol{H}(t)$ *satisfy the system* (3,10), *then* $\boldsymbol{G}(t, s)$, $\boldsymbol{H}(t)$ *is a Green couple of BVP* (2,1), (2,2).

Proof. Since (3,10) implies that (3,9) and consequently also (3,7) hold for any $\boldsymbol{x} \in W_n^p$, it is easy to see that then (3,6) is a solution to BVP (2,1), (2,2) for any couple

$\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in R(\mathscr{L})$. Furthermore, if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in W_n^p$ and $\mathscr{L}\boldsymbol{x}_1 = \mathscr{L}\boldsymbol{x}_2 = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix}$, then inserting $\boldsymbol{x} = \boldsymbol{x}_1$ and $\boldsymbol{x} = \boldsymbol{x}_2$ into (3,7) we obtain

$$\boldsymbol{x}_1(t) \equiv \int_0^1 \boldsymbol{G}(t, s)\, \boldsymbol{f}(s)\, \mathrm{d}s + \boldsymbol{H}(t)\, \boldsymbol{r} \equiv \boldsymbol{x}_2(t) \qquad \text{on } [0, 1],$$

i.e. $\dim N(\mathscr{L}) = 0$. If $m = n$, then by 2.9 $\operatorname{codim} R(\mathscr{L}) = \dim N(\mathscr{L}) = 0$. Thus $R(\mathscr{L}) = L_n^p \times R_m$ (cf. 1.9) and this completes the proof.

Let $\mathscr{L}^*\colon L_n^q \times R_m^* \to L_n^q \times R_n^*$ denote again the analytical representation of the adjoint operator to $\mathscr{L}$ given by 2.6.

3.9. Lemma. *If* (3,5) *holds, then* $\dim N(\mathscr{L}^*) = 0$ *and* $R(\mathscr{L}^*) = L_n^q \times R_m^*$.

Proof. By 2.9 (3,5) implies $0 = \dim N(\mathscr{L}) = \operatorname{codim} R(\mathscr{L}^*) = \dim N(\mathscr{L}^*)$ and the proof will be completed by means of 1.9.

Lemma 3.9 together with the Bounded Inverse Theorem I.3.4 yields

3.10. Proposition. *The operator* $\mathscr{L}^*\colon L_n^q \times R_n^* \to L_n^q \times R_n^*$ *defined by* 2.9 *possesses a bounded inverse.*

3.11. Theorem. *Let us assume* 2.1 *with* $\boldsymbol{P}(t, 1) = \boldsymbol{0}$ *a.e. on* $[0, 1]$ *and* (3,5). *Then there exist functions* $\boldsymbol{G}\colon [0, 1] \times [0, 1] \to L(R_n)$ *and* $\boldsymbol{H}\colon [0, 1] \to L(R_n)$ *which verify the system* (3,10) *for any* $t \in [0, 1]$. *Moreover,*

(i) *given* $t \in [0, 1]$, $\|\boldsymbol{G}(t, .)\|_{L^q} < \infty$ $(q = p/(p-1)$ *if* $p > 1$, $q = \infty$ *if* $p = 1)$,
(ii) *there exists* $\beta \in R$ *such that*

$$\|\boldsymbol{G}(t, .)\|_{L^q} + |\boldsymbol{H}(t)| \le \beta < \infty \qquad \text{for any} \quad t \in [0, 1],$$

(iii) *if* $\tilde{\boldsymbol{G}}\colon [0, 1] \times [0, 1] \to L(R_n)$ *and* $\tilde{\boldsymbol{H}}\colon [0, 1] \to L(R_n)$ *also fulfil* (3,10) *for any* $t \in [0, 1]$, (i) *and* (ii), *then* $\tilde{\boldsymbol{G}}(t, s) = \boldsymbol{G}(t, s)$ *and* $\tilde{\boldsymbol{H}}(t) = \boldsymbol{H}(t)$ *for all* $t \in [0, 1]$ *and for a.e.* $s \in [0, 1]$.

Proof. Let $\boldsymbol{\delta}_j^*(t, s)$ and $\mathbf{e}_j^*$ $(j = 1, 2, \ldots, n)$ be the rows of $\Delta(t, s)$ and $\boldsymbol{I}$, respectively. By 3.10 any equation from the system

$$\mathscr{L}^*(\boldsymbol{g}^*, \boldsymbol{h}^*) = (\boldsymbol{\delta}_j^*(t, .), \mathbf{e}_j^*), \qquad t \in [0, 1], \quad j = 1, 2, \ldots, n \tag{3,11}$$

has a unique solution $(\boldsymbol{g}_j^*(t, .), \boldsymbol{h}_j^*(t))$ in $L_n^q \times R_n^*$ and

$$\|\boldsymbol{g}_j^*(t, .)\|_{L^q} + |\boldsymbol{h}_j^*(t)| \le \varkappa(\|\boldsymbol{\delta}_j^*(t, .)\|_{L^q} + |\mathbf{e}_j^*|) \tag{3,12}$$
$$\text{for any} \quad t \in [0, 1] \quad \text{and} \quad j = 1, 2, \ldots, n,$$

where $\varkappa = \|(\mathscr{L}^*)^{-1}\| < \infty$. Let us put

$$\begin{aligned} \boldsymbol{G}(t,s) &= [\boldsymbol{g}_1(t,s), \boldsymbol{g}_2(t,s), \ldots, \boldsymbol{g}_n(t,s)]^* \qquad \text{on } [0,1] \times [0,1], \\ \boldsymbol{H}(t) &= [\boldsymbol{h}_1(t), \quad \boldsymbol{h}_2(t), \quad \ldots, \boldsymbol{h}_n(t)]^* \qquad \text{on } [0,1]. \end{aligned}$$

Then, given $t \in [0,1]$, the couple $(\boldsymbol{G}(t,s), \boldsymbol{H}(t))$ verifies (3,10). By (3,12)

$$\|\boldsymbol{G}(t,.)\|_{L^q} + |\boldsymbol{H}(t)| \leq n\varkappa < \infty \qquad \text{for any} \quad t \in [0,1]$$

whence (ii) follows. The assertion (iii) is a consequence of the uniqueness of solutions to the equations (3,11).

3.12. Corollary. *Under the assumptions of* 3.11 *the given operator* $\mathscr{L}$ *possesses a bounded inverse*

$$\mathscr{L}^{-1}\colon \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix} \in L_n^p \times R_m \rightarrow \int_0^1 \boldsymbol{G}(t,s)\, \boldsymbol{f}(s)\, \mathrm{d}s + \boldsymbol{H}(t)\, \boldsymbol{r} \in W_n^p .$$

3.13. Theorem. *Let us assume* 2.1 *with* $\boldsymbol{P}(t,1) = \boldsymbol{0}$ *and* (3,5). *Then the couple* $\boldsymbol{G}(t,s)$, $\boldsymbol{H}(t)$ *given by* 3.11 *is a Green couple of BVP* (3,1). *If* $\tilde{\boldsymbol{G}}(t,s)$, $\tilde{\boldsymbol{H}}(t)$ *is also a Green couple to* (3,1), *then* $\tilde{\boldsymbol{G}}(t,s) = \boldsymbol{G}(t,s)$ *and* $\tilde{\boldsymbol{H}}(t) = \boldsymbol{H}(t)$ *for all* $t \in [0,1]$ *and almost all* $s \in [0,1]$.

Proof follows from 3.7 and 3.11.

3.14. Remark. Let $\boldsymbol{r} \in R_n$. According to the definition 3.1 of $\boldsymbol{X}$, $\boldsymbol{x} \in W_n^p$ is a solution to

$$\text{(3,13)} \qquad \boldsymbol{Dx} - \boldsymbol{Px} = \boldsymbol{0}, \qquad \boldsymbol{Sx} = \boldsymbol{r}$$

if and only if $\boldsymbol{x}(t) = \boldsymbol{X}(t)\,\boldsymbol{c}$ on $[0,1]$, where $\boldsymbol{c} \in R_l$ fulfils $(\boldsymbol{SX})\,\boldsymbol{c} = \boldsymbol{r}$. In particular, if we assume (3,5), then by 1.8 $l = n$ and by 3.2 $\det(\boldsymbol{SX}) \neq 0$, i.e. $\boldsymbol{x} \in W_n^p$ verifies (3,13) if and only if $\boldsymbol{x}(t) = \tilde{\boldsymbol{H}}(t)\,\boldsymbol{r}$ on $[0,1]$, where

$$\tilde{\boldsymbol{H}}(t) = \boldsymbol{X}(t)\,(\boldsymbol{SX})^{-1} \qquad \text{on } [0,1].$$

On the other hand, if $\boldsymbol{G}(t,s)$, $\boldsymbol{H}(t)$ is the Green couple of BVP (2,1), (2,2), then $\boldsymbol{x}(t) = \boldsymbol{H}(t)\,\boldsymbol{r}$ on $[0,1]$ is for any $\boldsymbol{r} \in R_m$ the unique solution of (3,13) on W_n^p. Hence $(\boldsymbol{H}(t) - \tilde{\boldsymbol{H}}(t))\,\boldsymbol{r} = \boldsymbol{0}$ on $[0,1]$ for any $\boldsymbol{r} \in R_n$ or

$$\boldsymbol{H}(t) = \boldsymbol{X}(t)\,(\boldsymbol{SX})^{-1} \qquad \text{on } [0,1].$$

Let us notice that the columns of $\boldsymbol{X}$ being elements of W_n^p, the columns of $\boldsymbol{H}(t)$ are also elements of W_n^p.

4. Generalized Green's couples

If $\boldsymbol{P}\colon [0,1] \times [0,1] \rightarrow L(R_n)$ is an $L^2[BV]$-kernel, then obviously

$$\int_0^1 |\boldsymbol{P}(\tau,s)|^2\, \mathrm{d}\tau + \int_0^1 |\boldsymbol{P}(t,\sigma)|^2\, \mathrm{d}\sigma < \infty$$

for almost all $t, s \in [0,1]$ (cf. 1.1). Moreover, according to the assumptions (1,1) and (1,2) (where $p = 2$)

$$\int_0^1 \left(\int_0^1 |\mathbf{P}(t,s)|^2 \, ds\right) dt \leq \int_0^1 \varrho^2(t) \, dt < \infty .$$

By the Tonelli-Hobson Theorem I.4.36 this implies that if an $L^2[BV]$-kernel $\mathbf{P}(t,s)$ is measurable in (t,s) on $[0,1] \times [0,1]$, then

$$|||\mathbf{P}||| = \iint\limits_{[0,1]\times[0,1]} |\mathbf{P}(t,s)|^2 \, dt \, ds = \int_0^1 \left(\int_0^1 |\mathbf{P}(t,s)|^2 \, ds\right) dt < \infty . \tag{4,1}$$

4.1. L^2-kernels. The function $\mathbf{P}$: $[0,1] \times [0,1] \to L(R_n)$ is said to be an *L^2-kernel* if it is measurable in (t,s) on $[0,1] \times [0,1]$ and fulfils (4,1). Given an L^2-kernel $\mathbf{P}$, $|||\mathbf{P}|||$ is defined by (4,1).

Let us recall some basic properties of L^2-kernels and of Fredholm integral equations for $\mathbf{u} \in L_n^2$

$$\mathbf{u}(t) - \int_0^1 \mathbf{P}(t,s) \, \mathbf{u}(s) \, ds = \mathbf{g}(t) \tag{4,2}$$

with an L^2-kernel $\mathbf{P}$. (For the proofs see e.g. Dunford, Schwartz [1] or Smithies [1].)

Let $\mathbf{P}$: $[0,1] \times [0,1] \to L(R_n)$ be an L^2-kernel. Then for any $\mathbf{u} \in L_n^2$, the n-vector valued function

$$\mathbf{g}(t) = \int_0^1 \mathbf{P}(t,s) \, \mathbf{u}(s) \, ds , \qquad t \in [0,1]$$

is L^2-integrable on $[0,1]$ and the mapping $\mathbf{u} \in L_n^2 \to \mathbf{g} \in L_n^2$ is linear and bounded. (This may be shown easily by making use of the Cauchy inequality and the Tonelli-Hobson Theorem I.4.36.) Moreover, a linear operator Θ: $L_n^2 \to L_n^2$ is compact if and only if there exists an L^2-kernel $\mathbf{T}$: $[0,1] \times [0,1] \to L(R_n)$ such that

$$\Theta\colon \mathbf{u} \in L_n^2 \to \int_0^1 \mathbf{T}(t,s) \, \mathbf{u}(s) \, ds \in L_n^2 .$$

If $|||\mathbf{P}||| < 1$, then the equation (4,2) possesses for any $\mathbf{g} \in L_n^2$ a unique solution $\mathbf{u}$ in L_n^2 and there exists an L^2-kernel $\mathbf{R}$: $[0,1] \times [0,1] \to L(R_n)$ such that for any $\mathbf{g} \in L_n^2$ the unique solution $\mathbf{u} \in L_n^2$ of (4,2) is given by

$$\mathbf{u}(t) = \mathbf{g}(t) + \int_0^1 \mathbf{R}(t,s) \, \mathbf{g}(s) \, ds , \qquad t \in [0,1] .$$

$\mathbf{R}$ is called the *resolvent kernel* corresponding to $\mathbf{P}$.

Finally, given an L^2-kernel $\mathbf{P}$, there exist a natural number n', functions $\mathbf{P}_1$: $[0,1] \to L(R_{n'}, R_n)$ and $\mathbf{P}_2$: $[0,1] \to L(R_n, R_{n'})$ L^2-integrable on $[0,1]$ and an L^2-kernel $\mathbf{P}_0$: $[0,1] \times [0,1] \to L(R_n)$ such that

$$|||\mathbf{P}_0||| < 1 \quad \text{and} \quad \mathbf{P}(t,s) = \mathbf{P}_0(t,s) + \mathbf{P}_1(t) \, \mathbf{P}_2(s) \quad \text{on } [0,1] \times [0,1] . \tag{4,3}$$

Let us turn our attention to BVP (2,1), (2,2) fulfilling 2.1 with $p = q = 2$ and $\mathbf{P}(t, 1) = \mathbf{0}$ a.e. on $[0, 1]$. ($\mathbf{P}(t, s)$ is an $L^2[BV]$-kernel, $\mathbf{K}$ is L^2-integrable on $[0, 1]$ and $\mathbf{f} \in L^2_{n'}$.)

A function $\mathbf{x} \in W^2_n$ is a solution to BVP (2,1), (2,2) if and only if

$$\mathbf{x} = \boldsymbol{\Phi}\mathbf{c} + \boldsymbol{\Psi}\mathbf{u} + \boldsymbol{\Psi}\mathbf{f},$$

where

$$\text{(4,4)} \qquad \boldsymbol{\Phi}\colon \mathbf{c} \in R_n \to \mathbf{z}(t) \equiv \mathbf{c} \in W^2_n, \qquad \boldsymbol{\Psi}\colon \mathbf{u} \in L^2_n \to \int_0^t \mathbf{u}(\tau)\,\mathrm{d}\tau \in W^2_n$$

and the couple $\begin{pmatrix} \mathbf{u} \\ \mathbf{c} \end{pmatrix} \in L^2_n \times R_n$ verifies the system

$$\text{(4,5)} \qquad \mathbf{u} - \mathbf{P}\boldsymbol{\Phi}\mathbf{c} - \mathbf{P}\boldsymbol{\Psi}\mathbf{u} = \mathbf{P}\boldsymbol{\Psi}\mathbf{f},$$

$$\text{(4,6)} \qquad \mathbf{S}\boldsymbol{\Phi}\mathbf{c} + \mathbf{S}\boldsymbol{\Psi}\mathbf{u} = \mathbf{r} - \mathbf{S}\boldsymbol{\Psi}\mathbf{f}.$$

In fact, if $\mathbf{x} \in W^2_n$ is a solution to BVP (2,1), (2,2), then $\mathbf{x} = \boldsymbol{\Phi}\,\mathbf{x}(0) + \boldsymbol{\Psi}\mathbf{P}\mathbf{x} + \boldsymbol{\Psi}\mathbf{f}$ and $\mathbf{S}\mathbf{x} = \mathbf{S}\boldsymbol{\Phi}\,\mathbf{x}(0) + \mathbf{S}\boldsymbol{\Psi}\mathbf{P}\mathbf{x} + \mathbf{S}\boldsymbol{\Psi}\mathbf{f} = \mathbf{r}$. Consequently, $\mathbf{u} = \mathbf{P}\mathbf{x}$ and $\mathbf{c} = \mathbf{x}(0)$ satisfy (4,5) and (4,6). (Clearly $\mathbf{u} \in L^2_{n'}$.) On the other hand, if $\begin{pmatrix} \mathbf{u} \\ \mathbf{c} \end{pmatrix} \in L^2_n \times R_n$ is a solution to the system (4,5), (4,6) and $\mathbf{x} = \boldsymbol{\Phi}\mathbf{c} + \boldsymbol{\Psi}\mathbf{u} + \boldsymbol{\Psi}\mathbf{f}$, then $\mathbf{x}(0) = \mathbf{c}$, $\mathbf{P}\mathbf{x} = \mathbf{P}\boldsymbol{\Phi}\mathbf{c} + \mathbf{P}\boldsymbol{\Psi}\mathbf{u} + \mathbf{P}\boldsymbol{\Psi}\mathbf{f} = \mathbf{u}$ and hence $\mathbf{x} - \boldsymbol{\Phi}\,\mathbf{x}(0) - \boldsymbol{\Psi}\mathbf{P}\mathbf{x} = \boldsymbol{\Psi}\mathbf{f}$ and $\mathbf{S}\mathbf{x} = \mathbf{r}$.

