23. Special decompositions of groups, generated by subgroups

In: Otakar Borůvka (author): Foundations of the Theory of Groupoids and Groups. (English). Berlin: VEB Deutscher Verlag der Wissenschaften, 1974. pp. 169--179.

Persistent URL: http://dml.cz/dmlcz/401562

22.4. Exercises

1. The order of any group consisting of permutations of a finite set of order n is a divisor of $n!$.
2. In every finite Abelian group of order N the number of elements inverse of themselves is a divisor of N.

23. Special decompositions of groups, generated by subgroups

23.1. Semi-coupled and coupled left decompositions

Consider the subgroups $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{C} \supset \mathfrak{D}$ of $\mathfrak{G}$. Their fields are denoted by A, B, C, D.

We first ask under what conditions the left decompositions $\mathfrak{A}/_l\mathfrak{B}$, $\mathfrak{C}/_l\mathfrak{D}$ are semi-coupled or coupled.

Since the intersection $A \cap B$ contains the unit of $\mathfrak{G}$ and therefore is not empty, it is obvious, with respect to 4.1, that the mentioned decompositions are semi-coupled if and only if

$$\mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C} = \mathfrak{C}/_l\mathfrak{D} \sqcap \mathfrak{A}.$$

In accordance with 21.2.1, this may be written

$$(\mathfrak{A} \cap \mathfrak{C})/_l\,(\mathfrak{C} \cap \mathfrak{B}) = (\mathfrak{A} \cap \mathfrak{C})/_l\,(\mathfrak{A} \cap \mathfrak{D}).$$

This equality is evidently true if and only if

$$\mathfrak{A} \cap \mathfrak{D} = \mathfrak{C} \cap \mathfrak{B}. \tag{1}$$

Thus we have verified that *the left decompositions* $\mathfrak{A}/_l\mathfrak{B}$, $\mathfrak{C}/_l\mathfrak{D}$ *are semi-coupled if and only if the subgroups* $\mathfrak{A} \cap \mathfrak{D}$, $\mathfrak{C} \cap \mathfrak{B}$ *coincide, i.e., if* $\mathfrak{A} \cap \mathfrak{D} = \mathfrak{C} \cap \mathfrak{B}$.

Now suppose the left decompositions $\mathfrak{A}/_l\mathfrak{B}$, $\mathfrak{C}/_l\mathfrak{D}$ are coupled. Then (by 4.1; 20.3.2) we have, besides (1), even:

$$A = (A \cap C)B, \quad C = (C \cap A)D,$$

from which it follows (19.7.8) that $\mathfrak{A} \cap \mathfrak{C}$ is interchangeable with both $\mathfrak{B}$ and $\mathfrak{D}$ and so:

$$\mathfrak{A} = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}, \quad \mathfrak{C} = (\mathfrak{C} \cap \mathfrak{A})\,\mathfrak{D}. \tag{2}$$

Conversely, if (1) and (2) simultaneously apply, then with respect to 4.1 and 21.2.1, the left decompositions $\mathfrak{A}/_l\mathfrak{B}$, $\mathfrak{C}/_l\mathfrak{D}$ are coupled.

Consequently, *the left decompositions* $\mathfrak{A}/_l\mathfrak{B}$, $\mathfrak{C}/_l\mathfrak{D}$ *are coupled if and only if there simultaneously holds*:

$$\mathfrak{A} \cap \mathfrak{D} = \mathfrak{C} \cap \mathfrak{B},$$
$$\mathfrak{A} = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}, \quad \mathfrak{C} = (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}.$$

23.2. The general five-group theorem

Consider arbitrary subgroups $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{C} \supset \mathfrak{D}$ of $\mathfrak{G}$.

Suppose the subgroups $\mathfrak{A} \cap \mathfrak{D}$, $\mathfrak{C} \cap \mathfrak{B}$ are interchangeable. Moreover, let $\mathfrak{U}$ be a subgroup of $\mathfrak{G}$ such that

$$\mathfrak{A} \cap \mathfrak{C} \supset \mathfrak{U} \supset (\mathfrak{A} \cap \mathfrak{D})(\mathfrak{C} \cap \mathfrak{B})$$

and let $\mathfrak{A} \cap \mathfrak{C}$ and $\mathfrak{U}$ be interchangeable with both $\mathfrak{B}$ and $\mathfrak{D}$.

Then there holds *the general five-group theorem*:

The left decompositions $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/_l\mathfrak{U}\mathfrak{B}$, $(\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/_l\mathfrak{U}\mathfrak{D}$ *are coupled and therefore equivalent, whence*

$$(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/_l\mathfrak{U}\mathfrak{B} \simeq (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/_l\mathfrak{U}\mathfrak{D}.$$

Moreover, there holds:

$$(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B} \cap \mathfrak{U}\mathfrak{D} = \mathfrak{U} = (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D} \cap \mathfrak{U}\mathfrak{B}. \tag{1}$$

Proof. Denote $\mathfrak{A}' = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}$, $\mathfrak{C}' = (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}$ and, furthermore, $\bar{A} = \mathfrak{A}'/_l\mathfrak{B}$, $\bar{C} = \mathfrak{C}'/_l\mathfrak{D}$. Then we have $\mathfrak{A}' \supset \mathfrak{B}$, $\mathfrak{C}' \supset \mathfrak{D}$ and, moreover (20.3.2): $\bar{A} = \bar{C} \sqsubset \bar{A}$, $\bar{C} = \bar{A} \sqsubset \bar{C}$.

Consider the decompositions:

$$\bar{A} \sqcap \mathfrak{C}' = \mathfrak{A}'/_l\mathfrak{B} \sqcap \mathfrak{C}' = (\mathfrak{A}' \cap \mathfrak{C}')/_l(\mathfrak{C}' \cap \mathfrak{B}) = (\mathfrak{A} \cap \mathfrak{C})/_l(\mathfrak{C} \cap \mathfrak{B}),$$
$$\bar{C} \sqcap \mathfrak{A}' = \mathfrak{C}'/_l\mathfrak{D} \sqcap \mathfrak{A}' = (\mathfrak{C}' \cap \mathfrak{A}')/_l(\mathfrak{A}' \cap \mathfrak{D}) = (\mathfrak{C} \cap \mathfrak{A})/_l(\mathfrak{A} \cap \mathfrak{D})$$

and apply the construction described in 4.1 and leading to the coupled coverings $\mathring{A}$, $\mathring{C}$ of the decompositions $\bar{A}$, $\bar{C}$.

