17. Series of factoroids

In this chapter we shall develop a theory of the so-called series of factoroids. This theory is based on the properties we have verified for series of decompositions on sets, in chapter 10. But now our deliberations will be extended by including algebraic situations resulting from the multiplication. We shall often come across concepts connected with the properties of α-grade groupoidal structures.

17.1. Basic concepts

Let $\overline{\mathfrak{A}} \geqq \overline{\mathfrak{B}}$ denote arbitrary factoroids on $\mathfrak{G}$.

By a *series of factoroids on* $\mathfrak{G}$ *from* $\overline{\mathfrak{A}}$ *to* $\overline{\mathfrak{B}}$, briefly, a *series from* $\overline{\mathfrak{A}}$ *to* $\overline{\mathfrak{B}}$ we mean a finite α-membered ($\alpha \geqq 1$) sequence of factoroids $\overline{\mathfrak{A}}_1, \ldots, \overline{\mathfrak{A}}_\alpha$ on $\mathfrak{G}$ with the following properties: a) The first factoroid is $\overline{\mathfrak{A}}$, the last $\overline{\mathfrak{B}}$; hence $\overline{\mathfrak{A}}_1 = \overline{\mathfrak{A}}$, $\overline{\mathfrak{A}}_\alpha = \overline{\mathfrak{B}}$; b) every factoroid is a refinement of the preceding one and so:

$$(\overline{\mathfrak{A}} =)\, \overline{\mathfrak{A}}_1 \geqq \cdots \geqq \overline{\mathfrak{A}}_\alpha\, (= \overline{\mathfrak{B}}).$$

Such a series is briefly denoted by $(\overline{\mathfrak{A}})$. The factoroids $\overline{\mathfrak{A}}_1, \ldots, \overline{\mathfrak{A}}_\alpha$ are called *members of the series* $(\overline{\mathfrak{A}})$. $\overline{\mathfrak{A}}_1$ is the *initial*, $\overline{\mathfrak{A}}_\alpha$ the *final member of* $(\overline{\mathfrak{A}})$. By the *length of* $(\overline{\mathfrak{A}})$ we understand the number α of its members.

For example, the factoroid $\overline{\mathfrak{A}}$ is a series of length 1; the initial and the final member of this series coincide with the factoroid $\overline{\mathfrak{A}}$ itself.

The fields of the individual members of an arbitrary series $(\overline{\mathfrak{A}})$ on $\mathfrak{G}$ form a series of generating decompositions on $\mathfrak{G}$, $(\bar{A})$. The concepts and results valid for the series $(\bar{A})$ may directly be applied to the series $(\overline{\mathfrak{A}})$. In this way we may, for example, define the length of $(\overline{\mathfrak{A}})$ as the length of $(\bar{A})$. Naturally, as regards the theory of series of factoroids, we are particularly interested in situations connected with the multiplication.

The concepts, adopted by the theory of the series of factoroids in this way, will not be explicitely studied here, their meaning is obvious. For example: essential members, reduced length, shortening and lengthening, refinement of $(\overline{\mathfrak{A}})$, as well as the concepts of modular and complementary series of factoroids, etc.

17.2. Local chains

The following considerations are based on the concept of a local chain; it has also been adopted from the theory of series of decompositions (10.2) but will, however, be introduced here owing to its importance.

Let $\big((\overline{\mathfrak{A}}) =\big)\, \overline{\mathfrak{A}}_1 \geqq \cdots \geqq \overline{\mathfrak{A}}_\alpha$ be a series of factoroids on $\mathfrak{G}$, of an arbitrary length $\alpha \geqq 1$.

Let $\bar{a} \in \overline{\mathfrak{A}}_\alpha$ be an arbitrary element and $\bar{a}_\gamma$ denote that element of $\overline{\mathfrak{A}}_\gamma$ for which $\bar{a} \subset \bar{a}_\gamma$ $(\gamma = 1, \ldots, \alpha)$. Then we have:

$$\bar{a}_1 \supset \cdots \supset \bar{a}_\alpha \ (\bar{a}_\alpha = \bar{a}).$$

The intersection

$$\overline{K}_\gamma = \bar{a}_\gamma \sqcap \overline{\mathfrak{A}}_{\gamma+1}$$

coincides with the closure $\bar{a}_\gamma \sqsubset \overline{\mathfrak{A}}_{\gamma+1}$ and is a decomposition of the element $\bar{a}_\gamma$. It is a complex in $\overline{\mathfrak{A}}_{\gamma+1}$ such that $\bar{a}_{\gamma+1} \in \overline{K}_\gamma$ $(\bar{a}_{\alpha+1} = \bar{a}_\alpha)$.