Let us mention that in virtue of I.4.33, the composed operator $\mathbf{P}\boldsymbol{\Psi}\colon L^2_n \to L^2_n$ is given by

$$\text{(4,7)} \qquad \mathbf{P}\boldsymbol{\Psi}\colon \mathbf{u} \in L^2_n \to -\int_0^1 \mathbf{P}(t, s)\,\mathbf{u}(s)\,\mathrm{d}s \in L^2_n.$$

Now, let a natural number n', an L^2-kernel $\mathbf{P}_0\colon [0, 1] \times [0, 1] \to L(R_n)$ and L^2-integrable functions $\mathbf{P}_1\colon [0, 1] \to L(R_{n'}, R_n)$ and $\mathbf{P}_2\colon [0, 1] \to L(R_n, R_{n'})$ be such that (4,3) holds. Furthermore, let $\mathbf{R}_0\colon [0, 1] \times [0, 1] \to L(R_n)$ be the resolvent kernel corresponding to $\mathbf{P}_0$. The symbols $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{R}_0$ will denote the linear operators

$$\text{(4,8)} \qquad \begin{aligned} \mathbf{P}_0\colon \mathbf{u} \in L^2_n &\longrightarrow -\int_0^1 \mathbf{P}_0(t, s)\,\mathbf{u}(s)\,\mathrm{d}s \in L^2_n, \\ \mathbf{P}_1\colon \mathbf{d} \in R_{n'} &\longrightarrow \quad -\mathbf{P}_1(t)\,\mathbf{d} \in L^2_n, \\ \mathbf{P}_2\colon \mathbf{u} \in L^2_n &\longrightarrow \quad \int_0^1 \mathbf{P}_2(s)\,\mathbf{u}(s)\,\mathrm{d}s \in R_{n'}, \\ \mathbf{R}_0\colon \mathbf{u} \in L^2_n &\longrightarrow -\int_0^1 \mathbf{R}_0(t, s)\,\mathbf{u}(s)\,\mathrm{d}s \in L^2_n, \end{aligned}$$

as well. All of them are obviously compact.

By (4,3) and (4,8) we may write

$$\mathbf{P}\boldsymbol{\Psi} = \mathbf{P}_0 + \mathbf{P}_1\mathbf{P}_2$$

and the equation (4,5) becomes

$$\boldsymbol{u} - \boldsymbol{P}_0\boldsymbol{u} = \boldsymbol{P}\boldsymbol{\Phi}\boldsymbol{c} + \boldsymbol{P}_1\boldsymbol{P}_2\boldsymbol{u} + \boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{f}\,.$$

Accordingly

$$\boldsymbol{u} - [\boldsymbol{I} + \boldsymbol{R}_0]\,(\boldsymbol{P}\boldsymbol{\Phi}\boldsymbol{c} + \boldsymbol{P}_1\boldsymbol{P}_2\boldsymbol{u}) = [\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{f}\,. \tag{4,9}$$

Let us denote

$$\boldsymbol{d} = \boldsymbol{P}_2\boldsymbol{u}\,.$$

Then the equation (4,9) reduces to

$$\boldsymbol{u} = [\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{P}\boldsymbol{\Phi}\boldsymbol{c} + [\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{P}_1\boldsymbol{d} + [\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{f}\,. \tag{4,10}$$

Applying $\boldsymbol{P}_2$ to (4,10) and inserting (4,10) into (4,6) we reduce the system (4,5), (4,6) to the system of equations for $\boldsymbol{c} \in R_n$ and $\boldsymbol{d} \in R_{n'}$

$$\boldsymbol{B}\begin{pmatrix}\boldsymbol{c}\\ \boldsymbol{d}\end{pmatrix} = \begin{pmatrix}\boldsymbol{F}_1\boldsymbol{f}\\ \boldsymbol{r} - \boldsymbol{F}_2\boldsymbol{f}\end{pmatrix} \tag{4,11}$$

where

$$\boldsymbol{B}\colon \begin{pmatrix}\boldsymbol{c}\\ \boldsymbol{d}\end{pmatrix} \in R_{n+n'} \to \begin{pmatrix}-\boldsymbol{P}_2[\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{P}\boldsymbol{\Phi}\boldsymbol{c} + (\boldsymbol{I} - \boldsymbol{P}_2[\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{P}_1)\,\boldsymbol{d}\\ \boldsymbol{S}(\boldsymbol{I} - \boldsymbol{\Psi}[\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{P})\,\boldsymbol{\Phi}\boldsymbol{c} + \boldsymbol{S}\boldsymbol{\Psi}[\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{P}_1\boldsymbol{d}\end{pmatrix} \in R_{m+n'} \tag{4,12}$$

and

$$\begin{aligned} &\boldsymbol{F}_1\colon \boldsymbol{f} \in L_n^2 \to \boldsymbol{P}_2[\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{f} \in R_{n'}\,,\\ &\boldsymbol{F}_2\colon \boldsymbol{f} \in L_n^2 \to \boldsymbol{S}\boldsymbol{\Psi}(\boldsymbol{I} + [\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{P}\boldsymbol{\Psi})\,\boldsymbol{f} \in R_m\,. \end{aligned} \tag{4,13}$$

The operator $\boldsymbol{B}$ may be represented by a uniquely determined $(m+n') \times (n+n')$-matrix. Let us denote this matrix again by $\boldsymbol{B}$.

Thus BVP (2,1), (2,2) possesses a solution $\boldsymbol{x} \in W_n^2$ if and only if the system (4,11) possesses a solution $\begin{pmatrix}\boldsymbol{c}\\ \boldsymbol{d}\end{pmatrix} \in R_{n+n'}$ and $\boldsymbol{x}$ is then given by

$$\boldsymbol{x} = (\boldsymbol{\Phi} + \boldsymbol{\Psi}[\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{P}\boldsymbol{\Phi})\,\boldsymbol{c} + \boldsymbol{\Psi}[\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{P}_0\boldsymbol{d} + \boldsymbol{\Psi}[\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{P}\boldsymbol{\Psi}\boldsymbol{f} + \boldsymbol{\Psi}\boldsymbol{f}\,. \tag{4,14}$$

Let $\Delta_{1,1} \in L(R_{n'}, R_n)$, $\Delta_{1,2} \in L(R_m, R_n)$, $\Delta_{2,1} \in L(R_{n'})$ and $\Delta_{2,2} \in L(R_m, R_{n'})$ be chosen in such a way that

$$\boldsymbol{B}^+ = \begin{bmatrix}\Delta_{1,1}, & \Delta_{1,2}\\ \Delta_{2,1}, & \Delta_{2,2}\end{bmatrix} \in L(R_{m+n'}, R_{n+n'})$$

fulfils $\boldsymbol{B}\boldsymbol{B}^+\boldsymbol{B} = \boldsymbol{B}$ (e.g. $\boldsymbol{B}^+ = \boldsymbol{B}^\#$). Then if (4,11) has a solution, the couple

$$\begin{aligned} \boldsymbol{c} &= [\Delta_{1,1}\boldsymbol{F}_1 - \Delta_{1,2}\boldsymbol{F}_2]\,\boldsymbol{f} + \Delta_{1,2}\boldsymbol{r} \in R_n\,,\\ \boldsymbol{d} &= [\Delta_{2,1}\boldsymbol{F}_1 - \Delta_{2,2}\boldsymbol{F}_2]\,\boldsymbol{f} + \Delta_{2,2}\boldsymbol{r} \in R_{n'}\,, \end{aligned} \tag{4,15}$$

is also its solution.

Inserting (4,15) into (4,14) we obtain that if BVP (2,1), (2,2) has a solution, then

(4,16) $$\boldsymbol{x} = \boldsymbol{\Phi}[\boldsymbol{G}_1 \boldsymbol{f} + \boldsymbol{H}_1 \boldsymbol{r}] + \boldsymbol{\Psi}[\boldsymbol{I} + \boldsymbol{R}_0](\boldsymbol{G}_2 \boldsymbol{f} + \boldsymbol{H}_2 \boldsymbol{r}) + \boldsymbol{\Psi} \boldsymbol{f}$$

with

(4,17) $$\boldsymbol{G}_1 = \boldsymbol{\Delta}_{1,1}\boldsymbol{F}_1 - \boldsymbol{\Delta}_{1,2}\boldsymbol{F}_2, \qquad \boldsymbol{H}_1 = \boldsymbol{\Delta}_{1,2},$$
$$\boldsymbol{G}_2 = \boldsymbol{P\Phi}(\boldsymbol{\Delta}_{1,1}\boldsymbol{F}_1 - \boldsymbol{\Delta}_{1,2}\boldsymbol{F}_2) + \boldsymbol{P}_1(\boldsymbol{\Delta}_{2,1}\boldsymbol{F}_1 - \boldsymbol{\Delta}_{2,2}\boldsymbol{F}_2) + \boldsymbol{P\Psi},$$
$$\boldsymbol{H}_2 = \boldsymbol{P\Phi\Delta}_{1,2} + \boldsymbol{P}_1\boldsymbol{\Delta}_{2,2}$$

is also its solution. As $\boldsymbol{G}_1: L_n^2 \to R_n$ is a linear bounded n-vector valued functional on L_n^2 and $[\boldsymbol{I} + \boldsymbol{R}_0]\boldsymbol{G}_2 \in K(L_n^2)$, there exist an L^2-integrable function $\boldsymbol{G}_1: [0,1] \to L(R_n)$ and an L^2-kernel $\boldsymbol{G}_2: [0,1] \times [0,1] \to L(R_n)$ such that

(4,18) $$\boldsymbol{G}_1: \boldsymbol{f} \in L_n^2 \to \int_0^1 \boldsymbol{G}_1(s)\,\boldsymbol{f}(s)\,\mathrm{d}s \in R_n,$$
$$[\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{G}_2: \boldsymbol{f} \in L_n^2 \to \int_0^1 \boldsymbol{G}_2(t,s)\,\boldsymbol{f}(s)\,\mathrm{d}s \in L_n^2.$$

Applying the Tonelli-Hobson Theorem I.4.36 we may show that

$$\int_0^t \left(\int_0^1 \boldsymbol{G}_2(\tau,s)\,\boldsymbol{f}(s)\,\mathrm{d}s\right)\mathrm{d}\tau = \int_0^1 \left(\int_0^t \boldsymbol{G}_2(\tau,s)\,\mathrm{d}\tau\right)\boldsymbol{f}(s)\,\mathrm{d}s$$

for any $\boldsymbol{f} \in L_n^2$ and $t \in [0,1]$, i.e.

(4,20) $$\boldsymbol{\Psi}[\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{G}_2: \boldsymbol{f} \in L_n^2 \to \int_0^1 \left(\int_0^t \boldsymbol{G}_2(\tau,s)\,\mathrm{d}\tau\right)\boldsymbol{f}(s)\,\mathrm{d}s \in W_n^2.$$

Furthermore, by (4,3), (4,4) and (4,8) there exist an L^2-integrable function $\tilde{\boldsymbol{H}}_2: [0,1] \to L(R_m, R_n)$ such that

$$\boldsymbol{H}_2 = \boldsymbol{P\Phi\Delta}_{1,2} + \boldsymbol{P}_1\boldsymbol{\Delta}_{2,2}: \boldsymbol{r} \in R_m \to \tilde{\boldsymbol{H}}_2(t)\,\boldsymbol{r} \in L_n^2.$$

Consequently,

(4,21) $$\boldsymbol{\Psi}[\boldsymbol{I} + \boldsymbol{R}_0]\,\boldsymbol{H}_2: \boldsymbol{r} \in R_m \to \left(\int_0^t \boldsymbol{H}_2(\tau)\,\mathrm{d}\tau\right)\boldsymbol{r},$$

where

$$\boldsymbol{H}_2(t) = \tilde{\boldsymbol{H}}_2(t) + \int_0^1 \boldsymbol{R}_0(t,\tau)\,\tilde{\boldsymbol{H}}_2(\tau)\,\mathrm{d}\tau, \qquad t \in [0,1]$$

is also L^2-integrable on $[0,1]$. Inserting (4,18), (4,20) and (4,21) into (4,16) we obtain that if BVP (2,1), (2,2) has a solution, then also

(4,22) $$\boldsymbol{x}(t) = \int_0^1 \boldsymbol{G}_0(t,s)\,\boldsymbol{f}(s)\,\mathrm{d}s + \boldsymbol{H}_0(t)\,\boldsymbol{r}, \qquad t \in [0,1],$$

with

$$(4,23) \qquad \mathbf{G}_0(t,s) = \mathbf{G}_1(s) + \int_0^t \mathbf{G}_2(\tau,s)\,\mathrm{d}\tau + \Delta(t,s) \qquad \text{on } [0,1]\times[0,1],$$
$$\Delta(t,s) = \mathbf{0} \quad \text{if} \quad t < s, \qquad \Delta(t,s) = \mathbf{I} \quad \text{if} \quad t \geq s,$$
$$\mathbf{H}_0(t) = \mathbf{H}_1 + \int_0^t \mathbf{H}_2(\tau)\,\mathrm{d}\tau \qquad \text{on } [0,1]$$

is a solution to BVP (2,1), (2,2). It follows from the definition of the functions $\mathbf{G}_0(t,s)$ and $\mathbf{H}_0(t)$, that the linear operator

$$(4,24) \qquad \mathscr{L}^+ : \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} \in L_n^2 \times R_m \to \int_0^1 \mathbf{G}_0(t,s)\,\mathbf{f}(s)\,\mathrm{d}s + \mathbf{H}_0(t)\,\mathbf{r} \in W_n^2$$

is bounded. The results obtained are summarized in the following theorem.

4.2. Theorem. *Let the assumptions* 2.1 *with* $p = q = 2$ *be fulfilled and, moreover, let* $\mathbf{P}(t,s)$ *be measurable in* (t,s) *on* $[0,1]\times[0,1]$. *Then there exist functions* $\mathbf{G}_0\colon [0,1]\times[0,1] \to L(R_n)$ *and* $\mathbf{H}_0\colon [0,1] \to L(R_m, R_n)$ *such that for any* $\mathbf{f} \in L_n^2$ *and* $\mathbf{r} \in R_m$ *the function* $\mathbf{x}(t)$ *given by* (4,22) *belongs to* W_n^2 *and the linear operator* $\mathscr{L}^+$ *given by* (4,24) *is bounded. Furthermore, if BVP* (2,1), (2,2) *possesses a solution, then* (4,22) (*i.e.* $\mathbf{x} = \mathscr{L}^+ \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix}$) *is also its solution.*

4.3. Remark. According to the definition IV.3.10 we may say that $\mathbf{G}_0(t,s)$, $\mathbf{H}_0(t)$ is a generalized Green's couple of BVP (2,1), (2,2). The operator $\mathscr{L}^+$ given by (4,24) fulfils the relation $\mathscr{L}\mathscr{L}^+\mathscr{L} = \mathscr{L}$.