The least common covering of $\bar{A} \sqcap \mathfrak{C}'$, $\bar{C} \sqcap \mathfrak{A}'$ is the left decomposition $(\mathfrak{A} \cap \mathfrak{C})/_l(\mathfrak{A} \cap \mathfrak{D})(\mathfrak{C} \cap \mathfrak{B})$ (21.5). The decomposition $\bar{B} = (\mathfrak{A} \cap \mathfrak{C})/_l\mathfrak{U}$ is, with respect to $\mathfrak{A} \cap \mathfrak{C} \supset \mathfrak{U} \supset (\mathfrak{A} \cap \mathfrak{D})(\mathfrak{C} \cap \mathfrak{B})$, a covering of the least common covering of the decompositions $\bar{A} \sqcap \mathfrak{C}'$, $\bar{C} \sqcap \mathfrak{A}'$ (21.3) and therefore a common covering of the latter. In accordance with the mentioned construction, we now define the decomposition $\bar{\bar{A}}$ ($\bar{\bar{C}}$) on $\bar{A}$ ($\bar{C}$) as follows: Each element of $\bar{\bar{A}}$ ($\bar{\bar{C}}$) is the set of all elements of $\bar{A}$ (C) that are incident with the same element of $\bar{B}$. Then the mentioned coupled coverings $\mathring{A}$, $\mathring{C}$ are the coverings of $\bar{A}$, $\bar{C}$, enforced by $\bar{\bar{A}}$, $\bar{\bar{C}}$.

Now let $\bar{\bar{a}} \in \bar{\bar{A}}$ be an arbitrary element. $\bar{\bar{a}}$ is the set of all elements $\bar{a} \in \bar{A}$ incident with an element $\bar{b} \in \bar{B}$. Simultaneously we have $\bar{b} = x\mathfrak{U}$ where $x \in \mathfrak{A} \cap \mathfrak{C}$ is a point of $\mathfrak{A} \cap \mathfrak{C}$. Obviously, there holds $\bar{\bar{a}} = x\mathfrak{U} \sqsubset \bar{A}$ and, moreover (with regard to 20.3.2),

$$\mathring{a} = \mathbf{s}\bar{\bar{a}} = x\mathfrak{U}\mathfrak{B} \in \mathring{A}.$$

Thus we have verified that the elements of the decomposition $\mathring{A}$ are the left cosets, generated by $\mathfrak{U}\mathfrak{B}$, of the points lying in $\mathfrak{A} \cap \mathfrak{C}$. The sum of these cosets is evidently $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{U}\mathfrak{B} = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}$. Hence:

$$\mathring{A} = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/_l\mathfrak{U}\mathfrak{B}.$$

Analogously, we obtain $\mathring{C} = (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/_l\mathfrak{U}\mathfrak{D}$. It follows that the left decompositions $(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/_l\mathfrak{U}\mathfrak{B}$, $(\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/_l\mathfrak{U}\mathfrak{D}$ are coupled. In accordance with the second equivalence theorem (6.8), they are also equivalent.

Moreover, (by 4.1) we have: $\mathring{A} \sqcap \mathring{C} = \bar{B}$ and therefore (by 2.3):

$$(\mathring{A} \sqcap \mathbf{s}\mathring{C}) \sqcap (\mathring{C} \sqcap \mathbf{s}\mathring{A}) = \bar{B};$$

furthermore, (by 4.1):

$$\mathring{A} \sqcap \mathbf{s}\mathring{C} = \mathring{C} \sqcap \mathbf{s}\mathring{A}.$$

Thus we have arrived at the formulae $\mathring{A} \sqcap \mathbf{s}\mathring{C} = \mathring{C} \sqcap \mathbf{s}\mathring{A} = \bar{B}$ or

$$\begin{aligned} &\big((\mathfrak{A} \cap \mathfrak{C})\mathfrak{B} \cap (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}\big)/_l\big((\mathfrak{C} \cap \mathfrak{A})\mathfrak{D} \cap \mathfrak{U}\mathfrak{B}\big) \\ &= \big((\mathfrak{C} \cap \mathfrak{A})\mathfrak{D} \cap (\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}\big)/_l\big((\mathfrak{A} \cap \mathfrak{C})\mathfrak{B} \cap \mathfrak{U}\mathfrak{D}\big) = (\mathfrak{A} \cap \mathfrak{C})/_l\mathfrak{U} \end{aligned}$$

from which (1) immediately follows.

Remark. Under the same assumption there, naturally, holds an analogous statement about the right decompositions and so, in particular,

$$(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/_r\mathfrak{U}\mathfrak{B} \simeq (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/_r\mathfrak{U}\mathfrak{D}.$$

Especially for $\mathfrak{U} = (\mathfrak{A} \cap \mathfrak{D})(\mathfrak{C} \cap \mathfrak{B})$ we have *the general four-group theorem*:

Let $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{C} \supset \mathfrak{D}$ be arbitrary subgroups of $\mathfrak{G}$. Suppose that the subgroup $\mathfrak{A} \cap \mathfrak{D}$ is interchangeable with $\mathfrak{C} \cap \mathfrak{B}$, the subgroups $\mathfrak{A} \cap \mathfrak{C}$, $\mathfrak{A} \cap \mathfrak{D}$ are interchangeable with $\mathfrak{B}$ and $\mathfrak{C} \cap \mathfrak{A}$, $\mathfrak{C} \cap \mathfrak{B}$ with $\mathfrak{D}$. Then the left decompositions

$$(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/_l(\mathfrak{A} \cap \mathfrak{D})\mathfrak{B}, \quad (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/_l(\mathfrak{C} \cap \mathfrak{B})\mathfrak{D}$$

are coupled and therefore equivalent so that

$$(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/_l(\mathfrak{A} \cap \mathfrak{D})\mathfrak{B} \simeq (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/_l(\mathfrak{C} \cap \mathfrak{B})\mathfrak{D}.$$

Moreover, there holds:

$$(\mathfrak{C} \cap \mathfrak{A})\mathfrak{B} \cap (\mathfrak{C} \cap \mathfrak{B})\mathfrak{D} = (\mathfrak{A} \cap \mathfrak{D})(\mathfrak{C} \cap \mathfrak{B}) \\ = (\mathfrak{A} \cap \mathfrak{C})\mathfrak{D} \cap (\mathfrak{A} \cap \mathfrak{D})\mathfrak{B}.$$

An analogous statement applies to the right decompositions and so we have, in particular,

$$(\mathfrak{A} \cap \mathfrak{C})\mathfrak{B}/_r(\mathfrak{A} \cap \mathfrak{D})\mathfrak{B} \simeq (\mathfrak{C} \cap \mathfrak{A})\mathfrak{D}/_r(\mathfrak{C} \cap \mathfrak{B})\mathfrak{D}.$$

23.3. Adjoint left decompositions

Let again $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{C} \supset \mathfrak{D}$ be arbitrary subgroups of $\mathfrak{G}$ and $A \supset B$, $C \supset D$ their fields.