The chain of decompositions in $\mathfrak{G}$ from $\bar{a}_1$ to $\bar{a}_{\alpha+1}$:

$$([\overline{K}] =)\ \overline{K}_1 \to \cdots \to \overline{K}_\alpha$$

is called the *local chain of the series* $(\overline{\mathfrak{A}})$ *corresponding to the base* $\bar{a}$, briefly, the *local chain with the base* $\bar{a}$. Notation as above or more accurately:

$$([\overline{K}\bar{a}] =)\ \overline{K}_1\bar{a} \to \cdots \to \overline{K}_\alpha\bar{a}.$$

In connection with the multiplication in $\mathfrak{G}$ it may happen that the base $\bar{a}$ and therefore even the elements $\bar{a}_\gamma \in \overline{\mathfrak{A}}_\gamma$ $(\gamma = 1, \ldots, \alpha)$ are groupoidal subsets (14.5.1). In that case the decompositions $\overline{K}_\gamma$ are generating (14.4.1). Such a local chain is called *groupoidal.* The factoroids $\overline{\mathfrak{K}}_\gamma$ in $\mathfrak{G}$, belonging to the individual generating decompositions $\overline{K}_\gamma$, form the *local chain of factoroids of the series* $(\overline{\mathfrak{A}})$, *corresponding to the base* $\bar{a}$, briefly, the *local chain of factoroids with the base* $\bar{a}$. Notation: $[\overline{\mathfrak{K}}]$ or $[\overline{\mathfrak{K}}\bar{a}]$.

17.3. The groupoid of local chains

Suppose that $\big((\overline{\mathfrak{A}}) =\big)\ \overline{\mathfrak{A}}_1 \geqq \cdots \geqq \overline{\mathfrak{A}}_\alpha$ $(\alpha \geqq 1)$ is an arbitrary series of factoroids on $\mathfrak{G}$.

To every element $\bar{a} \in \overline{\mathfrak{A}}_\alpha$ there corresponds a local chain $[\overline{K}\bar{a}]$ of the series $(\overline{\mathfrak{A}})$, with the base $\bar{a}$.

The set consisting of the local chains corresponding to the individual elements of the factoroid $\overline{\mathfrak{A}}_\alpha$ forms the *manifold of the local chains,* $\tilde{A}$, *corresponding to the series* $(\overline{\mathfrak{A}})$. It is obviously an α-grade structure with regard to the sequence of factoroids $\overline{\mathfrak{A}}_2, \ldots, \overline{\mathfrak{A}}_{\alpha+1}$ $(\overline{\mathfrak{A}}_{\alpha+1} = \overline{\mathfrak{A}}_\alpha)$.

The multiplication in $\tilde{A}$ may be defined as follows: The product $[\overline{K}\bar{a}]\,[\overline{K}\bar{b}]$ of every two elements $[\overline{K}\bar{a}], [\overline{K}\bar{b}] \in \tilde{A}$ is given by the formula:

$$[\overline{K}\bar{a}]\,[\overline{K}\bar{b}] = [\overline{K}\bar{a} \circ \bar{b}].$$

The manifold $\tilde{A}$ together with this multiplication forms a groupoid $\tilde{\mathfrak{A}}$, called the *groupoid of local chains, corresponding to the series* $(\overline{\mathfrak{A}})$.

Let us, first, show that the groupoid $\tilde{\mathfrak{A}}$ is an α-grade groupoidal structure with regard to the sequence of factoroids $\overline{\mathfrak{A}}_2, \ldots, \overline{\mathfrak{A}}_{\alpha+1}$ $(\overline{\mathfrak{A}}_{\alpha+1} = \overline{\mathfrak{A}}_\alpha)$.

In fact, every element of $\tilde{\mathfrak{A}}$ is an α-membered sequence each element of which, with an arbitrary index γ $(= 1, \ldots, \alpha)$, is a decomposition in $\mathfrak{G}$ and is a complex in the factoroid $\overline{\mathfrak{A}}_{\gamma+1}$.