4.4. Proposition. *The functions* $\mathbf{G}_0(t,s)$ *and* $\mathbf{H}_0(t)$ *defined by* (4,23) *have the following properties*

(i) $\mathbf{H}_0$ *possesses a.e. on* $[0,1]$ *a derivative which is* L^2*-integrable on* $[0,1]$,
(ii) $\mathbf{G}_0$ *is an* L^2*-kernel,* $\mathbf{G}_0(.,s)$ *is of bounded variation on* $[0,1]$ *for a.e.* $s \in [0,1]$,
(iii) $\gamma(s) = |\mathbf{G}_0(0,s)| + \operatorname{var}_0^1 \mathbf{G}_0(.,s) \in L^2$,
(iv) *for almost every* $s \in [0,1]$ *the columns of* $\mathbf{G}_0(.,s) - \Delta(.,s)$ *belong to the space* W_n^2.

Proof follows from the construction of the functions $\mathbf{G}_0(t,s)$ and $\mathbf{H}_0(t)$ ($\mathbf{G}_0(0,s) = \mathbf{G}_1(s)$, $\operatorname{var}_0^1 \Delta(.,s) \leq 1$ and hence

$$\gamma(s) \leq |\mathbf{G}_1(s)| + \int_0^1 |\mathbf{G}_2(\tau,s)|\,\mathrm{d}\tau + 1 \qquad \text{a.e. on } [0,1].)$$

4.5. Remark. If $k = \dim N(\mathscr{L}) > 0$, let $\mathbf{X}_0$ denote the $n \times k$-matrix function whose columns form a basis in $N(\mathscr{L})$. If $k^* = \dim N(\mathscr{L}^*) > 0$, let $\mathbf{Y}_0\colon [0,1] \to L(R_n, R_{k^*})$

and $\Lambda_0 \in L(R_m, R_{k^*})$ be such that the couples $(\mathbf{y}_j^*, \lambda_j^*)$ $(j = 1, 2, \ldots, k^*)$ of their rows form a basis in $N(\mathscr{L}^*)$. Then evidently for any L^2-integrable function $\Theta_1: [0, 1] \to L(R_n, R_k)$, any matrix $\Theta_2 \in L(R_m, R_k)$ and any function $\Sigma: [0, 1] \to L(R_{k^*}, R_n)$ of bounded variation on $[0, 1]$

$$\text{(4,25)} \qquad \begin{aligned} \mathbf{G}(t, s) &= \mathbf{G}_0(t, s) + \mathbf{X}_0(t)\,\Theta_1(s) + \Sigma(t)\,\mathbf{Y}_0(s)\,, \qquad t, s \in [0, 1]\,, \\ \mathbf{H}(t) &= \mathbf{H}_0(t) + \mathbf{X}_0(t)\,\Theta_2 + \Sigma(t)\,\Lambda_0 \end{aligned}$$

is also a generalized Green's couple of BVP (2,1), (2,2) and fulfils (i)–(iv) from 4.4 in place of $\mathbf{G}_0(t, s)$ $\mathbf{H}_0(t)$.

4.6. Definition. Generalized Green's couples of the form (4,25) will be called *standard generalized Green's couples.*

4.7. Remark. It is easy to verify that given a standard generalized Green couple $\mathbf{G}(t, s)$, $\mathbf{H}(t)$, the operator

$$\text{(4,26)} \qquad \mathscr{L}^+: \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} \in L_n^2 \times R_m \to \int_0^1 \mathbf{G}(t, s)\,\mathbf{f}(s)\,\mathrm{d}s + \mathbf{H}(t)\,\mathbf{r}$$

is bounded and fulfils the relation $\mathscr{L}\mathscr{L}^+\mathscr{L} = \mathscr{L}$.*)

4.8. Remark. Making use of the equivalence between BVP (2,1), (2,2) and the linear algebraic equation (4,11) we could obtain (under the assumptions 2.1 with $p = q = 2$) the basic results of the Section V.2 in a more elementary way. An analogous procedure can be applied also to BVP

$$\text{(4,27)} \qquad \mathbf{x}'(t) - \mathbf{A}(t)\,\mathbf{x}(t) - \mathbf{C}(t)\,\mathbf{x}(0) - \mathbf{D}(t)\,\mathbf{x}(1) - \int_0^1 \mathrm{d}_s[\mathbf{R}(t, s)]\,\mathbf{x}(s) = \mathbf{f}(t)$$
$$\text{a.e. on } [0, 1]\,,$$

$$\text{(4,28)} \qquad \mathbf{M}\,\mathbf{x}(0) + \int_0^1 \mathbf{K}(t)\,\mathbf{x}'(t)\,\mathrm{d}t = \mathbf{r}\,,$$

where $\mathbf{A}$ is supposed to be only L-integrable on $[0, 1]$ and $\mathbf{K}$ is measurable and essentially bounded on $[0, 1]$. (In general BVP (4,27), (4,28) cannot be rewritten as the system of the form (2,1), (2,2) fulfilling the assumptions of this section.) If $\mathbf{X}(t)$ denotes the fundamental matrix solution of the equation $\mathbf{x}'(t) - \mathbf{A}(t)\,\mathbf{x}(t) = \mathbf{0}$, then BVP (4,27), (4,28) will be transferred to a system of integro-algebraical equations

*) Since in general we may not assume that $\mathbf{X}_0(t)$ has a full rank on $[0, 1]$ (cf. 1.10), we may not apply the procedure from IV.3.12 to show that $\mathscr{L}^+ \in B(L_n^2 \times R_m, W_n^2)$ fulfils $\mathscr{L}\mathscr{L}^+\mathscr{L} = \mathscr{L}$ if and only if $\mathscr{L}^+$ is given by (4,26), where $\mathbf{G}(t, s)$, $\mathbf{H}(t)$ is a standard generalized Green's couple.

for $\boldsymbol{u} \in L_n^2$ and $\boldsymbol{c} \in R_n$ of the form (4,5), (4,6) (with an L^2-kernel) by means of the substitution

$$\boldsymbol{u}(t) = \boldsymbol{C}(t)\,\boldsymbol{x}(0) + \boldsymbol{D}(t)\,\boldsymbol{x}(1) + \int_0^1 \mathrm{d}_s[\boldsymbol{R}(t,s)]\,\boldsymbol{x}(s)$$

$$\boldsymbol{c} = \boldsymbol{x}(0)\,.$$

On the other hand,

$$\boldsymbol{x}(t) = \boldsymbol{X}(t)\,\boldsymbol{c} + \boldsymbol{X}(t)\int_0^t \boldsymbol{X}^{-1}(s)\,\boldsymbol{u}(s)\,\mathrm{d}s + \boldsymbol{X}(t)\int_0^t \boldsymbol{X}^{-1}(s)\,\boldsymbol{f}(s)\,\mathrm{d}s\,,$$

i.e. $\boldsymbol{x} = \boldsymbol{U}\boldsymbol{c} + \boldsymbol{V}\boldsymbol{u} + \boldsymbol{V}\boldsymbol{f}$.

5. Best approximate solutions

We still assume that $\boldsymbol{P}\colon [0,1] \times [0,1] \to L(R_n)$ *is a measurable* $L^2[BV]$-*kernel*, $\boldsymbol{P}(t,1) = \boldsymbol{0}$ *a.e. on* $[0,1]$, *the columns of* $\boldsymbol{K}\colon [0,1] \to L(R_n, R_m)$ *belong to* L_n^2, $\boldsymbol{f} \in L_n^2$ *and* $\boldsymbol{r} \in R_m$. Given $\boldsymbol{x}, \boldsymbol{u} \in W_n^2$, let us put

$$(5{,}1) \qquad (\boldsymbol{x}, \boldsymbol{u})_X = \int_0^1 \boldsymbol{u}^*(t)\,\boldsymbol{x}(t)\,\mathrm{d}t \in R\,.$$

Clearly, $\boldsymbol{x}, \boldsymbol{u} \in W_n^2 \to (\boldsymbol{x}, \boldsymbol{u})_X \in R$ is a bilinear form on $W_n^2 \times W_n^2$, while $(\boldsymbol{x}, \boldsymbol{u})_X = (\boldsymbol{u}, \boldsymbol{x})_X$ for all $\boldsymbol{x}, \boldsymbol{u} \in W_n^2$ and $(\boldsymbol{x}, \boldsymbol{x})_X = 0$ if and only if $\boldsymbol{x}(t) \equiv \boldsymbol{0}$ on $[0,1]$. It means that $(.,.)_X$ is an inner product and $\boldsymbol{x} \in W_n^2 \to \|\boldsymbol{x}\|_X = (\boldsymbol{x}, \boldsymbol{x})_X^{1/2}$ is a norm on W_n^2.

Analogously,

$$(5{,}2) \qquad \boldsymbol{\varphi} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix},\ \boldsymbol{\psi} = \begin{pmatrix} \boldsymbol{g} \\ \boldsymbol{q} \end{pmatrix} \in L_n^2 \times R_m \to (\boldsymbol{\varphi}, \boldsymbol{\psi})_Y = \langle \boldsymbol{\varphi}, \boldsymbol{\psi}^* \rangle_{L^2 \times R}$$

$$= \int_0^1 \boldsymbol{g}^*(t)\,\boldsymbol{f}(t)\,\mathrm{d}t + \boldsymbol{q}^*\boldsymbol{r} \in R$$

is an inner product on $L_n^2 \times R_m$ and $\boldsymbol{\varphi} \in L_n^2 \times R_m \to \|\boldsymbol{\varphi}\|_Y = (\boldsymbol{\varphi}, \boldsymbol{\varphi})_Y^{1/2}$ is a norm on $L_n^2 \times R_m$. Moreover, as $|\boldsymbol{c}| \le |\boldsymbol{c}|_e = (\boldsymbol{c}^*\boldsymbol{c})^{1/2} \le n|\boldsymbol{c}|$ for any $\boldsymbol{c} \in R_n$,

$$\left\|\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix}\right\|_{L^2 \times R}^2 = \left(\left(\int_0^1 |\boldsymbol{f}(t)|^2\,\mathrm{d}t\right)^{1/2} + |\boldsymbol{r}|\right)^2 \ge \frac{1}{n^2}\left(\left(\int_0^1 |\boldsymbol{f}(t)|_e^2\,\mathrm{d}t\right)^{1/2} + |\boldsymbol{r}|_e\right)^2 \ge \left(\frac{1}{n^2}\right)\left\|\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{r} \end{pmatrix}\right\|_Y^2$$

$$\text{for all} \quad \boldsymbol{f} \in L_n^2 \quad \text{and} \quad \boldsymbol{r} \in R_m\,.$$

On the other hand,

$$\int_0^1 |\boldsymbol{f}(t)|^2\,\mathrm{d}t + |\boldsymbol{r}|^2 - 2\left(\int_0^1 |\boldsymbol{f}(t)|^2\,\mathrm{d}t\right)^{1/2} |\boldsymbol{r}| \ge 0$$

and hence

$$2\left\|\begin{pmatrix}\mathbf{f}\\ \mathbf{r}\end{pmatrix}\right\|_Y^2 \geq 2\left(\int_0^1 |\mathbf{f}(t)|^2\,\mathrm{d}t + |\mathbf{r}|^2\right) \geq \left(\left(\int_0^1 |\mathbf{f}(t)|^2\,\mathrm{d}t\right)^{1/2} + |\mathbf{r}|\right)^2 = \left\|\begin{pmatrix}\mathbf{f}\\ \mathbf{r}\end{pmatrix}\right\|_{L^2\times R}^2,$$

i.e.

$$\frac{1}{n}\,\|\varphi\|_Y \leq \|\varphi\|_{L^2\times R} \leq \sqrt{(2)}\,\|\varphi\|_Y \qquad \text{for each} \quad \varphi \in L_n^2 \times R_m\,. \tag{5,3}$$

It follows immediately that the space $L_n^2 \times R_m$ endowed with the norm $\|\,.\,\|_Y$ is complete, i.e. it is a Hilbert space.

5.1. Notation. In the subsequent text X stands for the inner product space of elements of W_n^2 with the inner product (5,1) and the corresponding norm $\|\,.\,\|_X$. Y denotes the Hilbert space of elements of $L_n^2 \times R_m$ equipped with the inner product (5,2) and the corresponding norm $\|\,.\,\|_Y$. The operator $\mathbf{x} \in X \to \mathscr{L}\mathbf{x} \in Y$ (cf. (2,3)) is denoted by $\mathscr{A}$.

5.2. Remark. Evidently $\mathscr{A} \in L(X, Y)$, $R(\mathscr{A}) = R(\mathscr{L})$ and $N(\mathscr{A}) = N(\mathscr{L})$. It follows easily from (5,3) and 2.9 that $R(\mathscr{A})$ is closed in Y.

5.3. Remark. Let us notice that in general $\mathscr{A}$ is unbounded.

5.4. Notation. If $k = \dim N(\mathscr{L}) > 0$, then $\mathbf{X}_0$ denotes the $n \times k$-matrix valued function whose columns form a basis in $N(\mathscr{L})$. If $k^* = \dim N(\mathscr{L}^*) > 0$ and $(\mathbf{y}_j, \boldsymbol{\lambda}_j^*) \in L_n^2 \times R_m^*$ $(j = 1, 2, \ldots, k^*)$ is a basis in $N(\mathscr{L}^*)$, let us put $\mathbf{Y}_0^*(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t), \ldots, \mathbf{y}_{k^*}(t)]$ on $[0, 1]$ and $\Lambda_0^* = [\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_{k^*}]$.

5.5. Lemma. *If $k^* > 0$, then the $k^* \times k^*$-matrix*

$$\mathbf{C} = \int_0^1 \mathbf{Y}_0(t)\,\mathbf{Y}_0^*(t)\,\mathrm{d}t + \Lambda_0\Lambda_0^*$$

is regular. If we put

$$\boldsymbol{\Pi}_1\colon \begin{pmatrix}\mathbf{f}\\ \mathbf{r}\end{pmatrix} \in Y \to \begin{pmatrix}\mathbf{f}(t)\\ \mathbf{r}\end{pmatrix} - \begin{bmatrix}\mathbf{Y}_0^*(t)\\ \Lambda_0^*\end{bmatrix}\mathbf{C}^{-1}\left[\int_0^1 \mathbf{Y}_0(s)\,\mathbf{f}(s)\,\mathrm{d}s + \Lambda_0\mathbf{r}\right] \in Y \qquad \text{if} \quad k^* > 0, \tag{5,4}$$

$$\boldsymbol{\Pi}_1 = \mathbf{I} \qquad \textit{if} \quad k^* = 0,$$

then $\boldsymbol{\Pi}_1$ is an orthogonal bounded projection of Y onto $R(\mathscr{A})$.

Proof. If there were $\boldsymbol{\delta}^*\mathbf{C} = \mathbf{0}$ for some $\boldsymbol{\delta} \in R_{k^*}$, then it would be also $0 = \boldsymbol{\delta}^*\mathbf{C}\boldsymbol{\delta}$, i.e.

$$0 = \int_0^1 (\boldsymbol{\delta}^*\,\mathbf{Y}_0(t))\,(\mathbf{Y}_0^*(t)\,\boldsymbol{\delta})\,\mathrm{d}t + (\boldsymbol{\delta}^*\Lambda_0)\,(\Lambda_0^*\boldsymbol{\delta}) = \|(\mathbf{Y}_0^*(t)\,\boldsymbol{\delta},\, \Lambda_0^*\boldsymbol{\delta})\|_Y^2\,.$$

This may hold if and only if $\delta^*[\mathbf{Y}_0(t), \Lambda_0] = \mathbf{0}$ a.e. on $[0, 1]$. Hence $\delta^*\mathbf{C} = \mathbf{0}$ implies $\delta^* = \mathbf{0}$.

Furthermore, it follows easily from 2.12 that $\Pi_1\varphi \in R(\mathscr{A})$ for any $\varphi \in Y$ and $\Pi_1\varphi = \varphi$ if $\varphi \in R(\mathscr{A})$. Finally, given $\varphi \in \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} \in Y$ and $\psi = \begin{pmatrix} \mathbf{g} \\ \mathbf{q} \end{pmatrix} \in R(\mathscr{A})$, we have by 2.12

$$(\varphi - \Pi_1\varphi, \psi)_Y = \left[\int_0^1 \mathbf{g}^*(t)\, \mathbf{Y}_0^*(t)\,\mathrm{d}t + \mathbf{q}^*\Lambda_0^*\right] \mathbf{C}^{-1} \left[\int_0^1 \mathbf{Y}_0(s)\, \mathbf{f}(s)\,\mathrm{d}s + \Lambda_0 \mathbf{r}\right] = 0\,.$$

The boundedness of Π_1 is obvious.