Our object now is to find out the circumstances under which the left decompositions $\mathfrak{A}/_l\mathfrak{B}$, $\mathfrak{C}/_l\mathfrak{D}$ are adjoint with respect to B, D.

The question is answered by the following theorem:

The left decompositions $\mathfrak{A}/_l\mathfrak{B}$, $\mathfrak{C}/_l\mathfrak{D}$ are adjoint with respect to B, D if and only if the subgroups $\mathfrak{A} \cap \mathfrak{D}$, $\mathfrak{C} \cap \mathfrak{B}$ are interchangeable. Then there holds:

$$(\mathfrak{A} \cap \mathfrak{D})\mathfrak{B} \cap \mathfrak{C} = (\mathfrak{C} \cap \mathfrak{B})\mathfrak{D} \cap \mathfrak{A} = (\mathfrak{A} \cap \mathfrak{D})(\mathfrak{C} \cap \mathfrak{B}). \tag{1}$$

Proof. By 2.6.5 we have

$$D \sqsubset \mathfrak{A}/_l\mathfrak{B} \sqcap C = (D \sqsubset \mathfrak{A}/_l\mathfrak{B}) \sqcap C = D \sqsubset (\mathfrak{A}/_l\mathfrak{B} \sqcap \mathfrak{C}),$$
$$B \sqsubset \mathfrak{C}/_l\mathfrak{D} \sqcap A = (B \sqsubset \mathfrak{C}/_l\mathfrak{D}) \sqcap A = B \sqsubset (\mathfrak{C}/_l\mathfrak{D} \sqcap \mathfrak{A}).$$

Consequently, with regard to 21.2.1, there holds:

$$\mathbf{s}(D \sqsubset \mathfrak{A}/_l\mathfrak{B} \sqcap C) = \mathbf{s}(D \sqsubset \mathfrak{A}/_l\mathfrak{B}) \cap C = \mathbf{s}\big(D \sqsubset (\mathfrak{A} \cap \mathfrak{C})/_l(\mathfrak{C} \cap \mathfrak{B})\big),$$
$$\mathbf{s}(B \sqsubset \mathfrak{C}/_l\mathfrak{D} \sqcap A) = \mathbf{s}(B \sqsubset \mathfrak{C}/_l\mathfrak{D}) \cap A = \mathbf{s}\big(B \sqsubset (\mathfrak{C} \cap \mathfrak{A})/_l(\mathfrak{A} \cap \mathfrak{D})\big)$$

which (by 20.3.2) may be expressed in the form:

$$\begin{aligned} \mathbf{s}(D \sqsubset \mathfrak{A}/_l\mathfrak{B} \sqcap C) &= (A \cap D)B \cap C = (A \cap D)(C \cap B), \\ \mathbf{s}(B \sqsubset \mathfrak{C}/_l\mathfrak{D} \sqcap A) &= (C \cap B)D \cap A = (C \cap B)(A \cap D). \end{aligned} \tag{2}$$

a) Suppose $\mathfrak{A}/_l\mathfrak{B}$, $\mathfrak{C}/_l\mathfrak{D}$ are adjoint with respect to B, D. Then we have, by (2),

$$(A \cap D)(C \cap B) = (C \cap B)(A \cap D).$$

We see that $\mathfrak{A} \cap \mathfrak{D}$, $\mathfrak{C} \cap \mathfrak{B}$ are interchangeable. Consequently, the product $(\mathfrak{A} \cap \mathfrak{D})(\mathfrak{C} \cap \mathfrak{B})$ is a subgroup of $\mathfrak{G}$. Moreover, from (2) we conclude that the field of this subgroup coincides with either of the sets $(A \cap D)B \cap C$, $(C \cap B)D \cap A$, a fact expressed by the formulae (1). Note that neither $\mathfrak{A} \cap \mathfrak{D}$, $\mathfrak{B}$ nor $\mathfrak{C} \cap \mathfrak{B}$, $\mathfrak{D}$ are necessarily interchangeable.

b) Suppose the subgroups $\mathfrak{A} \cap \mathfrak{D}$, $\mathfrak{C} \cap \mathfrak{B}$ are interchangeable. Then, by (2), there holds $\boldsymbol{s}(D \sqsubset \mathfrak{A}/_l\mathfrak{B} \sqcap C) = \boldsymbol{s}(B \sqsubset \mathfrak{C}/_l\mathfrak{D} \sqcap A)$ and we observe that the decompositions $\mathfrak{A}/_l\mathfrak{B}$, $\mathfrak{C}/_l\mathfrak{D}$ are adjoint with respect to B, D. This accomplishes the proof.

Analogously, for the right decompositions there holds:

The right decompositions $\mathfrak{A}/_r\mathfrak{B}$, $\mathfrak{C}/_r\mathfrak{D}$ *are adjoint if and only if the subgroups* $\mathfrak{A} \cap \mathfrak{D}$, $\mathfrak{C} \cap \mathfrak{D}$ *are interchangeable. In that case:*

$$\mathfrak{B}(\mathfrak{A} \cap \mathfrak{D}) \cap \mathfrak{C} = \mathfrak{D}(\mathfrak{C} \cap \mathfrak{B}) \cap \mathfrak{A} = (\mathfrak{A} \cap \mathfrak{D})\,(\mathfrak{C} \cap \mathfrak{B}).$$

23.4. Series of subgroups

In this chapter we shall describe the properties of the series of subgroups on the basis of our theory of series of decompositions, developed in Chapter 10. This new theory will prove extremely useful in connection with invariant subgroups (24.6) considered in the classical theory of groups.