The multiplication in $\tilde{\mathfrak{A}}$ is such that for any two elements:

$$[\bar{K}\bar{a}] = \bar{K}_1\bar{a} \to \cdots \to \bar{K}_\alpha\bar{a}, \quad [\bar{K}\bar{b}] = \bar{K}_1\bar{b} \to \cdots \to \bar{K}_\alpha\bar{b} \in \tilde{\mathfrak{A}}$$

and their product

$$[\bar{K}\bar{a}]\,[\bar{K}\bar{b}] = [\bar{K}\bar{a} \circ \bar{b}] = \bar{K}_1\bar{a} \circ \bar{b} \to \cdots \to \bar{K}_\alpha\bar{a} \circ \bar{b} \in \tilde{\mathfrak{A}},$$

there holds (15.4.2):

$$\bar{K}_1\bar{a} \circ \bar{K}_1\bar{b} \subset \bar{K}_1\bar{a} \circ \bar{b}, \ldots, \bar{K}_\alpha\bar{a} \circ \bar{K}_\alpha\bar{b} \subset \bar{K}_\alpha\bar{a} \circ \bar{b}.$$

Associating, with every point $a \in \mathfrak{G}$, the local chain $[\bar{K}\bar{a}] \in \tilde{\mathfrak{A}}$ with the base $\bar{a} = \bar{a}_\alpha \in \overline{\mathfrak{A}}_\alpha$ containing the point a $(a \in \bar{a})$, we obtain a mapping $\boldsymbol{d}$ of $\mathfrak{G}$ onto the groupoid of local chains $\tilde{\mathfrak{A}}$; $\boldsymbol{d}$ is obviously a deformation. It is called the *natural deformation of* $\mathfrak{G}$ *onto* $\tilde{\mathfrak{A}}$. The factoroid corresponding to the deformation $\boldsymbol{d}$ coincides with the factoroid $\overline{\mathfrak{A}}_\alpha$. By the *local chain of* $(\overline{\mathfrak{A}})$, *corresponding to the point* a, we mean the local chain $[\bar{K}\bar{a}]$.

Now let:

$$\big((\overline{\mathfrak{A}}) =\big)\ \overline{\mathfrak{A}}_1 \geqq \cdots \geqq \overline{\mathfrak{A}}_\alpha,$$
$$\big((\overline{\mathfrak{B}}) =\big)\ \overline{\mathfrak{B}}_1 \geqq \cdots \geqq \overline{\mathfrak{B}}_\beta$$

be arbitrary series of factoroids on $\mathfrak{G}$ such that their end-members $\overline{\mathfrak{A}}_\alpha$, $\overline{\mathfrak{B}}_\beta$ coincide, hence $\overline{\mathfrak{A}}_\alpha = \overline{\mathfrak{B}}_\beta$.

Consider the groupoids of local chains, $\tilde{\mathfrak{A}}$, $\tilde{\mathfrak{B}}$, corresponding to $(\overline{\mathfrak{A}})$, $(\overline{\mathfrak{B}})$, respectively.

Associating, with every element $[\bar{K}\bar{a}] \in \tilde{\mathfrak{A}}$, the element $[\bar{L}\bar{a}] \in \tilde{\mathfrak{B}}$ with the same base $\bar{a}$, we get a simple mapping of $\tilde{\mathfrak{A}}$ onto $\tilde{\mathfrak{B}}$. This mapping is obviously isomorphic and is called the *co-basal isomorphism.*

We observe that *the groupoids of local chains corresponding to two series of factoroids with coinciding end-members are isomorphic, the deformation being the co-basal isomorphism.*

17.4. Chain-isomorphic series of factoroids

Assume

$$\big((\overline{\mathfrak{A}}) =\big)\ \overline{\mathfrak{A}}_1 \geqq \cdots \geqq \overline{\mathfrak{A}}_\alpha,$$
$$\big((\overline{\mathfrak{B}}) =\big)\ \overline{\mathfrak{B}}_1 \geqq \cdots \geqq \overline{\mathfrak{B}}_\alpha$$

to be arbitrary series of factoroids on $\mathfrak{G}$ of the same length α $(\geqq 1)$.

Let $\tilde{\mathfrak{A}}$, $\tilde{\mathfrak{B}}$ stand for the groupoids of local chains, corresponding to the above series.

The series $(\overline{\mathfrak{B}})$ is said to be *chain-isomorphic with* $(\overline{\mathfrak{A}})$ if the groupoid $\tilde{\mathfrak{B}}$ is strongly isomorphic with $\tilde{\mathfrak{A}}$.