5.6. Lemma. *If $k > 0$, then the $k \times k$-matrix*

$$\mathbf{D} = \int_0^1 \mathbf{X}_0^*(t)\, \mathbf{X}_0(t)\,\mathrm{d}t$$

is regular. The mapping

$$(5{,}5) \qquad \Pi_2\colon \mathbf{x} \in X \to \mathbf{X}_0(t)\, \mathbf{D}^{-1} \left(\int_0^1 \mathbf{X}_0^*(s)\, \mathbf{x}(s)\,\mathrm{d}s\right) \in X \qquad \text{if} \quad k > 0\,,$$

$$\Pi_2 = \mathbf{0} \qquad \text{if} \quad k = 0$$

is an orthogonal bounded projection of X onto $N(\mathscr{A})$.

Proof. The regularity of $\mathbf{D}$ follows analogously as the regularity of $\mathbf{C}$. Obviously $R(\Pi_2) \subset N(\mathscr{A})$. Furthermore, if $\delta \in R_k$ and $\mathbf{x}(t) \equiv \mathbf{X}_0(t)\,\delta$ on $[0, 1]$ (i.e. $\mathbf{x} \in N(\mathscr{A})$), then

$$(\Pi_2\mathbf{x})(t) = \mathbf{X}_0(t)\, \mathbf{D}^{-1} \left(\int_0^1 \mathbf{X}_0^*(s)\, \mathbf{X}_0(s)\,\mathrm{d}s\right) \delta = \mathbf{X}_0(t)\,\delta = \mathbf{x}(t)\,.$$

Consequently $R(\Pi_2) = N(\mathscr{A})$ and $\Pi_2^2 = \Pi_2$. Finally, given $\mathbf{x} \in X$, $\delta \in R_k$ and $\mathbf{u}(t) = \mathbf{X}_0(t)\,\delta$,

$$(\mathbf{x} - \Pi_2\mathbf{x}, \mathbf{u})_X$$
$$= \delta^* \left(\int_0^1 \mathbf{X}_0^*(t)\, \mathbf{x}(t)\,\mathrm{d}t\right) - \delta^* \left(\int_0^1 \mathbf{X}_0^*(t)\, \mathbf{X}(t)\,\mathrm{d}t\right) \mathbf{D}^{-1} \left(\int_0^1 \mathbf{X}_0^*(t)\, \mathbf{x}(t)\,\mathrm{d}t\right) = 0\,.$$

5.7. Definition. A function $\mathbf{u}_0 \in W_n^2$ is said to be a *least square solution* or a *best approximate solution of BVP* (2,1), (2,2), if it is a least square solution or a best approximate solution of the operator equation

$$(5{,}6) \qquad \mathscr{A}\mathbf{x} = \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix},$$

respectively.

Let us assume 2.1 *with $p = q = 2$ and let $\mathbf{P}(t, s)$ be measurable in (t, s) on $[0, 1] \times [0, 1]$. Then there exist a standard generalized Green's couple $\mathbf{G}(t, s)$,*

$\mathbf{H}(t)$ *of BVP* (2,1), (2,2) *such that for any* $\mathbf{f} \in L_n^2$ *and* $\mathbf{r} \in R_m$

$$(5,7) \qquad \mathbf{u}_0(t) = \int_0^1 \mathbf{G}(t,s)\,\mathbf{f}(s)\,\mathrm{d}s + \mathbf{H}(t)\,\mathbf{r} \qquad \textit{on } [0,1]$$

is the unique best approximate solution of BVP (2,1), (2,2).

Proof. Let $\mathbf{G}_0(t,s)$, $\mathbf{H}_0(t)$ be the generalized Green's couple of BVP (2,1), (2,2) given by 4.2 and let $\mathscr{L}^+ : L_n^2 \times R_m \to W_n^2$ be the corresponding generalized inverse operator to $\mathscr{L}$ given by (4,24). Let us define $\mathscr{A}^+ : \begin{pmatrix}\mathbf{f}\\ \mathbf{r}\end{pmatrix} \in Y \to \mathscr{L}^+ \begin{pmatrix}\mathbf{f}\\ \mathbf{r}\end{pmatrix} \in X$ and

$$(5,8) \qquad \mathscr{A}^{\#} : \begin{pmatrix}\mathbf{f}\\ \mathbf{r}\end{pmatrix} \in Y \to (\mathbf{I} - \boldsymbol{\Pi}_2)\,\mathscr{A}^+ \boldsymbol{\Pi}_1 \begin{pmatrix}\mathbf{f}\\ \mathbf{r}\end{pmatrix} \in X\,,$$

where $\boldsymbol{\Pi}_1 \in B(Y)$ and $\boldsymbol{\Pi}_2 \in B(X)$ are given by (5,4) and (5,5), respectively. Then $\mathscr{A}^+ \in L(Y,X)$, $\mathscr{A}^{\#} \in L(Y,X)$, $\mathscr{A}\mathscr{A}^+\mathscr{A} = \mathscr{A}$ and according to I.3.28 and I.3.29 $\mathbf{u}_0 = \mathscr{A}^{\#}\begin{pmatrix}\mathbf{f}\\ \mathbf{r}\end{pmatrix}$ is the unique best approximate solution of (5,6) for every $\mathbf{f} \in L_n^2$ and $\mathbf{r} \in R_m$. Taking into account 4.4, (5,4), (4,24) and making use of I.4.36 we obtain that for any $\mathbf{f} \in L_n^2$ and $\mathbf{r} \in R_m$

$$(5,9) \qquad \mathscr{A}^+ \boldsymbol{\Pi}_1 \begin{pmatrix}\mathbf{f}\\ \mathbf{r}\end{pmatrix}(t) = \int_0^1 \tilde{\mathbf{G}}(t,s)\,\mathbf{f}(s)\,\mathrm{d}s + \tilde{\mathbf{H}}(t)\,\mathbf{r} \qquad \text{on } [0,1]\,,$$

where

$$(5,10) \qquad \tilde{\mathbf{G}}(t,s) = \mathbf{G}_0(t,s) - \left(\int_0^1 \mathbf{G}_0(t,\sigma)\,\mathbf{Y}_0^*(\sigma)\,\mathrm{d}\sigma\right)\mathbf{C}^{-1}\,\mathbf{Y}_0(s) \qquad \text{on } [0,1] \times [0,1]\,,$$

$$\tilde{\mathbf{H}}(t) = \mathbf{H}_0(t) - \mathbf{H}_0(t)\,\boldsymbol{\Lambda}_0^*\mathbf{C}^{-1}\boldsymbol{\Lambda}_0 \qquad \text{on } [0,1]\,.$$

Obviously, $\tilde{\mathbf{G}}(t,s)$ is an L^2-kernel and $\tilde{\mathbf{G}}(t,s)$, $\tilde{\mathbf{H}}(t)$ is a standard generalized Green's couple of BVP (2,1), (2,2). By 4.2 and I.4.36 we have

$$\int_0^1 \mathbf{X}_0^*(\tau)\left(\int_0^1 \tilde{\mathbf{G}}(\tau,s)\,\mathbf{f}(s)\,\mathrm{d}s\right)\mathrm{d}\tau = \int_0^1 \left(\int_0^1 \mathbf{X}_0^*(\tau)\,\tilde{\mathbf{G}}(\tau,s)\,\mathrm{d}\tau\right)\mathbf{f}(s)\,\mathrm{d}s\,.$$

Consequently, putting

$$(5,11) \qquad \mathbf{G}(t,s) = \tilde{\mathbf{G}}(t,s) - \mathbf{X}_0(t)\,\mathbf{D}^{-1}\int_0^1 \mathbf{X}_0^*(\tau)\,\tilde{\mathbf{G}}(\tau,s)\,\mathrm{d}\tau \qquad \text{on } [0,1] \times [0,1]\,,$$

$$\mathbf{H}(t) = \tilde{\mathbf{H}}(t) - \mathbf{X}_0(t)\,\mathbf{D}^{-1}\int_0^1 \mathbf{X}_0^*(\tau)\,\tilde{\mathbf{H}}(\tau)\,\mathrm{d}\tau \qquad \text{on } [0,1]\,,$$

we obtain

$$\mathbf{u}_0(t) = \int_0^1 \mathbf{G}(t,s)\,\mathbf{f}(s)\,\mathrm{d}s + \mathbf{H}(t)\,\mathbf{r} \qquad \text{on } [0,1]\,.$$

5.9. Remark. Let us notice that $\mathbf{v} \in X$ is a least square solution to BVP (2,1), (2,2) if and only if

$$(5,12) \qquad 0 = \left(\mathscr{A}\mathbf{x}, \mathscr{A}\mathbf{v} - \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix}\right)_Y \quad \text{for any} \quad \mathbf{x} \in X .$$

Since by the definition (5,2)

$$\left(\mathscr{A}\mathbf{x}, \mathscr{A}\mathbf{v} - \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix}\right)_Y = \left\langle \mathscr{L}\mathbf{x}, \left(\mathscr{L}\mathbf{v} - \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix}\right)^* \right\rangle_{L \times R}$$
$$= \langle \mathbf{x}, \mathscr{L}^*(\mathscr{L}\mathbf{v})^* - \mathscr{L}^*(\mathbf{f}^*, \mathbf{r}^*)\rangle_{L \times R} \qquad \text{for any} \quad \mathbf{x} \in W_n^2 ,$$

the condition (5,12) is equivalent to

$$\mathscr{L}^*(\mathscr{L}\mathbf{v})^* = \mathscr{L}^*(\mathbf{f}^*, \mathbf{r}^*)$$

or

$$(5,13) \qquad \mathbf{v}'(t) + \left[\mathbf{P}(t,0) + \int_0^1 \mathbf{P}^*(\sigma,t)\,\mathbf{P}(\sigma,0)\,\mathrm{d}\sigma + \mathbf{K}^*(t)\,\mathbf{M}\right]\mathbf{v}(0)$$
$$+ \int_0^1 \left[\mathbf{P}(t,s) + \mathbf{P}^*(s,t) + \int_0^1 \mathbf{P}^*(\sigma,t)\,\mathbf{P}(\sigma,s)\,\mathrm{d}\sigma + \mathbf{K}^*(t)\,\mathbf{K}(s)\right]\mathbf{v}'(s)\,\mathrm{d}s$$
$$= \mathbf{f}(t) + \int_0^1 \mathbf{P}^*(s,t)\,\mathbf{f}(s)\,\mathrm{d}s + \mathbf{K}^*(t)\,\mathbf{r} \qquad \text{a.e. on } [0,1] ,$$
$$\left[\mathbf{M}^*\mathbf{M} + \int_0^1 \mathbf{P}^*(\sigma,0)\,\mathbf{P}(\sigma,0)\,\mathrm{d}\sigma\right]\mathbf{v}(0)$$
$$+ \int_0^1 \left[\mathbf{M}^*\,\mathbf{K}(s) + \mathbf{P}^*(s,0) + \int_0^1 \mathbf{P}^*(\sigma,0)\,\mathbf{P}(\sigma,s)\,\mathrm{d}\sigma\right]\mathbf{v}'(s)\,\mathrm{d}s$$
$$= \mathbf{M}^*\mathbf{r} + \int_0^1 \mathbf{P}^*(s,0)\,\mathbf{f}(s)\,\mathrm{d}s .$$

Let us notice that the system (5,13) of equations for $\mathbf{u} = \mathbf{v}' \in L_n^2$ and $\mathbf{c} = \mathbf{v}(0) \in R_n$ may be treated in the same way as the system (4,5), (4,6) (cf. also Lemma 3.1 in Tvrdý, Vejvoda [1]). If $\mathbf{P}(.,s)$ and $\mathbf{K}$ are of bounded variation on $[0,1]$, then the system (5,13) may be reduced to the form (2,1), (2,2).

5.10. Remark. Let $\mathbf{r} \in R_m$ be fixed and let us define

$$D_r = \{\mathbf{x} \in W_n^2;\ \mathbf{S}\mathbf{x} = \mathbf{r}\} \quad \text{and} \quad \mathscr{L}_r\colon \mathbf{x} \in D_r \to \mathbf{D}\mathbf{x} - \mathbf{P}\mathbf{x} \in L_n^2 .$$

Then $R(\mathscr{L}_r)$ is closed in L_n^2 (cf. 2.16). Hence if $D_r \neq \emptyset$, then by the Classical Projection Theorem (Luenberger [1], p. 64) $R(\mathscr{L}_r)$ contains a unique element $\mathbf{y}$ of minimum L^2-norm and $\mathbf{y} \in R(\mathscr{L}_0)^\perp$. It follows from 2.17 that $\|\mathbf{y}\|_{L^2} \leq \|\mathbf{f}\|_{L^2}$ for all $\mathbf{f} \in R(\mathscr{L}_r)$

if and only if there exists $\lambda^* \in R_m^*$ such that $(\mathbf{y}^*, \lambda^*) \in N(\mathscr{L}^*)$. Thus $\mathbf{u} \in D_r$ fulfils $\|\mathbf{Du} - \mathbf{Pu}\|_{L^2} \leq \|\mathbf{Dx} - \mathbf{Px}\|_{L^2}$ for all $\mathbf{x} \in D_r$ if and only if there exists $\lambda^* \in R_m^*$ such that

$$\mathscr{L}^*((\mathbf{Du} - \mathbf{Pu})^*, \lambda^*) = \mathbf{0}.$$

6. Volterra-Stieltjes integro-differential operator

Let $\mathbf{P}$: $[0,1] \times [0,1] \to L(R_n)$ be an $L^p[BV]$-kernel and let for a.e. $t \in [0,1]$, $\mathbf{P}(t,s) = \mathbf{P}(t,t)$ if $0 \leq t \leq s \leq 1$. Then

$$\mathbf{P}\colon \mathbf{x} \in W_n^p \to \int_0^1 \mathrm{d}_s[\mathbf{P}(t,s)]\,\mathbf{x}(s) = \int_0^t \mathrm{d}_s[\mathbf{P}(t,s)]\,\mathbf{x}(s) \in L_n^p$$

and the Fredholm-Stieltjes integro-differential operator $\mathscr{L} = \mathbf{D} - \mathbf{P}$ defined in 1.5 reduces to a *Volterra-Stieltjes integro-differential operator*

$$\mathscr{L} = \mathbf{D} - \mathbf{P}\colon \mathbf{x} \in W_n^p \to \mathbf{x}'(t) - \int_0^t \mathrm{d}_s[\mathbf{P}(t,s)]\,\mathbf{x}(s) \in L_n^p. \tag{6,1}$$

If $\mathbf{P}(t,s) = \mathbf{P}(t,t) = \mathbf{0}$ for $0 \leq t \leq s \leq 1$, then by I.4.38

$$\int_0^t \left(\int_0^\tau \mathrm{d}_s[\mathbf{P}(\tau,s)]\,\mathbf{x}(s)\right)\mathrm{d}\tau = \int_0^t \left(\int_0^t \mathrm{d}_s[\mathbf{P}(\tau,s)]\,\mathbf{x}(s)\right)\mathrm{d}\tau$$

$$= \int_0^t \mathrm{d}_s\left[\int_0^t \mathbf{P}(\tau,s)\,\mathrm{d}\tau\right]\mathbf{x}(s) = \int_0^t \mathrm{d}_s\left[\int_s^t \mathbf{P}(\tau,s)\,\mathrm{d}\tau\right]\mathbf{x}(s).$$

Thus, if $\mathbf{f} \in L_n^p$, then by integrating the Volterra-Stieltjes integro-differential equation for $\mathbf{x} \in W_n^p$

$$\mathbf{x}'(t) - \int_0^t \mathrm{d}_s[\mathbf{P}(t,s)]\,\mathbf{x}(s) = \mathbf{f}(t) \qquad \text{a.e. on } [0,1] \tag{6,2}$$

we obtain

6.1. Proposition. *If* $\mathbf{P}(t,s) = \mathbf{0}$ *for* $0 \leq t \leq s \leq 1$, *then a function* $\mathbf{x} \in BV_n$ *is a solution to* (6,2) *if and only if*

$$\mathbf{x}(t) - \int_0^t \mathrm{d}_s[\mathbf{Q}(t,s)]\,\mathbf{x}(s) = \mathbf{x}(0) + \int_0^t \mathbf{f}(\tau)\,\mathrm{d}\tau \qquad \text{on } [0,1], \tag{6,3}$$

where

$$\mathbf{Q}(t,s) = \int_s^t \mathbf{P}(\tau,s)\,\mathrm{d}\tau \quad \text{if } 0 \leq s \leq t \leq 1, \qquad \mathbf{Q}(t,s) = \mathbf{0} \quad \text{if } 0 \leq t \leq s \leq 1. \tag{6,4}$$

(Obviously, if $\mathbf{x} \in BV_n$ fulfils (6,3), then $\mathbf{x} \in W_n^p$.)