1. *Basic notions.* Let $\mathfrak{A} \supset \mathfrak{B}$ denote arbitrary subgroups of $\mathfrak{G}$. By a *series of subgroups of the group* $\mathfrak{G}$, *from* $\mathfrak{A}$ *to* $\mathfrak{B}$, briefly, a *series from* $\mathfrak{A}$ *to* $\mathfrak{B}$, we mean a finite $\alpha(\geqq 1)$-membered sequence of subgroups $\mathfrak{A}_1, \ldots, \mathfrak{A}_\alpha$ of $\mathfrak{G}$ such that: a) The first and the last member of the sequence is $\mathfrak{A}$ and $\mathfrak{B}$, respectively, i.e., $\mathfrak{A}_1 = \mathfrak{A}$, $\mathfrak{A}_\alpha = \mathfrak{B}$; b) each subsequent member is a subgroup of the subgroup immediately preceding it, thus:

$$(\mathfrak{A} =)\ \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \cdots \supset \mathfrak{A}_\alpha\ (= \mathfrak{B}).$$

Such a series is briefly denoted by $(\mathfrak{A})$. The subgroups $\mathfrak{A}_1, \ldots, \mathfrak{A}_\alpha$ are the *members of* $(\mathfrak{A})$. $\mathfrak{A}_1$ is the *initial* and $\mathfrak{A}_\alpha$ the *final member of* $(\mathfrak{A})$. By the *length of* $(\mathfrak{A})$ we mean the number of its members.

Each subgroup $\mathfrak{A}$ of $\mathfrak{G}$, for example, is a series of length 1 whose initial as well as final member coincides with $\mathfrak{A}$.

Now let $\big((\mathfrak{A}) =\big)\ \mathfrak{A}_1 \supset \cdots \supset \mathfrak{A}_\alpha$ be a series from $\mathfrak{A}$ to $\mathfrak{B}$.

A member of $(\mathfrak{A})$ is called *essential* if it is either the initial member $\mathfrak{A}_1$ or a proper subgroup (19.4.1) of the member immediately preceding it; otherwise it is said to be *inessential.* If there occurs in $(\mathfrak{A})$ at least one inessential member $\mathfrak{A}_{\gamma+1}$, then we say (since $\mathfrak{A}_{\gamma+1} = \mathfrak{A}_\gamma$) that $(\mathfrak{A})$ is a *series with iteration.* If all the members of $(\mathfrak{A})$ are essential, then $(\mathfrak{A})$ is a *series without iteration.* The number α' of the essential members of $(\mathfrak{A})$ is the *reduced length of* $(\mathfrak{A})$. Obviously, there holds:

$1 \leqq \alpha' \leqq \alpha$, where $\alpha' = \alpha$, characterizes the series without iteration. Analogously as in case of series of decompositions (10.1), $(\mathfrak{A})$ may be reduced by omitting all the inessential members (if there are any) or lengthened by inserting further subgroups. The notion of a *partial series* or a *part of* $(\mathfrak{A})$ does not need any further explanation.

By a *refinement of* $(\mathfrak{A})$ we mean a series of subgroups of $\mathfrak{G}$ containing $(\mathfrak{A})$ as its own part. Every refinement of $(\mathfrak{A})$ has therefore the following form:

$$\begin{gathered}\mathfrak{A}_{1,1} \supset \cdots \supset \mathfrak{A}_{1,\beta_1-1} \supset \mathfrak{A}_{1,\beta_1} \supset \mathfrak{A}_{2,1} \supset \cdots \\ \cdots \supset \mathfrak{A}_{2,\beta_2-1} \supset \mathfrak{A}_{2,\beta_2} \supset \cdots \supset \mathfrak{A}_{\alpha,\beta_\alpha} \supset \mathfrak{A}_{\alpha+1,1} \supset \cdots \supset \mathfrak{A}_{\alpha+1,\beta_{\alpha+1}-1},\end{gathered}$$

where $\mathfrak{A}_{\gamma,\beta_\gamma} = \mathfrak{A}_\gamma$, $\gamma = 1, \ldots, \alpha$, and the symbols $\beta_1, \ldots, \beta_{\alpha+1}$ stand for positive integers; if $\beta_\delta = 1$, then the members $\mathfrak{A}_{\delta,1} \supset \cdots \supset \mathfrak{A}_{\delta,\beta_\delta-1}$ are not read.

2. *Associated series of left and right decompositions.* Let

$$\big((\mathfrak{A}) =\big)\ \mathfrak{A}_1 \supset \cdots \supset \mathfrak{A}_\alpha$$

be a series of subgroups of $\mathfrak{G}$.

Associate, with $(\mathfrak{A})$, the following series of left and right decompositions:

$$\begin{gathered}\big((\mathfrak{G}/_l\mathfrak{A}) =\big)\ \mathfrak{G}/_l\mathfrak{A}_1 \geqq \cdots \geqq \mathfrak{G}/_l\mathfrak{A}_\alpha, \\ \big((\mathfrak{G}/_r\mathfrak{A}) =\big)\ \mathfrak{G}/_r\mathfrak{A}_1 \geqq \cdots \geqq \mathfrak{G}/_r\mathfrak{A}_\alpha.\end{gathered}$$

Then we speak about *series of left or right decompositions associated with or corresponding to* $(\mathfrak{A})$. It is obvious that the series $(\mathfrak{G}/_l\mathfrak{A})$ or $(\mathfrak{G}/_r\mathfrak{A})$ is obtained by replacing each member $\mathfrak{A}_\gamma$ $(\gamma = 1, \ldots, \alpha)$ of $(\mathfrak{A})$ by $\mathfrak{G}/_l\mathfrak{A}_\gamma$ or $\mathfrak{G}/_r\mathfrak{A}_\gamma$, respectively.

Consider, for example, the series of left decompositions $(\mathfrak{G}/_l\mathfrak{A})$. In the same way we could, of course, consider the series of right decompositions $(\mathfrak{G}/_r\mathfrak{A})$.

First, the statements set out below are evidently correct:

The series $(\mathfrak{A})$ *and* $(\mathfrak{G}/_l\mathfrak{A})$ *have the same length* α.

The series $(\mathfrak{A})$ *and* $(\mathfrak{G}/_l\mathfrak{A})$ *are simultaneously without or with iteration and have the same reduced length* α' $(\leqq \alpha)$.

The series of left decompositions $(\mathfrak{G}/_l\mathring{\mathfrak{A}})$ *associated with an arbitrary refinement* $(\mathring{\mathfrak{A}})$ *of* $(\mathfrak{A})$ *is a refinement of* $(\mathfrak{G}/_l\mathfrak{A})$.

By means of the notion of associated series of left and right decompositions we may study the properties of the series of subgroups on the basis of the theory of the series of decompositions. All we have to do is to apply the considerations relative to the series of decompositions to the series of subgroups. But we must make sure to apply only those properties as are common to both the left and the right decompositions. The importance of this remark will be realized later.