If $(\overline{\mathfrak{B}})$ is chain-isomorphic with $(\overline{\mathfrak{A}})$, then $(\overline{\mathfrak{A}})$ has the same property with respect to $(\overline{\mathfrak{B}})$ (16.3.1). Taking account of this symmetry, we sometimes use the term *chain-isomorphic series* $(\overline{\mathfrak{A}})$, $(\overline{\mathfrak{B}})$.

By the above definition, $(\overline{\mathfrak{B}})$ is chain-isomorphic with $(\overline{\mathfrak{A}})$ if there exists a strong isomorphism of $\tilde{\mathfrak{A}}$ onto $\tilde{\mathfrak{B}}$ (16.3.2). If, in particular, the end-members $\overline{\mathfrak{A}}_\alpha$, $\overline{\mathfrak{B}}_\alpha$ of the series $(\overline{\mathfrak{A}})$, $(\overline{\mathfrak{B}})$, respectively, coincide and the co-basal mapping of $\tilde{\mathfrak{A}}$ onto $\tilde{\mathfrak{B}}$ is a strong isomorphism, then $(\overline{\mathfrak{B}})$ is said to be *co-basally chain-isomorphic with* $(\overline{\mathfrak{A}})$ and we speak about *co-basally chain-isomorphic series* $(\overline{\mathfrak{A}})$, $(\overline{\mathfrak{B}})$.

Suppose that the series $(\overline{\mathfrak{A}})$, $(\overline{\mathfrak{B}})$ are chain-isomorphic.

This situation can briefly be described as follows:

There exists an isomorphic mapping $\boldsymbol{i}$ of $\tilde{\mathfrak{A}}$ onto $\tilde{\mathfrak{B}}$ and, moreover, a permutation $\boldsymbol{p}$ of the set $\{1, \ldots, \alpha\}$ with the following effect:

The permutation $\boldsymbol{p}$ determines, for every element $[\bar{K}]$ and its image $\boldsymbol{i}[\bar{K}]$ under the isomorphism $\boldsymbol{i}$, a simple function associating, with every member $\bar{K}_\gamma$ of the local chain $[\bar{K}]$ $(\gamma = 1, \ldots, \alpha)$, a member $\bar{L}_\delta$ of $\boldsymbol{i}[\bar{K}]$ with the index $\delta = \boldsymbol{p}\gamma$. Furthermore, to $\bar{K}_\gamma$ there corresponds a simple mapping $\boldsymbol{a}_\gamma$ of the set $\bar{K}_\gamma$ onto $\bar{L}_\delta$. The simple mappings $\boldsymbol{a}_\gamma$, $\boldsymbol{b}_\gamma$, $\boldsymbol{c}_\gamma$ corresponding to the members $\bar{K}_\gamma\bar{a}$, $\bar{K}_\gamma\bar{b}$ of arbitrary local chains $[\bar{K}\bar{a}]$, $[\bar{K}\bar{b}]$ and to the member $\bar{K}_\gamma\bar{a} \circ \bar{b}$ of the product $[\bar{K}\bar{a}]\,[\bar{K}\bar{b}] = [\bar{K}\bar{a} \circ \bar{b}]$ are of homomorphic character, i.e., for any elements $a \in \bar{K}_\gamma\bar{a}$, $b \in \bar{K}_\gamma\bar{b}$ there holds:

$$\boldsymbol{c}_\gamma(a \circ b) = (\boldsymbol{a}_\gamma a) \circ (\boldsymbol{b}_\gamma b).$$

It is obvious that $(\overline{\mathfrak{A}})$, $(\overline{\mathfrak{B}})$ are chain-equivalent so that our considerations concerning chain-equivalent series of decompositions of sets (10.5) may be applied to them. We observe, moreover, that $(\overline{\mathfrak{A}})$ *and* $(\overline{\mathfrak{B}})$ *are of the same reduced length.*

17.5. Semi-joint and joint series of factoroids

Considerations similar to those by which we have arrived at the notion of chain-isomorphic series of factoroids lead to semi-joint and joint series of factoroids.

Let us employ the same notation as above.

The series $(\overline{\mathfrak{B}})$ is said to be *semi-joint* or *loosely joint* (*joint*) with the series $(\overline{\mathfrak{A}})$ if the groupoid $\tilde{\mathfrak{B}}$ is isomorphic and semi-coupled (isomorphic and coupled) with the groupoid $\tilde{\mathfrak{A}}$.

If $(\overline{\mathfrak{B}})$ is loosely joint (joint) with $(\overline{\mathfrak{A}})$, then $(\overline{\mathfrak{A}})$ has the same property with regard to $(\overline{\mathfrak{B}})$. Accordingly, we also use the expression *semi-joint* or *loosely joint* (*joint*) *series* $(\overline{\mathfrak{A}})$, $(\overline{\mathfrak{B}})$.