6.2. Remark. Let us notice that if $\mathbf{P}_0(t,s) = \mathbf{P}(t,s) - \mathbf{P}(t,t)$ on $[0,1] \times [0,1]$, then $\mathbf{P}_0(t,s) = \mathbf{0}$ for $0 \le t \le s \le 1$ and

$$\int_0^t \mathrm{d}_s[\mathbf{P}_0(t,s)]\,\mathbf{x}(s) = \int_0^t \mathrm{d}_s[\mathbf{P}(t,s)]\,\mathbf{x}(s) \qquad \text{for any} \quad t \in [0,1] \quad \text{and} \quad \mathbf{x} \in C_n\,.$$

This means that the assumption $\mathbf{P}(t,t) = \mathbf{0}$ for every $t \in [0,1]$ does not cause any loss of generality.

6.3. Proposition. $\mathrm{v}_{[0,1]\times[0,1]}(\mathbf{Q}) < \infty$, $\mathbf{Q}(0,s) = \mathbf{0}$ *on* $[0,1]$ *and* $\mathbf{Q}(t,t-) = \mathbf{Q}(t,t) = \mathbf{0}$ *for any* $t \in (0,1]$.

Proof. Let a net-type subdivision $\{0 = t_0 < t_1 < \ldots < t_k = 1;\ 0 = s_0 < s_1 < \ldots < s_k = 1\}$ be given. Then

$$m_{i,j}(\mathbf{Q}) = \left|\int_{t_{i-1}}^{t_i} (\mathbf{P}(\tau,s_j) - \mathbf{P}(\tau,s_{j-1}))\,\mathrm{d}\tau\right| \le \int_{t_{i-1}}^{t_i} |\mathbf{P}(\tau,s_j) - \mathbf{P}(\tau,s_{j-1})|\,\mathrm{d}\tau\,.$$

Hence

$$\sum_{i=1}^{k}\sum_{j=1}^{k} m_{i,j}(\mathbf{Q}) \le \int_0^1 \sum_{j=1}^{k} |\mathbf{P}(\tau,s_j) - \mathbf{P}(\tau,s_{j-1})|\,\mathrm{d}\tau \le \int_0^1 \varrho(\tau)\,\mathrm{d}\tau$$

and consequently $\mathrm{v}_{[0,1]\times[0,1]}(\mathbf{Q}) < \infty$. The other assertions of the lemma follow immediately from (6,4).

Making use of the results obtained for Volterra-Stieltjes integral equations in the Section II.3 we can deduce the variation-of-constants formula for Volterra-Stieltjes integro-differential equations.

6.4. Theorem. *Let* $\mathbf{P}$: $[0,1] \times [0,1] \to L(R_n)$ *be an* $L^p[BV]$*-kernel such that for a.e.* $t \in [0,1]$ $\mathbf{P}(t,s) = \mathbf{0}$ *if* $0 \le t \le s \le 1$. *Then for any* $\mathbf{c} \in R_n$ *and* $\mathbf{f} \in L^p_n$ *there exists a unique solution* $\mathbf{x}$ *of the equation* (6,2) *in* W^p_n *such that* $\mathbf{x}(0) = \mathbf{c}$.

Furthermore, there exists a uniquely determined function $\mathbf{U}$: $[0,1] \times [0,1] \to L(R_n)$ *such that for any* $\mathbf{f} \in L^1_n$ *and* $\mathbf{c} \in R_n$ *this solution is given by*

$$(6,5) \qquad \mathbf{x}(t) = \mathbf{U}(t,0)\,\mathbf{c} + \int_0^t \mathbf{U}(t,s)\,\mathbf{f}(s)\,\mathrm{d}s\,, \qquad t \in [0,1]\,.$$

The function $\mathbf{U}$ *satisfies the equation*

$$(6,6)\quad \frac{\partial}{\partial t}\mathbf{U}(t,s) = \int_s^t \mathrm{d}_r[\mathbf{P}(t,r)]\,\mathbf{U}(r,s) \qquad \textit{for any} \quad s \in [0,1] \quad \textit{and a.e.} \quad t \in [s,1]\,.$$

Moreover, $\mathrm{v}_{[0,1]\times[0,1]}(\mathbf{U}) + \mathrm{var}_0^1\,\mathbf{U}(0,.) < \infty$, $\mathbf{U}(.,s)$ *is absolutely continuous on* $[0,1]$ *for any* $s \in [0,1]$ *and* $\mathbf{U}(t,s) = \mathbf{I}$ *if* $0 \le t \le s \le 1$.

Proof. Let $\boldsymbol{\Gamma}$: $[0,1] \times [0,1] \to L(R_n)$ correspond to $\mathbf{Q}$ by II.3.10. In particular, the function $\mathbf{x}$: $[0,1] \to R_n$ given by

$$\mathbf{x}(t) = \mathbf{c} + \int_0^t \mathbf{f}(\tau)\,\mathrm{d}\tau + \int_0^t \mathrm{d}_s[\boldsymbol{\Gamma}(t,s)]\left(\mathbf{c} + \int_0^s \mathbf{f}(\tau)\,\mathrm{d}\tau\right)$$

is for any $\mathbf{c} \in R_n$ and $\mathbf{f} \in L_n^p$ a unique solution to (6,3) such that $\mathbf{x}(0) = \mathbf{c}$. Integration by parts yields

$$\text{(6,7)} \qquad \mathbf{x}(t) = [\mathbf{I} + \boldsymbol{\Gamma}(t,t)]\,\mathbf{c} + \int_0^t [\mathbf{I} + \boldsymbol{\Gamma}(t,t) - \boldsymbol{\Gamma}(t,s)]\,\mathbf{f}(s)\,\mathrm{d}s \qquad \text{on } [0,1].$$

Denoting

$$\text{(6,8)} \qquad \mathbf{U}(t,s) = \begin{cases} \mathbf{I} + \boldsymbol{\Gamma}(t,t) - \boldsymbol{\Gamma}(t,s) & \text{if } 0 \le s \le t \le 1, \\ \mathbf{I} & \text{if } 0 \le t \le s \le 1, \end{cases}$$

the expression (6,7) reduces to (6,5). (Recall that $\boldsymbol{\Gamma}(t,0) = \mathbf{0}$ for every $t \in [0,1]$.) In our case the function $\boldsymbol{\Gamma}$ satisfies for $0 \le s \le t \le 1$ the relation (cf. (II.3.29))

$$\text{(6,9)} \qquad \boldsymbol{\Gamma}(t,s) = \int_s^t \mathbf{P}(\tau,s)\,\mathrm{d}\tau - \int_0^t \mathbf{P}(\tau,0)\,\mathrm{d}\tau + \int_0^t \mathrm{d}_r\left[\int_r^t \mathbf{P}(\tau,r)\,\mathrm{d}\tau\right]\boldsymbol{\Gamma}(r,s).$$

Taking into account that $\mathbf{P}(\tau,r) = \mathbf{0}$ if $0 \le \tau \le r \le 1$ and $\boldsymbol{\Gamma}(r,s) = \boldsymbol{\Gamma}(r,r)$ if $0 \le r \le s \le 1$ and employing I.4.38 we obtain for $0 \le s \le t \le 1$

$$\begin{aligned} \int_0^t \mathrm{d}_r\left[\int_r^t \mathbf{P}(\tau,r)\,\mathrm{d}\tau\right]\boldsymbol{\Gamma}(r,s) &= \int_0^t \mathrm{d}_r\left[\int_0^t \mathbf{P}(\tau,r)\,\mathrm{d}\tau\right]\boldsymbol{\Gamma}(r,s) \\ = \int_0^t \left(\int_0^t \mathrm{d}_r[\mathbf{P}(\tau,r)]\,\boldsymbol{\Gamma}(r,s)\right)\mathrm{d}\tau &= \int_0^t \left(\int_0^\tau \mathrm{d}_r[\mathbf{P}(\tau,r)]\,\boldsymbol{\Gamma}(r,s)\right)\mathrm{d}\tau \\ = \int_s^t \left(\int_s^t \mathrm{d}_r[\mathbf{P}(\tau,r)]\,\boldsymbol{\Gamma}(r,s)\right)\mathrm{d}\tau &+ \int_0^t \left(\int_0^s \mathrm{d}_r[\mathbf{P}(\tau,r)]\,\boldsymbol{\Gamma}(r,r)\right)\mathrm{d}\tau. \end{aligned}$$

It is easy to verify (cf. also (6,8) and (6,9)) that

$$\mathbf{U}(t,s) = \mathbf{I} - \int_s^t \mathbf{P}(\tau,s)\,\mathrm{d}\tau - \int_s^t \left(\int_s^\tau \mathrm{d}_r[\mathbf{P}(\tau,r)]\,(\boldsymbol{\Gamma}(r,s) - \boldsymbol{\Gamma}(r,r))\right)\mathrm{d}\tau$$
$$\text{for } \quad 0 \le s \le t \le 1.$$

On the other hand, it follows from (6,8) that

$$\int_s^\tau \mathrm{d}_r[\mathbf{P}(\tau,r)]\,\mathbf{U}(r,s) = -\mathbf{P}(\tau,s) - \int_s^\tau \mathrm{d}_r[\mathbf{P}(\tau,r)]\,(\boldsymbol{\Gamma}(r,r) - \boldsymbol{\Gamma}(r,s))$$
$$\text{for } \quad 0 \le s \le \tau \le 1.$$

Thus $(\mathbf{U}(s,s) = \mathbf{I})$

$$\mathbf{U}(t,s) = \mathbf{U}(s,s) + \int_s^t \left(\int_s^\tau \mathrm{d}_r[\mathbf{P}(\tau,r)]\,\mathbf{U}(r,s)\right)\mathrm{d}\tau \qquad \text{if } \quad 0 \le s \le t \le 1,$$

which yields (6,6) immediately.

As $\mathrm{v}_{[0,1]\times[0,1]}(\boldsymbol{\Gamma}) < \infty$ (cf. II.3.10), also $\mathrm{v}_{[0,1]\times[0,1]}(\boldsymbol{U}) < \infty$. The other assertions of the theorem are evident.

6.5. Remark. Denoting for $\boldsymbol{c} \in R_n$, $\boldsymbol{f} \in L_n^p$ and $t \in [0,1]$

$$(6{,}10) \qquad (\boldsymbol{\Phi c})(t) = \boldsymbol{U}(t,0)\,\boldsymbol{c} \quad \text{and} \quad (\boldsymbol{\Psi f})(t) = \int_0^t \boldsymbol{U}(t,s)\,\boldsymbol{f}(s)\,\mathrm{d}s,$$

the variation of constants formula (6,5) for solutions of (6,2) becomes

$$(6{,}11) \qquad \boldsymbol{x}(t) = (\boldsymbol{\Phi c})(t) + (\boldsymbol{\Psi f})(t) \quad \text{on } [0,1] \quad (\boldsymbol{x} = \boldsymbol{\Phi c} + \boldsymbol{\Psi f}).$$

By 6.4 the functions $\boldsymbol{\Phi c}$ and $\boldsymbol{\Psi f}$ belong to W_n^p for every $\boldsymbol{c} \in R_n$ and $\boldsymbol{f} \in L_n^p$. Moreover, the linear operators $\boldsymbol{\Phi}$: $\boldsymbol{c} \in R_n \to \boldsymbol{\Phi c} \in W_n^p$ and $\boldsymbol{\Psi}$: $\boldsymbol{f} \in L_n^p \to \boldsymbol{\Psi f} \in W_n^p$ are bounded. Indeed, if $\boldsymbol{f} \in L_n^p$ and $\boldsymbol{\psi} = \boldsymbol{\Psi f}$, then in virtue of I.4.27, I.6.6 and 6.4 we have for a.e. $t \in [0,1]$

$$|\boldsymbol{\psi}'(t)| = \left|\int_0^t \mathrm{d}_s[\boldsymbol{P}(t,s)]\,\boldsymbol{\psi}(s) + \boldsymbol{f}(t)\right| \le \left|\int_0^1 \mathrm{d}_\tau[\boldsymbol{P}(t,\tau)]\left(\int_0^\tau \boldsymbol{U}(\tau,s)\,\boldsymbol{f}(s)\,\mathrm{d}s\right)\right| + |\boldsymbol{f}(t)|$$
$$\le \varrho(t) \sup_{t,s\in[0,1]} |\boldsymbol{U}(t,s)|\;\|\boldsymbol{f}\|_{L^1} + |\boldsymbol{f}(t)|.$$

Consequently

$$\|\boldsymbol{\Psi f}\|_{W^p} = \|\boldsymbol{\psi}'\|_{L^p} \le (1 + \|\varrho\|_{L^p}(\sup_{t,s\in[0,1]} |\boldsymbol{U}(t,s)|))\;\|\boldsymbol{f}\|_{L^p},$$

i.e. $\boldsymbol{\Psi} \in B(L_n^p, W_n^p)$. Analogously we could obtain $\boldsymbol{\Phi} \in B(R_n, W_n^p)$.

6.6. Corollary. *Let $\mathscr{U}$ be a linear normed space and let $\boldsymbol{\Theta} \in B(\mathscr{U}, L_n^p)$. If $\boldsymbol{P}$: $[0,1] \times [0,1] \to L(R_n)$ is an $L^p[BV]$-kernel such that for a.e. $t \in [0,1]$, $\boldsymbol{P}(t,s) = \boldsymbol{0}$ if $s \in [t,1]$, then for any $\boldsymbol{u} \in \mathscr{U}$, $\boldsymbol{f} \in L_n^p$ and $\boldsymbol{c} \in R_n$ there exists a unique solution $\boldsymbol{x} \in W_n^p$ of*

$$\boldsymbol{Dx} - \boldsymbol{Px} = \boldsymbol{\Theta u} + \boldsymbol{f}, \qquad \boldsymbol{x}(0) = \boldsymbol{c}.$$

This solution is given by $\boldsymbol{x} = \boldsymbol{\Phi c} + \boldsymbol{\Psi\Theta u} + \boldsymbol{\Psi f}$.