3. *The manifold of local chains.* Consider an arbitrary series $(\mathfrak{A})$ of subgroups of $\mathfrak{G}$:

$$\big((\mathfrak{A}) =\big)\ \mathfrak{A}_1 \supset \cdots \supset \mathfrak{A}_\alpha \qquad (\alpha \geqq 1)$$

and, furthermore, the corresponding series of left and right decompositions on $\mathfrak{G}$:

$$\big((\mathfrak{G}/_l\mathfrak{A}) =\big)\ \mathfrak{G}/_l\mathfrak{A}_1 \geqq \cdots \geqq \mathfrak{G}/_l\mathfrak{A}_\alpha,$$
$$\big((\mathfrak{G}/_r\mathfrak{A}) =\big)\ \mathfrak{G}/_r\mathfrak{A}_1 \geqq \cdots \geqq \mathfrak{G}/_r\mathfrak{A}_\alpha.$$

We know that to each element $\bar{a}$ of $(\mathfrak{G}/_l\mathfrak{A}_\alpha)$ or $(\mathfrak{G}/_r\mathfrak{A}_\alpha)$, respectively, there corresponds a local chain of the series $(\mathfrak{G}/_l\mathfrak{A})$ or $(\mathfrak{G}/_r\mathfrak{A})$ with the base $\bar{a}$. The set of the local chains belonging to the individual elements of $\mathfrak{G}/_l\mathfrak{A}_\alpha$ or $\mathfrak{G}/_r\mathfrak{A}_\alpha$, respectively, is the *left or the right manifold of local chains corresponding to* $(\mathfrak{A})$. Notation: $\tilde{\boldsymbol{A}}_l$, $\tilde{\boldsymbol{A}}_r$.

Our object now is to study the relationship between $\tilde{\boldsymbol{A}}_l$ and $\tilde{\boldsymbol{A}}_r$.

First, let us remark that to every left or right coset $\bar{a}$ with regard to a subgroup of $\mathfrak{G}$ there exists an inverse right or left coset $\bar{a}^{-1}$, respectively; $\bar{a}^{-1}$ consists of all the points inverse of the individual points lying in $\bar{a}$ (20.2.8).

Now consider two mutually inverse cosets $\bar{a} \in \mathfrak{G}/_l\mathfrak{A}_\alpha$, $\bar{a}^{-1} \in \mathfrak{G}/_r\mathfrak{A}_\alpha$ and the corresponding local chains of $(\mathfrak{G}/_l\mathfrak{A})$, $(\mathfrak{G}/_r\mathfrak{A})$ with the bases $\bar{a}$, $\bar{a}^{-1}$:

$$([\bar{K}\bar{a}] =)\ \bar{K}_1\bar{a} \to \cdots \to \bar{K}_\alpha\bar{a},$$
$$([\bar{K}\bar{a}^{-1}] =)\ \bar{K}_1\bar{a}^{-1} \to \cdots \to \bar{K}_\alpha\bar{a}^{-1}.$$

In the above formulae we have denoted the local chains $[\bar{K}\bar{a}]$, $[\bar{K}\bar{a}^{-1}]$ and their members $\bar{K}_\gamma\bar{a}$, $\bar{K}_\gamma\bar{a}^{-1}$ by the same symbol $\bar{K}$ although the local chains or the members of the series $(\mathfrak{G}/_l\mathfrak{A})$, $(\mathfrak{G}/_r\mathfrak{A})$ in question are generally different from one another. This simplification cannot cause any confusion, since the notation of the local chains and their members differs in the symbols of the bases $\bar{a}$, $\bar{a}^{-1}$. A similar simplification will be employed even in the further considerations.

Let $\bar{a}_\gamma$ be an element of the decomposition $\mathfrak{G}/_l\mathfrak{A}_\gamma$ whose subset is $\bar{a}$ $(\gamma = 1, \ldots, \alpha)$. Then the inverse coset $\bar{a}_\gamma^{-1}$ is an element of $\mathfrak{G}/_r\mathfrak{A}_\gamma$ whose subset is $\bar{a}^{-1}$. There evidently holds:

$$\bar{a}_1 \supset \cdots \supset \bar{a}_\alpha\ (= \bar{a}), \qquad \bar{a}_1^{-1} \supset \cdots \supset \bar{a}_\alpha^{-1}\ (= \bar{a}^{-1})$$

and, furthermore,

$$\bar{a}_\gamma = \bar{a}\mathfrak{A}_\gamma, \qquad \bar{K}_\gamma\bar{a} = \bar{a}_\gamma \sqcap \mathfrak{G}/_l\mathfrak{A}_{\gamma+1},$$
$$\bar{a}_\gamma^{-1} = \mathfrak{A}_\gamma\bar{a}^{-1}, \qquad \bar{K}_\gamma\bar{a}^{-1} = \bar{a}_\gamma^{-1} \sqcap \mathfrak{G}/_r\mathfrak{A}_{\gamma+1} \qquad (\mathfrak{A}_{\alpha+1} = \mathfrak{A}_\alpha).$$

Either of the decompositions $\bar{K}_\gamma\bar{a}$, $\bar{K}_\gamma\bar{a}^{-1}$ is mapped, under the extended inversion $\boldsymbol{n}$ of $\mathfrak{G}$, onto the other (21.8.5) and so $\boldsymbol{n}\bar{K}_\gamma\bar{a} = \bar{K}_\gamma\bar{a}^{-1}$, $\boldsymbol{n}\bar{K}_\gamma\bar{a}^{-1} = \bar{K}_\gamma\bar{a}$. With regard to this, any two members $\bar{K}_\gamma \bar{a}$, $\bar{K}_\gamma \bar{a}^{-1}$ with the same index γ $(= 1, \ldots, \alpha)$ are called *mutually inverse*; the same term is employed for the local chains $[\bar{K}\bar{a}]$, $[\bar{K}\bar{a}^{-1}]$. Two mutually inverse members of $[\bar{K}\bar{a}]$ and $[\bar{K}\bar{a}^{-1}]$ are equivalent sets (21.8.5).

It is easy to verify that *the manifolds of local chains, $\tilde{\boldsymbol{A}}_l$, $\tilde{\boldsymbol{A}}_r$, are strongly equivalent.*

Indeed, associating with every local chain $[\bar{K}\bar{a}] \in \tilde{\boldsymbol{A}}_l$ its inverse: $[\bar{K}\bar{a}^{-1}] \in \tilde{\boldsymbol{A}}_r$, we obtain a simple mapping $\boldsymbol{f}$ of the manifold $\tilde{\boldsymbol{A}}_l$ onto $\tilde{\boldsymbol{A}}_r$. The mapping $\boldsymbol{f}$ is a strong equivalence because every two mutually inverse members of $[\bar{K}\bar{a}]$ and $\boldsymbol{f}[\bar{K}\bar{a}] = [\bar{K}\bar{a}^{-1}]$ are equivalent sets.