By the above definition, $(\overline{\mathfrak{B}})$ is semi-joint (joint) with $(\overline{\mathfrak{A}})$ if there exists an isomorphism with loose coupling (an isomorphism with coupling) of $\tilde{\mathfrak{A}}$ onto $\tilde{B}$ (16.3.2). If, in particular, the end-members $\overline{\mathfrak{A}}_\alpha$, $\overline{\mathfrak{B}}_\alpha$ of $(\overline{\mathfrak{A}})$ and $(\overline{\mathfrak{B}})$, respectively, coincide and the co-basal mapping of $\tilde{\mathfrak{A}}$ onto $\tilde{\mathfrak{B}}$ is an isomorphism with loose coupling (isomorphism with coupling), then $(\overline{\mathfrak{B}})$ is said to be *co-basally semi-joint* or *co-basally loosely joint (co-basally joint) with* $(\overline{\mathfrak{A}})$; in that case we also speak about *co-basally semi-joint* or *co-basally loosely joint (co-basally joint) series* $(\overline{\mathfrak{A}})$, $(\overline{\mathfrak{B}})$.

This situation can briefly be described as follows:

There exists an isomorphic mapping $\boldsymbol{i}$ of $\tilde{\mathfrak{A}}$ onto $\tilde{\mathfrak{B}}$ and, moreover, a permutation $\boldsymbol{p}$ of the set $\{1, \ldots, \alpha\}$ with the following effect:

The permutation $\boldsymbol{p}$ determines, for every element $[\bar{K}] \in \tilde{\mathfrak{A}}$ and its image $\boldsymbol{i}[\bar{K}] \in \tilde{\mathfrak{B}}$ under the isomorphism $\boldsymbol{i}$, a simple function associating, with every member $\bar{K}_\gamma$ of the local chain $[K]$ $(\gamma = 1, \ldots, \alpha)$, a member $\bar{L}_\delta$ of the local chain $\boldsymbol{i}[\bar{K}]$, while $\delta = \boldsymbol{p}\gamma$. Furthermore, to the closure $\mathrm{H}\bar{K}_\gamma = \bar{L}_\delta \sqsubset \bar{K}_\gamma$ there belongs a simple mapping $\boldsymbol{a}_\gamma$, given by the incidence of elements, which maps the closure $\mathrm{H}\bar{K}_\gamma$ onto $\mathrm{H}\bar{L}_\delta = \bar{K}_\gamma \sqsubset \bar{L}_\delta$. The mappings $\boldsymbol{a}_\gamma$, $\boldsymbol{b}_\gamma$ which belong to the closures $\mathrm{H}\bar{K}_\gamma\bar{a}$, $\mathrm{H}\bar{K}_\gamma\bar{b}$ corresponding to arbitrary local chains $[\bar{K}\bar{a}]$, $[\bar{K}\bar{b}] \in \tilde{\mathfrak{A}}$ and the mapping $\boldsymbol{c}_\gamma$ which belongs to the closure $\mathrm{H}\bar{K}_\gamma\bar{a} \circ \bar{b}$ corresponding to the product $[\bar{K}\bar{a}]\,[\bar{K}\bar{b}] = [\bar{K}\bar{a} \circ \bar{b}]$ $\in \tilde{\mathfrak{A}}$ are of homomorphic character, i.e., for arbitrary elements $a \in \mathrm{H}\bar{K}_\gamma\bar{a}$, $b \in \mathrm{H}\bar{K}_\gamma\bar{b}$ there holds $\boldsymbol{c}_\gamma(a \circ b) = (\boldsymbol{a}_\gamma a) \circ (\boldsymbol{b}_\gamma b)$.

If, in particular, $(\overline{\mathfrak{A}})$ *and* $(\overline{\mathfrak{B}})$ *are joint, then they are chain-isomorphic and therefore of the same reduced length* (17.4).

17.6. Modular and complementary series of factoroids

Let

$$\big((\overline{\mathfrak{A}}) =\big)\ \overline{\mathfrak{A}}_1 \geqq \cdots \geqq \overline{\mathfrak{A}}_\alpha,$$
$$\big((\overline{\mathfrak{B}}) =\big)\ \overline{\mathfrak{B}}_1 \geqq \cdots \geqq \overline{\mathfrak{B}}_\beta$$

be modular series of factoroids on $\mathfrak{G}$ of lengths α, β $(\geqq 1)$, respectively.