6.7. Remark. Let $r > 0$ and let $\boldsymbol{P}$: $[0,1] \times [-r,1] \to L(R_n)$ be an $L^p[BV]$-kernel on $[0,1] \times [-r,1]$ such that $\boldsymbol{P}(t,s) = \boldsymbol{0}$ if $t \le s$ and $\boldsymbol{P}(t,s) = \boldsymbol{P}(t,t-r)$ if $s \le t-r$. Let $\boldsymbol{u} \in BV_n[-r,0]$ and $\boldsymbol{f} \in L_n^p$ be given and let us look for a function $\boldsymbol{x} \in BV_n[-r,1]$ absolutely continuous on $[0,1]$ and such that $\boldsymbol{x}'$ is L^p-integrable on $[0,1]$ and

$$(6{,}12) \qquad \boldsymbol{x}'(t) - \int_{t-r}^t \mathrm{d}_s[\boldsymbol{P}(t,s)]\,\boldsymbol{x}(s) = \boldsymbol{f}(t) \quad \text{a.e. on } [0,1],$$
$$\boldsymbol{x}(t) = \boldsymbol{u}(t) \quad \text{on } [-r,0].$$

If we put

$$\boldsymbol{\Theta}: \boldsymbol{u} \in BV_n[-r,0] \to \int_{-r}^0 \mathrm{d}_s[\boldsymbol{P}(t,s)]\,\boldsymbol{u}(s) \in L_n^p,$$

then $\boldsymbol{\Theta}$ is a linear compact operator (cf. 1.4). For any $\boldsymbol{x} \in BV_n[-r, 1]$ and $t \in [0, 1]$ we have

$$\int_{t-r}^{t} \mathrm{d}_s[\boldsymbol{P}(t, s)]\, \boldsymbol{x}(s) = \int_{-r}^{0} \mathrm{d}_s[\boldsymbol{P}(t, s)]\, \boldsymbol{x}(s) + \int_{0}^{t} \mathrm{d}_s[\boldsymbol{P}(t, s)]\, \boldsymbol{x}(s).$$

Thus our problem may be formulated in the form of the operator equation $\boldsymbol{Dx} - \boldsymbol{Px} = \boldsymbol{\Theta u} + \boldsymbol{f}$ and according to 6.6 (with $\mathscr{U} = BV_n[-r, 0]$) the equation (6,12) has for any $\boldsymbol{u} \in BV_n[-r, 0]$ and $\boldsymbol{x} \in L_n^p$ a unique solution $\boldsymbol{x} \in W_n^p$ such that $\boldsymbol{x}(t) = \boldsymbol{u}(t)$ on $[-r, 0]$. This solution is of the form $\boldsymbol{x} = \boldsymbol{\Phi}_0 \boldsymbol{u} + \boldsymbol{\Psi f}$, where $\boldsymbol{\Phi}_0$: $\boldsymbol{u} \in BV_n[-r, 0] \to \boldsymbol{\Phi}\, \boldsymbol{u}(0) + \boldsymbol{\Psi\Theta u} \in W_n^p$ is a linear compact operator. (Let us notice that in virtue of I.4.38

$$(\boldsymbol{\Phi}_0 \boldsymbol{u})(t) = \boldsymbol{U}(t, 0)\, \boldsymbol{u}(0) + \int_{-r}^{0} \mathrm{d}_s \left[\int_0^t \boldsymbol{U}(t, \tau)\, \boldsymbol{P}(\tau, s)\, \mathrm{d}\tau \right] \boldsymbol{u}(s) \qquad \text{on } [0, 1]$$

for any $\boldsymbol{u} \in BV_n[-r, 0]$.) Thus, the variation-of-constants formula for functional-differential equations of the retarded type (cf. Banks [1] or Hale [1]) is a consequence of Theorem 6.2.

Analogously we may show that if $0 < r_i \leq r$ $(i = 1, 2, \ldots, k)$, $\boldsymbol{A}_i$: $[0, 1] \to L(R_n)$ $i = 1, 2, \ldots, k)$ are measurable and essentially bounded on $[0, 1]$ and $\boldsymbol{A}_0$: $[0, 1] \times [-r, 0] \to L(R_n)$ is measurable and essentially bounded on $[0, 1] \times [-r, 0]$, then the system

$$\begin{aligned} (6{,}13) \qquad & \boldsymbol{x}'(t) - \sum_{i=1}^{k} \boldsymbol{A}_i(t) \begin{Bmatrix} \boldsymbol{0} & \text{if } t - r_i > 0 \\ \boldsymbol{u}(t - r_i) & \text{if } t - r_i < 0 \end{Bmatrix} \\ & - \int_{-r}^{0} \boldsymbol{A}_0(t, s) \begin{Bmatrix} \boldsymbol{0} & \text{if } t + s > 0 \\ \boldsymbol{u}(t + s) & \text{if } t + s < 0 \end{Bmatrix} \mathrm{d}s \\ & - \int_0^t \mathrm{d}_s[\boldsymbol{P}(t, s)]\, \boldsymbol{x}(s) = \boldsymbol{f}(t) \qquad \text{a.e. on } [0, 1] \end{aligned}$$

has for any $\boldsymbol{f} \in L_n^p[0, 1]$, $\boldsymbol{u} \in L_n^p[-r, 0]$ and $\boldsymbol{c} \in R_n$ a unique solution $\boldsymbol{x} \in L_n^p[-r, 1]$ such that $\boldsymbol{x}(t) = \boldsymbol{u}(t)$ a.e. on $[-r, 0]$, $\boldsymbol{u}(0) = \boldsymbol{c}$ and $\boldsymbol{x}|_{[0,1]} \in W_n^p$. This solution is of the form $\boldsymbol{x} = \boldsymbol{\Phi c} + \boldsymbol{\Psi\Theta u} + \boldsymbol{\Psi f}$, where

$$\begin{aligned} \boldsymbol{\Theta}\colon \boldsymbol{u} \in L_n^p[-r, 0] \to & \begin{Bmatrix} \sum_{i=1}^{k} \boldsymbol{A}_i(t)\, \boldsymbol{u}(t - r_i) & \text{if } t - r_i < 0 \\ \boldsymbol{0} & \text{if } t - r_i > 0 \end{Bmatrix} \\ & + \int_{-r}^{0} \boldsymbol{A}_0(t, s) \begin{Bmatrix} \boldsymbol{u}(t + s) & \text{if } t + s < 0 \\ \boldsymbol{0} & \text{if } t + s > 0 \end{Bmatrix} \mathrm{d}s \in L_n^p. \end{aligned}$$

(Functional-differential equations of the type (6,13) were studied in detail in Delfour-Mitter [1] and [2].)

6.8. Theorem. *Let Λ be a Banach space, $\mathbf{S} \in B(W_n^p, \Lambda)$ and let $\mathbf{P}$: $[0,1] \times [0,1] \to L(R_n)$ be an $L^p[BV]$-kernel such that for a.e. $t \in [0,1]$, $\mathbf{P}(t,s) = \mathbf{0}$ if $s \in [t,1]$. Then the linear bounded operator*

$$\mathscr{L}\colon \mathbf{x} \in W_n^p \to \begin{bmatrix} \mathbf{D}\mathbf{x} - \mathbf{P}\mathbf{x} \\ \mathbf{S}\mathbf{x} \end{bmatrix} \in L_n^p \times \Lambda$$

has a closed range.

Proof. By 6.5, $\begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} \in L_n^p \times \Lambda$ belongs to $R(\mathscr{L})$ if and only if $\mathbf{r} - \mathbf{S}\boldsymbol{\Psi}\mathbf{f} \in R(\mathbf{S}\boldsymbol{\Phi})$. As $\mathbf{W}\colon \begin{pmatrix} \mathbf{f} \\ \mathbf{r} \end{pmatrix} \in L_n^p \times \Lambda \to \mathbf{r} - \mathbf{S}\boldsymbol{\Psi}\mathbf{f} \in \Lambda$ is bounded and $R(\mathbf{S}\boldsymbol{\Phi})$ is a finite dimensional linear subspace in Λ $(\boldsymbol{\Phi} \in B(R_n, W_n^p))$, it follows that $R(\mathscr{L})$ is closed.

7. Fredholm-Stieltjes integral equations with linear constraints

This section is devoted to the system of equations for $\mathbf{x} \in BV_n$

$$\mathbf{x}(t) - \mathbf{x}(0) - \int_0^1 \mathrm{d}_s[\mathbf{P}(t,s) - \mathbf{P}(0,s)]\,\mathbf{x}(s) = \mathbf{f}(t) - \mathbf{f}(0) \qquad \text{on } [0,1], \tag{7,1}$$

$$\int_0^1 \mathrm{d}[\mathbf{K}(s)]\,\mathbf{x}(s) = \mathbf{r}\,. \tag{7,2}$$

The following hypotheses are pertinent.

7.1. Assumptions. $\mathbf{P}$: $[0,1] \times [0,1] \to L(R_n)$ *and there are* $t_0, s_0 \in [0,1]$ *such that*

$$\mathrm{v}_{[0,1]\times[0,1]}(\mathbf{P}) + \operatorname{var}_0^1 \mathbf{P}(t_0, .) + \operatorname{var}_0^1 \mathbf{P}(., s_0) < \infty\,, \tag{7,3}$$

$\mathbf{K}$: $[0,1] \to L(R_n, R_m)$ *is of bounded variation on* $[0,1]$, $\mathbf{f} \in BV_n$ *and* $\mathbf{r} \in R_m$.

7.2. Definition. Any function $\mathbf{P}$: $[0,1] \times [0,1] \to L(R_n)$ fulfilling (7,3) is called an *SBV-kernel.*

7.3. Remark. If $\mathbf{P}$: $[0,1] \times [0,1] \to L(R_n)$ is an *SBV*-kernel and

$$\mathbf{Q}(t,s) = \begin{cases} \mathbf{P}(t,s) - \mathbf{P}(0,s) & \text{for } t \in [0,1] \text{ and } s \in (0,1]\,, \\ \mathbf{P}(t,0) - \mathbf{P}(0,0) - \mathbf{I} & \text{for } t \in [0,1] \text{ and } s = 0\,, \end{cases} \tag{7,4}$$

then obviously $\mathbf{Q}(t,s)$ is an *SBV*-kernel and

$$\mathbf{x}(0) + \int_0^1 \mathrm{d}_s[\mathbf{P}(t,s) - \mathbf{P}(0,s)]\,\mathbf{x}(s) = \int_0^1 \mathrm{d}_s[\mathbf{Q}(t,s)]\,\mathbf{x}(s) \qquad \text{for any } \mathbf{x} \in BV_n \tag{7,5}$$

(cf. I.4.23). It means that the equation (7,1) is a special case of Fredholm-Stieltjes integral equations studied in Chapter II. Let us denote by $\mathbf{Q}$ the linear operator

$$(7,6) \qquad \mathbf{Q}\colon \mathbf{x} \in BV_n \to \mathbf{x}(0) + \int_0^1 \mathrm{d}_s[\mathbf{P}(t,s) - \mathbf{P}(0,s)]\,\mathbf{x}(s)\,.$$

By (7,5) and II.1.5 $R(\mathbf{Q}) \subset BV_n$ and $\mathbf{Q} \in L(BV_n)$ is compact.

The following assertion follows analogously as 1.8 from I.3.20 and 1.9.

7.4. Proposition. *If $\mathbf{P}\colon [0,1] \times [0,1] \to L(R_n)$ is an SBV-kernel and the operator $\mathbf{Q}$ is given by (7,6), then $n \leq \dim N(\mathbf{I} - \mathbf{Q}) < \infty$, while $\dim N(\mathbf{I} - \mathbf{Q}) = n$ if and only if the equation (7,1) has a solution $\mathbf{x} \in BV_n$ for any $\mathbf{f} \in BV_n$.*

Let us mention that the following additional hypotheses do not mean any loss of generality (cf. II.1.4).

7.5. Assumptions. *$\mathbf{P}(t, .)$ is right-continuous on $(0,1)$ and $\mathbf{P}(t,1) = \mathbf{0}$ for any $t \in [0,1]$ and $\mathbf{P}(0,s) = \mathbf{0}$ for any $s \in [0,1]$; $\mathbf{K}$ is right-continuous on $(0,1)$ and $\mathbf{K}(1) = \mathbf{0}$.*

Analogously as in the case of BVP (2,1), (2,2) for Fredholm-Stieltjes integro-differential operators we rewrite the system (7,1), (7,2) of equations for $\mathbf{x} \in BV_n$ as the system of operator equations for $\xi = \begin{pmatrix}\mathbf{x}\\ \mathbf{d}\end{pmatrix} \in BV_n \times R_m$

$$(7,7) \qquad (\mathbf{I} - \mathbf{T})\,\xi = \begin{pmatrix}\mathbf{x}\\ \mathbf{d}\end{pmatrix} - \begin{pmatrix}\mathbf{Qx}\\ \mathbf{d} - \mathbf{Sx}\end{pmatrix} = \begin{pmatrix}\mathbf{\Psi f}\\ \mathbf{r}\end{pmatrix},$$

where $\mathbf{Q} \in K(BV_n)$ is defined by (7,6),

$$(7,8) \qquad \mathbf{S}\colon \mathbf{x} \in BV_n \to \int_0^1 \mathrm{d}[\mathbf{K}(s)]\,\mathbf{x}(s) \in R_m\,,$$

$$(7,9) \qquad \mathbf{T}\colon \begin{pmatrix}\mathbf{x}\\ \mathbf{d}\end{pmatrix} \in BV_n \times R_m \to \begin{pmatrix}\mathbf{Qx}\\ \mathbf{d} - \mathbf{Sx}\end{pmatrix} \in BV_n \times R_m$$

and $\mathbf{\Psi}$ is now given by

$$(7,10) \qquad \mathbf{\Psi}\colon \mathbf{f} \in BV_n \to \mathbf{f}(t) - \mathbf{f}(0) \in BV_n\,.$$

7.6. Proposition. *If $\mathbf{x} \in BV_n$ is a solution to (7,1), (7,2), then $\xi = \begin{pmatrix}\mathbf{x}\\ \mathbf{d}\end{pmatrix}$ is a solution to (7,7) for any $\mathbf{d} \in R_m$. If $\mathbf{x} \in BV_n$ and there exists $\mathbf{d} \in R_m$ such that $\xi = \begin{pmatrix}\mathbf{x}\\ \mathbf{d}\end{pmatrix}$ verifies (7,7), then $\mathbf{x}$ is a solution of (7,1), (7,2).*

7.7. Proposition. *Under the assumptions 7.1 the operator $\mathbf{T} \in L(BV_n \times R_m)$ defined by (7,6), (7,8) and (7,9) is compact.*

Proof. As obviously $\mathbf{S} \in B(BV_n, R_m) = K(BV_n, R_m)$ (cf. I.3.21) and $\mathbf{Q} \in K(BV_n)$, it is easy to see that $\mathbf{T} \in K(BV_n \times R_m)$.

Our wish is now to establish the duality theory for problems of the form (7,1), (7,2). To this end it is necessary to choose a space $BV_n^{\backprime}$ of functions $[0, 1] \to R_n^*$ and an operator $\mathbf{T}^{\backprime} \in L(BV_n^{\backprime} \times R_m^*)$ in such a way that $(BV_n \times R_m, BV_n^{\backprime} \times R_m^*)$ is a dual pair with respect to some bilinear form $[.,.]$ (cf. I.3.1) and

$$(7,11) \qquad \left[\mathbf{T}\begin{pmatrix}\mathbf{x}\\ \mathbf{d}\end{pmatrix}, (\mathbf{z}^*, \boldsymbol{\lambda}^*)\right] = \left[\begin{pmatrix}\mathbf{x}\\ \mathbf{d}\end{pmatrix}, \mathbf{T}^{\backprime}(\mathbf{z}^*, \boldsymbol{\lambda}^*)\right]$$

$$\text{for all} \quad \begin{pmatrix}\mathbf{x}\\ \mathbf{d}\end{pmatrix} \in BV_n \times R_m \quad \text{and} \quad (\mathbf{z}^*, \boldsymbol{\lambda}^*) \in BV_n^{\backprime} \times R_m^* .$$

According to I.5.9 the spaces BV_n and NBV_n form a dual pair with respect to the bilinear form

$$\mathbf{x} \in BV_n,\ \varphi \in NBV_n \to \int_0^1 \mathrm{d}[\varphi^*(t)]\, \mathbf{x}(t) \in R .$$

For the purposes of this section a slightly different choice of the space $BV_n^{\backprime}$ is more suitable.

7.8. Definition. $BV_n^{\backprime}$ denotes the space of all functions $\mathbf{z}^*\colon [0, 1] \to R_n^*$ of bounded variation on $[0, 1]$, right-continuous on $(0, 1)$ and such that $\mathbf{z}^*(1) = \mathbf{0}$.

7.9. Proposition. *The space $BV_n^{\backprime}$ defined in* 7.8 *becomes a Banach space if it is endowed with the norm $\mathbf{z}^* \in BV_n^{\backprime} \to \|\mathbf{z}^*\|_{BV^{\backprime}} = |\mathbf{z}^*(0)| + \mathrm{var}_0^1\, \mathbf{z}^*$. Moreover, $(BV_n \times R_m, BV_n^{\backprime} \times R_m^*)$ is a dual pair with respect to the bilinear form*

$$(7,12) \qquad \begin{pmatrix}\mathbf{x}\\ \mathbf{d}\end{pmatrix} \in BV_n \times R_m\,,\ (\mathbf{z}^*, \boldsymbol{\lambda}^*) \in BV_n^{\backprime} \times R_m^*$$

$$\to \left[\begin{pmatrix}\mathbf{x}\\ \mathbf{d}\end{pmatrix}, (\mathbf{z}^*, \boldsymbol{\lambda}^*)\right] = \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{x}(t) + \boldsymbol{\lambda}^* \mathbf{d} \in R .$$

(For the proofs of analogous assertions for NBV_n see I.5.2 and I.5.9.)