4. *Pairs of series of subgroups.* Consider a pair of series of subgroups of $\mathfrak{G}$:

$$\big((\mathfrak{A}) =\big)\ \mathfrak{A}_1 \supset \cdots \supset \mathfrak{A}_\alpha \qquad (\alpha \geqq 1),$$
$$\big((\mathfrak{B}) =\big)\ \mathfrak{B}_1 \supset \cdots \supset \mathfrak{B}_\beta \qquad (\beta \geqq 1).$$

To $(\mathfrak{A})$ and $(\mathfrak{B})$ there correspond the following series of left decompositions of $\mathfrak{G}$:

$$\begin{aligned}\big((\mathfrak{G}/_l\mathfrak{A}) =\big)\ \mathfrak{G}/_l\mathfrak{A}_1 \geqq \cdots \geqq \mathfrak{G}/_l\mathfrak{A}_\alpha,\\ \big((\mathfrak{G}/_l\mathfrak{B}) =\big)\ \mathfrak{G}/_l\mathfrak{B}_1 \geqq \cdots \geqq \mathfrak{G}/_l\mathfrak{B}_\beta\end{aligned} \tag{1}$$

and the left manifolds of local chains: $\tilde{\boldsymbol{A}}_l$, $\tilde{\boldsymbol{B}}_l$.

Analogously, to $(\mathfrak{A})$, $(\mathfrak{B})$ there belong the series of right decompositions of $\mathfrak{G}$:

$$\begin{aligned}\big((\mathfrak{G}/_r\mathfrak{A}) =\big)\ \mathfrak{G}/_r\mathfrak{A}_1 \geqq \cdots \geqq \mathfrak{G}/_r\mathfrak{A}_\alpha,\\ \big((\mathfrak{G}/_r\mathfrak{B}) =\big)\ \mathfrak{G}/_r\mathfrak{B}_1 \geqq \cdots \geqq \mathfrak{G}/_r\mathfrak{B}_\beta\end{aligned} \tag{2}$$

and the right manifolds of local chains: $\tilde{\boldsymbol{A}}_r$, $\tilde{\boldsymbol{B}}_r$.

Under these circumstances there applies the theorem:

If the series (1) *or* (2) *are in any of the following four relations, then the series* (2) *or* (1), *respectively, are in the same relation*: *The series* (1) *or* (2), *respectively, are* a) *complementary,* b) *chain-equivalent,* c) *loosely joint or co-basally loosely joint,* d) *joint or co-basally joint.*

Proof. Suppose, for example, that the series (1) are complementary.

In that case each member $\mathfrak{G}/_l\mathfrak{A}_\mu$, of $(\mathfrak{G}/_l\mathfrak{A})$ is complementary to each member $\mathfrak{G}/_l\mathfrak{B}_\nu$ of $(\mathfrak{G}/_l\mathfrak{B})$ (10.8); $\mu = 1, \ldots, \alpha$; $\nu = 1, \ldots, \beta$. Consequently, each member $\mathfrak{A}_\mu$ of $(\mathfrak{A})$ is interchangeable with each member $\mathfrak{B}_\nu$ of $(\mathfrak{B})$ (21.6). Obviously, even each member $\mathfrak{G}/_r\mathfrak{A}_\mu$ of $(\mathfrak{G}/_r\mathfrak{A})$ is complementary to each member $\mathfrak{G}/_r\mathfrak{B}_\nu$ of $(\mathfrak{G}/_r\mathfrak{B})$ (21.6) so that the series (2) are complementary.

Let us now assume that the series (1), for example, are in one of the relations b), c), d). Then the series (1), (2) and therefore even the series $(\mathfrak{A})$, $(\mathfrak{B})$ have the same length $\alpha = \beta$ and in each of the mentioned cases there exists a simple mapping $\boldsymbol{f}_l$ (strong equivalence, equivalence connected with loose coupling, equivalence connected with coupling) of the manifolds $\tilde{\boldsymbol{A}}_l$ onto $\tilde{\boldsymbol{B}}_l$ which may be co-basal. By means of $\boldsymbol{f}_l$ we define a simple mapping $\boldsymbol{f}_r$ of $\tilde{\boldsymbol{A}}_r$ onto $\tilde{\boldsymbol{B}}_r$ by way of associating, with each element $[\bar{K}\bar{a}] \in \tilde{\boldsymbol{A}}_r$, the inverse local chain $[\bar{K}\bar{a}^{-1}] \in \tilde{\boldsymbol{A}}_l$ and, with $[\bar{K}\bar{a}]$, the local chain $\boldsymbol{f}_r[\bar{K}\bar{a}] = [\bar{K}\bar{b}^{-1}] \in \tilde{\boldsymbol{B}}_r$ inverse of $\boldsymbol{f}_l[\bar{K}\bar{a}^{-1}] = [\bar{K}\bar{b}] \in \tilde{\boldsymbol{B}}_l$. If the mapping $\boldsymbol{f}_l$ is co-basal, then $\bar{b} = \bar{a}^{-1}$ and therefore $\bar{b}^{-1} = \bar{a}$; consequently even $\boldsymbol{f}_r$ is co-basal.