There holds the following theorem:

The series $(\overline{\mathfrak{A}})$, $(\overline{\mathfrak{B}})$ *have co-basally loosely joint refinements* $(\mathring{\mathfrak{A}})$, $(\mathring{\mathfrak{B}})$, *respectively, with the same initial and final members.*

Denote

$$[\overline{\mathfrak{A}}_1, \overline{\mathfrak{B}}_1] = \overline{\mathfrak{U}}, \quad (\overline{\mathfrak{A}}_\alpha, \overline{\mathfrak{B}}_\beta) = \overline{\mathfrak{V}},$$
$$\overline{\mathfrak{A}}_0 = \overline{\mathfrak{B}}_0 = \overline{\mathfrak{G}}_{\max}; \quad \overline{\mathfrak{A}}_{\alpha+1} = \overline{\mathfrak{B}}_{\beta+1} = \overline{\mathfrak{V}}$$

and, moreover, for $\gamma, \mu = 1, \ldots, \alpha + 1$; $\delta, \nu = 1, \ldots, \beta + 1$,

$$\mathring{\mathfrak{A}}_{\gamma,\nu} = [\overline{\mathfrak{A}}_\gamma, (\overline{\mathfrak{A}}_{\gamma-1}, \overline{\mathfrak{B}}_\nu)] = (\overline{\mathfrak{A}}_{\gamma-1}, [\overline{\mathfrak{A}}_\gamma, \overline{\mathfrak{B}}_\nu]),$$
$$\mathring{\mathfrak{B}}_{\delta,\mu} = [\overline{\mathfrak{B}}_\delta, (\overline{\mathfrak{B}}_{\delta-1}, \overline{\mathfrak{A}}_\mu)] = (\overline{\mathfrak{B}}_{\delta-1}, [\overline{\mathfrak{B}}_\delta, \overline{\mathfrak{A}}_\mu]).$$

Then the above co-basally loosely joint refinements of $(\overline{\mathfrak{A}})$, $(\overline{\mathfrak{B}})$ *are expressed by the following formulae:*

$$\big((\mathring{\mathfrak{A}}) =\big)\ \overline{\mathfrak{U}} = \mathring{\mathfrak{A}}_{1,1} \geqq \cdots \geqq \mathring{\mathfrak{A}}_{1,\beta+1} \geqq \mathring{\mathfrak{A}}_{2,1} \geqq \cdots \geqq \mathring{\mathfrak{A}}_{2,\beta+1} \geqq \cdots$$
$$\geqq \mathring{\mathfrak{A}}_{\alpha+1,1} \geqq \cdots \geqq \mathring{\mathfrak{A}}_{\alpha+1,\beta+1} = \overline{\mathfrak{B}},$$
$$\big((\mathring{\mathfrak{B}}) =\big)\ \overline{\mathfrak{U}} = \mathring{\mathfrak{B}}_{1,1} \geqq \cdots \geqq \mathring{\mathfrak{B}}_{1,\alpha+1} \geqq \mathring{\mathfrak{B}}_{2,1} \geqq \cdots \geqq \mathring{\mathfrak{B}}_{2,\alpha+1} \geqq \cdots$$
$$\geqq \mathring{\mathfrak{B}}_{\beta+1,1} \geqq \cdots \geqq \mathring{\mathfrak{B}}_{\beta+1,\alpha+1} = \overline{\mathfrak{B}}$$

If $(\overline{\mathfrak{A}})$, $(\overline{\mathfrak{B}})$ *are complementary, then the refinements* $(\mathring{\mathfrak{A}})$, $(\mathring{\mathfrak{B}})$ *are co-basally joint.*

The correctness of this theorem follows from 10.7, 10.8.

17.7. Exercises

1. If any two factoroids lying on $\mathfrak{G}$ are complementary, then any two series of factoroids on $\mathfrak{G}$ have co-basally joint refinements.

18. Remarkable kinds of groupoids

The study of some remarkable kinds of groupoids closely ties up with our considerations in chapter 11.2. We have not dealt with them before because we wish to emphasize that the preceding deliberations apply to all groupoids regardless of any particular properties. Now we shall be concerned with the groupoids that are of most importance to our theory, namely, the associative groupoids, the groupoids with uniquely defined division and the groupoids with a unit element.

Moreover, we shall pay a brief attention to the Brandt groupoids though they do not belong exactly within the range of our study.