In the following the bilinear form $[.,.]$ is defined by (7,12).

7.10. Proposition. *If the hypotheses* 7.1 *are fulfilled,* $\mathbf{Q}\colon [0, 1] \times [0, 1] \to L(R_n)$ *is defined by* (7,4) *and*

$$(7,13) \qquad \mathbf{T}^{\backprime}\colon (\mathbf{z}^*, \boldsymbol{\lambda}^*) \in BV_n^{\backprime} \times R_m^* \to \begin{pmatrix} \displaystyle\int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\, \mathbf{Q}(t, s) - \boldsymbol{\lambda}^* \mathbf{K}(s) \\ \boldsymbol{\lambda}^* \end{pmatrix},$$

then (7,11) *holds. If* 7.5 *is also assumed, then* $R(\mathbf{T}^{\backprime}) \subset BV_n^{\backprime} \times R_m^*$ *and* $\mathbf{T}^{\backprime} \in K(BV_n^{\backprime} \times R_m)$.

Proof. Let us denote

$$\mathbf{Q}^{\backprime}: \mathbf{z} \in BV_n \to \int_0^1 \mathbf{Q}(t,s)\,\mathrm{d}[\mathbf{z}(t)]\,.$$

As $\mathbf{Q}(t,s)$ is an SBV-kernel, $\mathbf{Q}^{\backprime} \in K(BV_n)$ (cf. II.1.9). Moreover, by I.6.20

$$\int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\left(\int_0^1 \mathrm{d}_s[\mathbf{Q}(t,s)]\,\mathbf{x}(s)\right) + \boldsymbol{\lambda}^*\left(\mathbf{d} - \int_0^1 \mathrm{d}[\mathbf{K}(s)]\,\mathbf{x}(s)\right)$$

$$= \int_0^1 \mathrm{d}_s\left[\int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,\mathbf{Q}(t,s) - \boldsymbol{\lambda}^*\,\mathbf{K}(s)\right]\mathbf{x}(s) + \boldsymbol{\lambda}^*\mathbf{d}$$

for any $\mathbf{x} \in BV_n$, $\mathbf{d} \in R_m$, $\mathbf{z}^* \in BV_n^{\backprime}$ and $\boldsymbol{\lambda}^* \in R_m^*$. If $\mathbf{P}(t,.)$ is right-continuous on $(0,1)$, then according to I.6.16 and I.4.17 also $\mathbf{Q}^{\backprime}\mathbf{z} \in BV_n$ is right-continuous on $(0,1)$ for any $\mathbf{z} \in BV_n$. Consequently, $R(\mathbf{T}^{\backprime}) \subset BV_n^{\backprime} \times R_m^*$ provided that 7.5 is satisfied. The compactness of $\mathbf{T}^{\backprime} \in L(BV_n^{\backprime} \times R_m^*)$ follows readily from the compactness of $\mathbf{Q}^{\backprime}$.

The operators $\mathbf{T}$ and $\mathbf{T}^{\backprime}$ being compact,

$$\operatorname{ind}(\mathbf{I} - \mathbf{T}) = \operatorname{ind}(\mathbf{I} - \mathbf{T}^{\backprime}) = 0 \tag{7,14}$$

(cf. I.3.20) and we may apply Theorem I.3.2.

7.11. Theorem. *If the hypotheses* 7.1 *and* 7.5 *are satisfied, then the system* (7,1), (7,2) *has a solution* $\mathbf{x} \in BV_n$ *if and only if*

$$\int_0^1 \mathrm{d}[\mathbf{z}^*(s)]\,(\mathbf{f}(s) - \mathbf{f}(0)) + \boldsymbol{\lambda}^*\mathbf{r} = 0 \tag{7,15}$$

for any $\mathbf{z}^* \in BV_n^{\backprime}$ *and* $\boldsymbol{\lambda}^* \in R_m^*$ *such that*

$$\mathbf{z}^*(s) - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,\mathbf{P}(t,s) + \boldsymbol{\lambda}^*\,\mathbf{K}(s) = \mathbf{0} \quad on\ [0,1]\,, \qquad \mathbf{z}^*(0) = \mathbf{0}\,. \tag{7,16}$$

Proof. By I.3.2 the system (7,1), (7,2) has a solution if and only if (7,15) holds for any $\mathbf{z}^* \in BV_n^{\backprime}$ and $\boldsymbol{\lambda}^* \in R_m^*$ fulfilling the equation

$$\mathbf{z}^*(s) - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,\mathbf{Q}(t,s) + \boldsymbol{\lambda}^*\,\mathbf{K}(s) = \mathbf{0} \qquad \text{on } [0,1]\,, \tag{7,17}$$

i.e. $(\mathbf{I} - \mathbf{T}^{\backprime})(\mathbf{z}^*, \boldsymbol{\lambda}^*) = \mathbf{0}$ (cf. 7.9, 7.10 and (7,14)). Given $\mathbf{z}^* \in BV_n^{\backprime}$,

$$\int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,\mathbf{Q}(t,s) = \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,\mathbf{P}(t,s) - \begin{cases} \mathbf{z}^*(1) - \mathbf{z}^*(0) & \text{if } s = 0 \\ \mathbf{0} & \text{if } s > 0 \end{cases} \tag{7,18}$$

(cf. (7,4) and I.4.23). After the substitution (7,18), the equation (7,17) becomes

$$(7,19) \qquad \mathbf{z}^*(s) - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,\mathbf{P}(t,s) + \boldsymbol{\lambda}^*\,\mathbf{K}(0) = \mathbf{0} \qquad \text{on } (0,1],$$
$$-\int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,\mathbf{P}(t,0) + \boldsymbol{\lambda}^*\,\mathbf{K}(s) = \mathbf{0}.$$

According to 7.5 $\mathbf{P}(0,s) = \mathbf{0}$ on $[0,1]$. Thus the value of each of the integrals

$$\int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,\mathbf{P}(t,s) \quad (s \in [0,1]), \qquad \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,(\mathbf{f}(t) - \mathbf{f}(0))$$

does not depend on the value $\mathbf{z}^*(0)$ (cf. I.4.23). Consequently $(\mathbf{z}^*, \boldsymbol{\lambda}^*) \in BV_n^{\backprime} \times R_m^*$ is a solution to (7,19) if and only if $(\mathbf{z}_0^*, \boldsymbol{\lambda}^*)$ with $\mathbf{z}_0^*(s) = \mathbf{z}^*(s)$ on $(0,1]$ and $\mathbf{z}_0^*(0) = \mathbf{0}$ is also its solution. The proof is complete.

The following assertion is also a consequence of I.3.2.

7.12. Proposition. *Let* 7.1 *and* 7.5 *be satisfied and let* $\mathbf{h} \in BV_n^{\backprime}$. *Then there exist* $\mathbf{z}^* \in BV_n^{\backprime}$ *and* $\boldsymbol{\lambda}^* \in R_m^*$ *such that*

$$(7,20) \qquad \mathbf{z}^*(s) - \int_0^1 \mathrm{d}[\mathbf{z}^*(t)]\,\mathbf{Q}(t,s) + \boldsymbol{\lambda}^*\,\mathbf{K}(s) = \mathbf{h}^*(s) \qquad \text{on } [0,1]$$

$((\mathbf{I} - \mathbf{T}^{\backprime})(\mathbf{z}^*, \boldsymbol{\lambda}^*) = (\mathbf{h}^*, \mathbf{0}))$ *if and only if*

$$\int_0^1 \mathrm{d}[\mathbf{h}^*(t)]\,\mathbf{x}(t) = 0$$

holds for every $\mathbf{x} \in N(\mathscr{L})$, *where*

$$(7,21) \qquad \mathscr{L}\colon\ \mathbf{x} \in BV_n \to \begin{pmatrix} \mathbf{x} - \mathbf{Q}\mathbf{x} \\ \mathbf{S}\mathbf{x} \end{pmatrix} \in BV_n \times R_m.$$

7.13. Theorem. *Let us assume* 7.1 *and* 7.5 *and let* $\mathscr{L} \in B(BV_n, BV_n \times R_m)$ *be given by* (7,6), (7,8) *and* (7,21). *Then* $k = \dim N(\mathscr{L}) < \infty$ *and the system* (7,16) *has exactly* $k^* = k + m - n$ *linearly independent solutions in* $BV_n^{\backprime} \times R_m^*$.

Proof. By 7.4 $k = \dim N(\mathscr{L}) < \infty$. Obviously $\dim N(\mathbf{I} - \mathbf{T}) = k + m$. Since (7,14), it is by I.3.2 $\dim N(\mathbf{I} - \mathbf{T}^{\backprime}) = \dim N(\mathbf{I} - \mathbf{T}) = k + m$. The set $N^{\backprime}$ of all solutions to (7,16) consists of all $(\mathbf{z}^*, \boldsymbol{\lambda}^*) \in N(\mathbf{I} - \mathbf{T}^{\backprime})$ for which $\mathbf{z}^*(0) = \mathbf{0}$. So $\dim N^{\backprime} = \dim N(\mathbf{I} - \mathbf{T}^{\backprime}) - n = k + m - n$. The proof is complete.

In addition to 7.1 *and* 7.5 *we shall assume henceforth that*

$$(7,22) \qquad \mathbf{P}(t-, s) = \mathbf{P}(t, s) \quad \textit{for all} \quad (t,s) \in (0,1] \times [0,1],$$
$$\mathbf{P}(0+, s) = \mathbf{P}(0, s) \quad \textit{for all} \quad s \in [0,1].$$

In this case we may formulate the adjoint problem to (7,1), (7,2) in a form more similar to (7,1), (7,2).

Integrating by parts (I.4.33) we transfer the system (7,16) of equations for $(\mathbf{z}^*, \boldsymbol{\lambda}^*) \in BV_n^{\backprime} \times R_m^*$ to the form

$$(7,23) \qquad \mathbf{z}^*(s) + \int_0^1 \mathbf{z}^*(t)\, \mathrm{d}_t[\mathbf{P}(t,s)] + \boldsymbol{\lambda}^* \mathbf{K}(s) = \mathbf{0} \quad \text{on } [0,1],$$
$$\mathbf{z}^*(0) = \mathbf{z}^*(1) = \mathbf{0}.$$

As by (7,22) $\mathbf{P}(0+,s) = \mathbf{P}(0,s)$ and $\mathbf{P}(1-,s) = \mathbf{P}(1,s)$ for every $s \in [0,1]$, the value of each of the integrals

$$\int_0^1 \mathbf{z}^*(t)\, \mathrm{d}_t[\mathbf{P}(t,s)], \qquad s \in [0,1]$$

does not depend on the value $\mathbf{z}^*(0)$ and $\mathbf{z}^*(1)$. In particular, if $\mathbf{z}^* \in BV_n^{\backprime}$, $\mathbf{z}^*(0) = \mathbf{0}$, $\boldsymbol{\lambda}^* \in R_m^*$ and

$$(7,24) \quad \mathbf{y}^*(s) = \mathbf{z}^*(s) \text{ on } (0,1), \qquad \mathbf{y}^*(0) = \mathbf{z}^*(0+), \qquad \mathbf{y}^*(1) = \mathbf{z}^*(1-),$$

then the couple $(\mathbf{z}^*, \boldsymbol{\lambda}^*)$ solves (7,23) (i.e. (7,16)) if and only if

$$(7,25) \qquad \mathbf{y}^*(s) + \int_0^1 \mathbf{y}^*(t)\, \mathrm{d}_t[\mathbf{P}(t,s)] + \boldsymbol{\lambda}^* \mathbf{K}(s) = \mathbf{0} \quad \text{on } (0,1),$$
$$\mathbf{0} = \int_0^1 \mathbf{y}^*(t)\, \mathrm{d}[\mathbf{P}(t,0)] + \boldsymbol{\lambda}^* \mathbf{K}(0) \qquad (= \mathbf{z}^*(0)).$$

Applying I.6.16 and I.4.17 we obtain

$$\mathbf{y}^*(0) = \mathbf{y}^*(0+) = -\int_0^1 \mathbf{y}^*(t)\, \mathrm{d}_t[\mathbf{P}(t,0+) - \mathbf{P}(t,0)] - \boldsymbol{\lambda}^*[\mathbf{K}(0+) - \mathbf{K}(0)]$$

and

$$\mathbf{y}^*(1) = \mathbf{y}^*(1-) = -\int_0^1 \mathbf{y}^*(t)\, \mathrm{d}_t[\mathbf{P}(t,1-)] - \boldsymbol{\lambda}^* \mathbf{K}(1-)$$

for every $\mathbf{y} \in BV_n$ and $\boldsymbol{\lambda} \in R_m$ fulfilling (7,25). If for $t \in [0,1]$ we put

$$(7,26) \quad \mathbf{P}_0(t,s) = \begin{cases} \mathbf{P}(t,0+) & \text{if } s = 0, \\ \mathbf{P}(t,s) & \text{if } 0 < s < 1, \\ \mathbf{P}(t,1-) & \text{if } s = 1, \end{cases} \qquad \mathbf{K}_0(s) = \begin{cases} \mathbf{K}(0+) & \text{if } s = 0, \\ \mathbf{K}(s) & \text{if } 0 < s < 1, \\ \mathbf{K}(1-) & \text{if } s = 1, \end{cases}$$

$$\mathbf{C}(t) = \mathbf{P}(t,0+) - \mathbf{P}(t,0), \qquad \mathbf{D}(t) = -\mathbf{P}(t,1-),$$
$$\mathbf{M} = \mathbf{K}(0+) - \mathbf{K}(0), \qquad \mathbf{N} = -\mathbf{K}(1-),$$

then system (7,25) becomes

(7,27)

$$\mathbf{y}^*(s) = \mathbf{y}^*(1) - \int_0^1 \mathbf{y}^*(t)\, \mathrm{d}_t[\mathbf{P}_0(t,s) - \mathbf{P}_0(t,1)] - \boldsymbol{\lambda}^*[\mathbf{K}_0(s) - \mathbf{K}_0(1)] \quad \text{on } [0,1].$$

$$(7,28) \qquad \mathbf{y}^*(0) + \boldsymbol{\lambda}^*\mathbf{M} + \int_0^1 \mathbf{y}^*(t)\,\mathrm{d}[\mathbf{C}(t)] = \mathbf{0}\,,$$

$$(7,29) \qquad \mathbf{y}^*(1) - \boldsymbol{\lambda}^*\mathbf{N} - \int_0^1 \mathbf{y}^*(t)\,\mathrm{d}[\mathbf{D}(t)] = \mathbf{0}\,.$$

Given $\mathbf{z} \in BV_n$ with $\mathbf{z}(0) = \mathbf{z}(1) = \mathbf{0}$ and $\mathbf{y} \in BV_n$ such that (7,24) holds, we have in virtue of I.4.23

$$\int_0^1 \mathrm{d}[\mathbf{z}^*(s)]\,(\mathbf{f}(s) - \mathbf{f}(0)) = \int_0^1 \mathrm{d}[\mathbf{y}^*(s)]\,\mathbf{f}(s) - \mathbf{y}^*(1)\,\mathbf{f}(1) - \mathbf{y}^*(0)\,\mathbf{f}(0)\,.$$

This completes the proof of the following

7.14. Theorem. *If the hypotheses* 7.1, 7.5 *and* (7,22) *are satisfied, then the problem* (7,1), (7,2) *possesses a solution* $\mathbf{x} \in BV_n$ *if and only if*

$$(7,30) \qquad \mathbf{y}^*(1)\,\mathbf{f}(1) - \mathbf{y}^*(0)\,\mathbf{f}(0) - \int_0^1 \mathrm{d}[\mathbf{y}^*(s)]\,\mathbf{f}(s) = \boldsymbol{\lambda}^*\mathbf{r}$$

for any solution $\mathbf{y} \in BV_n$, $\boldsymbol{\lambda} \in R_m$ *of* (7,27)–(7,29), *where* $\mathbf{P}_0$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{K}_0$, $\mathbf{M}$ *and* $\mathbf{N}$ *are defined in* (7,26).