Now let $[\bar{K}\bar{a}]$, $\boldsymbol{f}_r[\bar{K}\bar{a}] = [\bar{K}\bar{b}^{-1}]$ be arbitrary elements of the manifolds $\tilde{\boldsymbol{A}}_r$, $\tilde{\boldsymbol{B}}_r$, representing the inverse image and the image under the mapping $\boldsymbol{f}_r$, respectively. Consider the corresponding inverse local chains $[\bar{K}\bar{a}^{-1}] \in \tilde{\boldsymbol{A}}_l$, $\boldsymbol{f}_l[\bar{K}\bar{a}^{-1}] = [\bar{K}\bar{b}] \in \tilde{\boldsymbol{B}}_l$:

$$
\begin{aligned}
([\bar{K}\bar{a}^{-1}] =)\ & \bar{K}_1\bar{a}^{-1} \to \cdots \to \bar{K}_\alpha\bar{a}^{-1}, \\
([\bar{K}\bar{b}] =)\ & \bar{K}_1\bar{b} \quad \to \cdots \to \bar{K}_\alpha\bar{b}.
\end{aligned}
$$

Since the series (1) are in one of the relations b), c), d), there exists a permutation $\boldsymbol{p}$ of the set $\{1, \ldots, \alpha\}$ such that every two members $\bar{K}_\gamma\bar{a}^{-1}$, $\bar{K}_\delta\bar{b}$ of the local chains $[\bar{K}\bar{a}^{-1}]$, $[\bar{K}\bar{b}]$ are equivalent or loosely coupled or coupled decompositions in $\mathfrak{G}$; at the same time $\delta = \boldsymbol{p}\gamma$. Let us apply the permutation $\boldsymbol{p}$ to the local chains $[\bar{K}\bar{a}] \in \tilde{A}_r$, $\boldsymbol{f}_r[\bar{K}\bar{a}] = [\bar{K}\bar{b}^{-1}] \in \tilde{\boldsymbol{B}}_r$ by associating, with each member $\bar{K}_\gamma\bar{a}$ of the first local chain, the member $\bar{K}_\delta\bar{b}^{-1}$ of the second. Every pair of such members $\bar{K}_\gamma\bar{a}$, $\bar{K}_\delta\bar{b}^{-1}$ represents decompositions in $\mathfrak{G}$ that are inverse of the equivalent or loosely coupled or coupled decompositions $\bar{K}_\gamma\bar{a}^{-1}$, $\bar{K}_\delta\bar{b}$. Hence even $\bar{K}_\gamma\bar{a}$, $\bar{K}_\delta\bar{b}^{-1}$ are equivalent or loosely coupled or coupled (7.3.4) and the proof is complete.

The symmetry we have just verified in the relations between the series of the left and the right decompositions corresponding to the series $(\mathfrak{A})$, $(\mathfrak{B})$, respectively, leads to the following definition:

The series of subgroups, $(\mathfrak{A})$ and $(\mathfrak{B})$, are called: a) *complementary* or *interchangeable*, b) *chain-equivalent* or *co-basally chain-equivalent*, c) *semi-joint* or *loosely joint*, or *co-basally semi-joint* or *co-basally loosely joint*, d) *joint* or *co-basally joint* if the series of the left decompositions of $\mathfrak{G}$, namely $(\mathfrak{G}/_l\mathfrak{A})$, $(\mathfrak{G}/_l\mathfrak{B})$, and therefore (by the above theorem) even the series of the right decompositions of $\mathfrak{G}$, namely, $(\mathfrak{G}/_r\mathfrak{A})$, $(\mathfrak{G}/_r\mathfrak{B})$ belonging to $(\mathfrak{A})$ and $(\mathfrak{B})$, have the corresponding property.

5. *Complementary series of subgroups.* Consider two complementary series of subgroups of $\mathfrak{G}$:

$$
\begin{aligned}
\big((\mathfrak{A}) =\big)\ & \mathfrak{A}_1 \supset \cdots \supset \mathfrak{A}_\alpha \qquad (\alpha \geqq 1), \\
\big((\mathfrak{B}) =\big)\ & \mathfrak{B}_1 \supset \cdots \supset \mathfrak{B}_\beta \qquad (\beta \geqq 1).
\end{aligned}
$$

To these series there belong the corresponding series of the left and the right decompositions of $\mathfrak{G}$, namely, $(\mathfrak{G}/_l\mathfrak{A})$, $(\mathfrak{G}/_l\mathfrak{B})$ and $(\mathfrak{G}/_r\mathfrak{A})$, $(\mathfrak{G}/_r\mathfrak{B})$, respectively, which are more accurately described by the formulae (1), (2).

There holds the following theorem:

The series $(\mathfrak{A})$, $(\mathfrak{B})$ *have co-basally joint refinements* $(\mathfrak{A}_*)$, $(\mathfrak{B}_*)$ *with coinciding initial and final members. The refinements are given by the construction described in part* a) *of the following proof.*

Proof. a) Under the above assumption, every two decompositions $\mathfrak{G}/_l\mathfrak{A}_\gamma$, $\mathfrak{G}/_l\mathfrak{B}_\delta$ $(\gamma = 1, \ldots, \alpha;\ \delta = 1, \ldots, \beta)$ are complementary, hence every two subgroups $\mathfrak{A}_\gamma$, $\mathfrak{B}_\delta$ are interchangeable (21.6). By 22.2.1 even the subgroups $\mathfrak{A}_\gamma$, $\mathfrak{A}_{\gamma-1} \cap \mathfrak{B}_\nu$ or $\mathfrak{B}_\delta$, $\mathfrak{B}_{\delta-1} \cap \mathfrak{A}_\mu$ $(\mathfrak{A}_0 = \mathfrak{B}_0 = \mathfrak{G};\ \mu = 1, \ldots, \alpha;\ \nu = 1, \ldots, \beta)$ are interchangeable and there holds:

$$\begin{aligned}(\mathfrak{A}_{\gamma,\nu} =)\ \mathfrak{A}_\gamma(\mathfrak{A}_{\gamma-1} \cap \mathfrak{B}_\nu) &= \mathfrak{A}_{\gamma-1} \cap \mathfrak{A}_\gamma\mathfrak{B}_\nu,\\ (\mathfrak{B}_{\delta,\mu} =)\ \mathfrak{B}_\delta(\mathfrak{B}_{\delta-1} \cap \mathfrak{A}_\mu) &= \mathfrak{B}_{\delta-1} \cap \mathfrak{B}_\delta\mathfrak{A}_\mu.\end{aligned} \tag{3}$$

Let us denote:

$$\begin{aligned}&\mathfrak{A}_1\mathfrak{B}_1 = \mathfrak{U}, \quad \mathfrak{A}_\alpha \cap \mathfrak{B}_\beta = \mathfrak{B},\\ &\mathfrak{A}_0 = \mathfrak{B}_0 = \mathfrak{G}, \quad \mathfrak{A}_{\alpha+1} = \mathfrak{B}_{\beta+1} = \mathfrak{B}.\end{aligned}$$

Then the formulae (3) are true for $\gamma, \mu = 1, \ldots, \alpha + 1$; $\delta, \nu = 1, \ldots, \beta + 1$.