7.15. Remark. If (7,22) holds and $\mathbf{f}(t-) = \mathbf{f}(t)$ on $(0,1]$, $\mathbf{f}(0+) = \mathbf{f}(0)$, then by I.6.16 and I.4.17 any solution $\mathbf{x} \in BV_n$ of (7,1), (7,2) is left-continuous on $(0,1]$ and right-continuous at 0. On the other hand, if $\mathbf{y} \in BV_n$ and $\boldsymbol{\lambda} \in R_m$ satisfy (7,27)–(7,29), then provided that 7.5 holds, $\mathbf{y}$ is right-continuous on $[0,1)$ and left-continuous at 1 (cf. 7.24).

7.16. Remark. Let $\mathbf{g} \in BV_n$ be right-continuous on $(0,1)$, $\mathbf{p}, \mathbf{q} \in R_n$. It is easy to see that $\mathbf{y} \in BV_n$ and $\boldsymbol{\lambda} \in R_m$ satisfy (7,27), (7,28), (7,29) with the right-hand sides $\mathbf{g}^*(s) - \mathbf{g}^*(1)$, $\mathbf{p}^*$ and $\mathbf{q}^*$, respectively, if and only if $\mathbf{y}$ is right-continuous on $(0,1)$ and the couple $(\mathbf{z}^*, \boldsymbol{\lambda}^*)$, $\mathbf{z}^*(s) = \mathbf{y}^*(s)$ on $(0,1)$, $\mathbf{z}^*(0) = \mathbf{z}^*(1) = \mathbf{0}$, fulfils (7,20), where $\mathbf{h}^*(s) = \mathbf{g}^*(s) - \mathbf{g}^*(1) + \boldsymbol{\chi}^*(s)$ on $[0,1]$, $\boldsymbol{\chi}^*(0) = \mathbf{q}^* - \mathbf{p}^*$, $\boldsymbol{\chi}^*(s) = \mathbf{q}^*$ on $(0,1)$ and $\boldsymbol{\chi}^*(1) = \mathbf{0}$. It follows immediately from 7.12 that the system (7,27), (7,28), (7,29) with the right-hand sides $\mathbf{g}^*(s) - \mathbf{g}^*(1)$, $\mathbf{p}^*$ and $\mathbf{q}^*$, respectively, has a solution $\mathbf{y} \in BV_n$, $\boldsymbol{\lambda} \in R_m$ if and only if (cf. (7,21))

$$\int_0^1 \mathrm{d}[\mathbf{g}^*(t)]\,\mathbf{x}(t) = \mathbf{q}^*\,\mathbf{x}(1) - \mathbf{p}^*\,\mathbf{x}(0) \qquad \text{for each} \quad \mathbf{x} \in N(\mathscr{L})\,.$$

7.17. Remark. If $\mathbf{P}$: $[0,1] \times [0,1] \rightarrow L(R_n)$ is an $L^1[BV]$-kernel ($|\mathbf{P}(t,0)| + \mathrm{var}_0^1\, \mathbf{P}(t,.)$ $= \varrho(t) < \infty$ a.e. on $[0,1]$ and $\varrho \in L^1$) and $\mathbf{f} \in L^1_n$, then $\mathbf{x}$: $[0,1] \rightarrow R_n$ is a solution to (2,1) on $[0,1]$ if and only if

$$\mathbf{x}(t) - \mathbf{x}(0) - \int_0^1 \mathrm{d}_s[\mathbf{R}(t,s)]\,\mathbf{x}(s) = \int_0^t \mathbf{f}(\tau)\,\mathrm{d}\tau \qquad \text{on } [0,1]\,,$$

where

$$\mathbf{R}(t,s) = \int_0^t \mathbf{P}(\tau,s)\,\mathrm{d}\tau \qquad \text{on } [0,1]\times[0,1].$$

Given a subdivision $\{0 = t_0 < t_1 < \ldots < t_k = 1;\ 0 = s_0 < s_1 < \ldots < s_k = 1\}$ of $[0,1]\times[0,1]$, we have

$$\sum_{i=1}^{k}\sum_{j=1}^{k} |\mathbf{R}(t_i,s_j) - \mathbf{R}(t_{i-1},s_j) - \mathbf{R}(t_i,s_{j-1}) + \mathbf{R}(t_{i-1},s_{j-1})|$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{k} \left| \int_{t_{i-1}}^{t_i} (\mathbf{P}(\tau,s_j) - \mathbf{P}(\tau,s_{j-1}))\,\mathrm{d}\tau \right| \le \int_0^1 \left(\sum_{j=1}^{k} |\mathbf{P}(\tau,s_j) - \mathbf{P}(\tau,s_{j-1})| \right) \mathrm{d}\tau$$

$$= \|\varrho\|_{L_1} < \infty .$$

Consequently $\mathrm{v}_{[0,1]\times[0,1]}(\mathbf{R}) < \infty$. Clearly $\mathrm{var}_0^1\,\mathbf{R}(.,1) < \infty$. (We may assume $\mathbf{P}(t,1) = \mathbf{0}$ a.e. on $[0,1]$.) As $\mathbf{R}(0,.) = \mathbf{0}$ on $[0,1]$, this implies that $\mathbf{R}$ is an SBV-kernel and the Fredholm-Stieltjes integro-differential equation (2,1) is a special case of the equation (7,1).

7.18. Remark. Let $\mathbf{A}\colon [0,1] \to L(R_n)$, $\mathrm{var}_0^1\,\mathbf{A} < \infty$, $\mathbf{M}$ and $\mathbf{N} \in L(R_n, R_m)$ and

$$\mathbf{P}(t,s) = \begin{cases} \mathbf{A}(0) \;\;\; - \mathbf{A}(t) & \text{if } 0 = s < t \le 1, \\ \mathbf{A}(s+) - \mathbf{A}(t) & \text{if } 0 < s < t \le 1, \\ \mathbf{0} & \text{if } 0 \le t \le s \le 1, \end{cases} \qquad \mathbf{K}(s) = \begin{cases} -\mathbf{M} - \mathbf{N} & \text{if } \quad s = 0, \\ -\mathbf{N} & \text{if } 0 < s < 1, \\ \mathbf{0} & \text{if } \quad s = 1. \end{cases}$$

It can be shown that $\mathrm{v}_{[0,1]\times[0,1]}(\mathbf{P}) \le \mathrm{var}_0^1\,\mathbf{A}$. Furthermore, $\mathbf{P}(0,.) = \mathbf{0}$ on $[0,1]$, $\mathrm{var}_0^1\,\mathbf{P}(.,0) = \mathrm{var}_0^1\,\mathbf{A}$ and $\mathrm{var}_0^1\,\mathbf{K} = |\mathbf{M}| + |\mathbf{N}|$. Since for any $t \in [0,1]$ $\mathbf{P}(t,.)$ and $\mathbf{K}$ are right-continuous on $(0,1)$, $\mathbf{K}(1) = \mathbf{0}$ and $\mathbf{P}(t,1) = \mathbf{0}$, the assumptions 7.1 and 7.5 are satisfied in this case. If, moreover, $\mathbf{A}$ is left-continuous on $(0,1]$ and right-continuous at 0, then $\mathbf{P}(t-,0) = \mathbf{A}(0) - \mathbf{A}(t-) = \mathbf{A}(0) - \mathbf{A}(t)$ for $0 < t \le 1$, $\mathbf{P}(t-,s) = \mathbf{A}(s+) - \mathbf{A}(t-) = \mathbf{A}(s+) - \mathbf{A}(t)$ for $0 < s < t \le 1$ and $\mathbf{P}(t-,s) = \mathbf{0}$ for $0 < t \le s \le 1$. Finally, $\mathbf{P}(0+,s) = \mathbf{0}$ for any $s \in [0,1]$. Thus $\mathbf{P}$ fulfils also (7,22). By 7.14 the system (7,1), (7,2) which is now reduced to BVP $\mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\,\mathbf{x} + \mathrm{d}\mathbf{f}$, $\mathbf{M}\,\mathbf{x}(0) + \mathbf{N}\,\mathbf{x}(1) = \mathbf{r}$ has a solution if and only if (7,30) holds for all $\mathbf{y} \in BV_n$ and $\lambda \in R_m$ satisfying (7,27), (7,29). In our case $\mathbf{P}_0(t,s) = \mathbf{P}(t,s)$, $\mathbf{C}(t) = \mathbf{D}(t) = \mathbf{0}$ and $\mathbf{K}_0(s) = -\mathbf{N}$. Moreover,

$$\int_0^1 \mathbf{y}^*(t)\,\mathrm{d}_t[\mathbf{P}(t,s)] = \int_s^1 \mathbf{y}^*(t)\,\mathrm{d}[\mathbf{B}(t)] \qquad \text{for any } \mathbf{y} \in BV_n \text{ and } s \in [0,1],$$

where $\mathbf{B}(s) = \mathbf{A}(s+)$ on $(0,1)$, $\mathbf{B}(0) = \mathbf{A}(0)$ and $\mathbf{B}(1) = \mathbf{A}(1)$. It follows that under the assumptions of this remark the adjoint system (7,27)–(7,29) to (7,1), (7,2) reduces to BVP (III.5,12), (III.5,13). Let us notice that now no assumptions on the regularity of the matrices $(\mathbf{I} + \Delta^+\mathbf{A}(t))$ are needed.

7.19. Remark. Let the matrix valued functions $\mathbf{A}: [0,1] \to L(R_n)$, $\mathbf{P}_1: [0,1] \to L(R_p, R_n)$, $\mathbf{P}_2: [0,1] \to L(R_n, R_p)$, $\mathbf{C}: [0,1] \to L(R_n)$, $\mathbf{D}: [0,1] \to L(R_n)$ and $\mathbf{K}: [0,1] \to L(R_n, R_m)$ be of bounded variation on $[0,1]$, $\mathbf{M}, \mathbf{N} \in L(R_n, R_m)$, $\mathbf{f} \in BV_n$ and $\mathbf{r} \in R_m$ and let us consider the system of equations for $\mathbf{x} \in BV_n$

$$(7,31) \quad \mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \mathrm{d}[\mathbf{A}(s)]\,\mathbf{x}(s) + (\mathbf{C}(t) - \mathbf{C}(0))\,\mathbf{x}(0) + (\mathbf{D}(t) - \mathbf{D}(0))\,\mathbf{x}(1)$$

$$+ (\mathbf{P}_1(t) - \mathbf{P}_1(0)) \int_0^1 \mathrm{d}[\mathbf{P}_2(s)]\,\mathbf{x}(s) + \mathbf{f}(t) - \mathbf{f}(0) \qquad \text{on } [0,1],$$

$$(7,32) \qquad \mathbf{M}\,\mathbf{x}(0) + \mathbf{N}\,\mathbf{x}(1) + \int_0^1 \mathrm{d}[\mathbf{K}(s)]\,\mathbf{x}(s) = \mathbf{r}.$$

Introducing new unknowns $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, $\boldsymbol{\chi}$ by the relations

$$\boldsymbol{\alpha}(t) = \int_0^t \mathrm{d}[\mathbf{K}(s)]\,\mathbf{x}(s), \qquad \boldsymbol{\beta}(t) = \int_0^t \mathrm{d}[\mathbf{P}_2(s)]\,\mathbf{x}(s),$$

$$\boldsymbol{\gamma}(t) = \mathbf{x}(0), \qquad \boldsymbol{\delta}(t) = \mathbf{x}(1), \qquad \boldsymbol{\chi}(t) = \boldsymbol{\beta}(1),$$

we reduce the given problem to the form

$$\mathrm{d}\mathbf{x} = \mathrm{d}[\mathbf{A}]\,\mathbf{x} + \mathrm{d}[\mathbf{P}_1]\,\boldsymbol{\beta} + \mathrm{d}[\mathbf{C}]\,\boldsymbol{\gamma} + \mathrm{d}[\mathbf{D}]\,\boldsymbol{\delta} + \mathrm{d}\mathbf{f},$$

$$\mathrm{d}\boldsymbol{\alpha} = \mathrm{d}[\mathbf{K}]\,\mathbf{x}, \qquad \mathrm{d}\boldsymbol{\beta} = \mathrm{d}[\mathbf{P}_2]\,\mathbf{x}, \qquad \mathrm{d}\boldsymbol{\gamma} = \mathbf{0}, \qquad \mathrm{d}\boldsymbol{\delta} = \mathbf{0}, \qquad \mathrm{d}\boldsymbol{\chi} = \mathbf{0},$$

$$\mathbf{M}\,\mathbf{x}(0) + \mathbf{N}\,\mathbf{x}(1) + \boldsymbol{\alpha}(1) = \mathbf{r}, \qquad \boldsymbol{\alpha}(0) = \mathbf{0}, \qquad \mathbf{x}(0) - \boldsymbol{\gamma}(0) = \mathbf{0},$$

$$\mathbf{x}(1) - \boldsymbol{\delta}(0) = \mathbf{0}, \qquad \boldsymbol{\beta}(0) = \mathbf{0}, \qquad \boldsymbol{\beta}(1) - \boldsymbol{\chi}(0) = \mathbf{0}$$

which may be expressed in the matrix version

$$\mathrm{d}\xi = \mathrm{d}[\mathfrak{A}]\,\xi + \mathrm{d}\varphi, \qquad \mathfrak{M}\,\varphi(0) + \mathfrak{N}\,\xi(1) = \varrho,$$

where $\xi^* = (\mathbf{x}^*, \boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*, \boldsymbol{\delta}^*, \boldsymbol{\chi}^*)$ and $\mathfrak{A}: [0,1] \to L(R_\nu)$ and $\mathfrak{M}. \mathfrak{N} \in L(R_\nu, R_\mu)$ are appropriately defined matrices, $\mu = 2m + 2n + 2p$, $\nu = m + 3n + 2p$, $\varphi = \begin{pmatrix} \mathbf{f} \\ \mathbf{0}_{\nu-n} \end{pmatrix}$ and $\varrho = \begin{pmatrix} \mathbf{r} \\ \mathbf{0}_{\mu-m} \end{pmatrix}$. By this $\mathrm{var}_0^1 \mathfrak{A} < \infty$. The complicated problem (7,31), (7,32) was transferred to the two-point boundary value problem for a linear generalized differential equation.

Notes

In the case $p = 1$ the compactness of the operator $\mathbf{P}$ and hence also the closedness of $R(\mathbf{L})$ (V.1.4 and V.1.7) were proved by Maksimov [1] and independently by Tvrdý [4]. Theorem V.1.8 is due to Maksimov and Rahmatullina [2]. Our proof follows a different idea. The proofs of the main theorems of Section V.2 (V.2.5, V.2.6 and V.2.12) are carried out in a similar way as the proofs of analogous results for ordinary differential operators in Wexler [1] (cf. also Tvrdý, Vejvoda [1], Tvrdý [3], Maksimov [1]).

For more detail concerning Green's couples see Tvrdý [6]. Systems of the form (4,27), (4,28) were treated in Tvrdý, Vejvoda [1]. Theorem V.6.4 follows also from the variation of constants formula for functional differential equations of the retarded type due to Banks [1]. Equations of the form (V.6,13) were introduced in Delfour, Mitter [1], [2]. Section V.7 is based on the paper Tvrdý [5]. The transformation similar to (7,33) was for the first time used in a simpler situation by Jones [1] and Taufer [1]. For more detail concerning the systems of the form (7,31), (7,32) (Green's function, Jones transformation, selfadjoint problems etc.) see Vejvoda, Tvrdý [1], Tvrdý [1] and Zimmerberg [1], [2].

The oldest papers on the subject seem to be Duhamel [1], Lichtenstein [1] and Tamarkin [1]. Further related references to particular sections are

V.1: Catchpole [1], [2];
V.2: Parhimovič [1]–[3], Lando [1]–[4], Krall [2], [5], Tvrdý [1];
V.3: Maksimov, Rahmatullina [1], [2];
V.6: Hale [1], Maksimov, Rahmatullina [1], Rahmatullina [1], Tvrdý [4];
V.7: Krall [6]–[8], Hönig [1], Tvrdý [2].

Related results may be found also in the papers by N. V. Azbelev and the members of his group (L. F. Rahmatullina, V. P. Maksimov, A. G. Terent'ev, T. S. Sulavko, S. M. Labovskij, G. G. Islamov a.o.) which have appeared mainly in Differencial'nye uravnenija and in the collections of papers published by the Moscow and Tambov institutes of the chemical machines construction.

In Lando [3], [4] and Kultyšev [1] the controllability of integro-differential operators is studied.