From the definition of the subgroups $\mathfrak{A}_{\gamma,\nu}$, $\mathfrak{B}_{\delta,\mu}$ there follows

$$\begin{aligned}&\mathfrak{A}_{\gamma-1} \supset \mathfrak{A}_{\gamma,\nu}, \quad \mathfrak{A}_{\gamma,\beta+1} = \mathfrak{A}_\gamma,\\ &\mathfrak{B}_{\delta-1} \supset \mathfrak{B}_{\delta,\mu}, \quad \mathfrak{B}_{\delta,\alpha+1} = \mathfrak{B}_\delta,\end{aligned}$$

moreover, for $\nu \leqq \beta$, $\mu \leqq \alpha$, we have

$$\mathfrak{A}_{\gamma,\nu} \supset \mathfrak{A}_{\gamma,\nu+1}, \quad \mathfrak{B}_{\delta,\mu} \supset \mathfrak{B}_{\delta,\mu+1}.$$

Thus we arrive at the following series of subgroups from $\mathfrak{A}_{\gamma,1}$ to $\mathfrak{A}_\gamma$ and from $\mathfrak{B}_{\delta,1}$ to $\mathfrak{B}_\delta$:

$$\begin{aligned}&\mathfrak{A}_{\gamma,1} \supset \cdots \supset \mathfrak{A}_{\gamma,\beta+1},\\ &\mathfrak{B}_{\delta,1} \supset \cdots \supset \mathfrak{B}_{\delta,\alpha+1}.\end{aligned}$$

Consequently, the series of the subgroups of $\mathfrak{G}$ set below are refinements of the series $(\mathfrak{A})$, $(\mathfrak{B})$:

$$\begin{aligned}\big((\mathfrak{A}_*) =\big)\ \mathfrak{U} = \mathfrak{A}_{1,1} \supset \cdots \supset \mathfrak{A}_{1,\beta+1} \supset \mathfrak{A}_{2,1} \supset&\\ \cdots \supset \mathfrak{A}_{2,\beta+1} \supset \cdots \supset \mathfrak{A}_{\alpha+1,1} \supset \cdots \supset \mathfrak{A}_{\alpha+1,\beta+1} &= \mathfrak{B},\\ \big((\mathfrak{B}_*) =\big)\ \mathfrak{U} = \mathfrak{B}_{1,1} \supset \cdots \supset \mathfrak{B}_{1,\alpha+1} \supset \mathfrak{B}_{2,1} \supset&\\ \cdots \supset \mathfrak{B}_{2,\alpha+1} \supset \cdots \supset \mathfrak{B}_{\beta+1,1} \supset \cdots \supset \mathfrak{B}_{\beta+1,\alpha+1} &= \mathfrak{B}.\end{aligned}$$

We observe that $(\mathfrak{A}_*)$, $(\mathfrak{B}_*)$ have the same length and that their initial and final members coincide:

$$(\mathfrak{U} =)\, \mathfrak{A}_{1,1} = \mathfrak{B}_{1,1}, \quad \mathfrak{A}_{\alpha+1,\beta+1} = \mathfrak{B}_{\beta+1,\alpha+1}\ (= \mathfrak{B}).$$

The series $(\mathfrak{A}_*)$, $(\mathfrak{B}_*)$ *are the mentioned co-basally joint refinements of the series* $(\mathfrak{A})$, $(\mathfrak{B})$.

b) Let us show that the series of the left decompositions, $(\mathfrak{G}/_l\mathfrak{A}_*)$, $(\mathfrak{G}/_l\mathfrak{B}_*)$, corresponding to $(\mathfrak{A}_*)$, $(\mathfrak{B}_*)$ are co-basally joint. These series are obtained by way of replacing each member $\mathfrak{A}_{\gamma,\nu}$ of $(\mathfrak{A}_*)$ by the left decomposition $\mathfrak{G}/_l\mathfrak{A}_{\gamma,\nu}$ and each member $\mathfrak{B}_{\delta,\mu}$ of $(\mathfrak{B}_*)$ by $\mathfrak{G}/_l\mathfrak{B}_{\delta,\mu}$.

Denote:

$$\bar{A}_\gamma = \mathfrak{G}/_l\mathfrak{A}_\gamma, \quad \bar{B}_\delta = \mathfrak{G}/_l\mathfrak{B}_\delta, \quad \mathring{A}_{\gamma,\nu} = \mathfrak{G}/_l\mathfrak{A}_{\gamma,\nu}, \quad \mathring{B}_{\delta,\mu} = \mathfrak{G}/_l\mathfrak{B}_{\delta,\mu}.$$

Then, on taking account of the formulae (3) and in accordance with 21.4 and 21.5, we have

$$\mathring{A}_{\gamma,\nu} = \left[\bar{A}_\gamma, (\bar{A}_{\gamma-1}, \bar{B}_\nu)\right] = \left(\bar{A}_{\gamma-1}, [\bar{A}_\gamma, \bar{B}_\nu]\right),$$
$$\mathring{B}_{\delta,\mu} = \left[\bar{B}_\delta, (\bar{B}_{\delta-1}, \bar{A}_\mu)\right] = \left(\bar{B}_{\delta-1}, [\bar{B}_\delta, \bar{A}_\mu]\right).$$

We see that the series of decompositions, $(\mathfrak{G}/_l\mathfrak{A}_*)$, $(\mathfrak{G}/_l\mathfrak{B}_*)$, corresponding to $(\mathfrak{A}_*)$, $(\mathfrak{B}_*)$ are formed from the complementary series $(\mathfrak{G}/_l\mathfrak{A})$, $(\mathfrak{G}/_l\mathfrak{B})$ by the construction described in 10.7, part a) of the proof. Hence, by 10.8, the series $(\mathfrak{G}/_l\mathfrak{A}_*)$, $(\mathfrak{G}/_l\mathfrak{B}_*)$ are co-basally joint and the proof is complete.

23.5. Exercises

1. Apply the five-group theorem to subgroups of $\mathfrak{Z}$ (18.5.1).
2. Let $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{C} \supset \mathfrak{D}$ be subgroups of $\mathfrak{G}$ and $A \supset B$, $C \supset D$ their fields. Suppose the left decompositions $(\bar{A} =)\mathfrak{A}/_l\mathfrak{B}$, $(\bar{C} =)\mathfrak{C}/_l\mathfrak{D}$ in $\mathfrak{G}$ are adjoint with regard to B, D. Realize the construction described in 4.2 and leading to the coupled coverings $\mathring{A}$, $\mathring{C}$ of the decompositions $\bar{A}_1 = C \sqsubset \mathfrak{A}/_l\mathfrak{B}$, $\bar{C}_1 = A \sqsubset \mathfrak{C}/_l\mathfrak{D}$.