Linear Differential Transformations of the Second Order

3 Conjugate numbers

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3 Conjugate numbers

Conjugate numbers, the concept and properties of which will be described in this paragraph, play a very important role in the theory of linear differential equations of the second order and particularly in the transformation theory of such differential equations. We will consider a differential equation (q).

3.1 The concept of conjugate numbers

Let $t \in j$ be arbitrary, and let u, v be arbitrary integrals of the differential equation (q), such that $u(t) = 0$, $v'(t) = 0$.

We describe a number $x \in j$ $(x \neq t)$ as being *conjugate* with the number t (with respect to the differential equation (q)), and more precisely of the

1st kind, 2nd kind, 3rd kind, 4th kind,

according as

$$u(x) = 0, \quad v'(x) = 0, \quad u'(x) = 0, \quad v(x) = 0.$$

If the number x is conjugate with t of the κ-th kind, then we call it κ-conjugate with t; $\kappa = 1, 2, 3, 4$. A number x which is κ-conjugate with t is called *left* or *right conjugate* (of the κ-th kind), according as $x < t$ or $x > t$. We see that the number x is a conjugate number with t of the 1st, 2nd, 3rd or 4th kind, according as it is a zero of the function u, v', u', v respectively. Let the number x be the n-th zero $(n \geqslant 1)$ of this function lying on the left or right of t; then we call it the n-th *left* or *right conjugate number with* t, of the corresponding kind.

Let x be the n-th left or right conjugate number with t of the 1st or 2nd kind, then t is the n-th right or left conjugate number with x of the 1st or 2nd kind. Because of this symmetry we can refer simply to 1st or 2nd kind conjugate numbers, and particularly in the case $n = 1$ to *neighbouring* 1st or 2nd conjugate numbers t, x. On the assumption that between two neighbouring zeros of an integral of (q), or of its derivative, there lies precisely one zero of this derivative or of the integral respectively, we can make the further statement: Let x be the n-th left or right conjugate number with t of the 3rd or 4th kind; then t is respectively the n-th right or left conjugate number with x of the 4th or 3rd kind.

We know that two integrals of the differential equation (q) which have a common zero, or whose derivatives have a common zero, are merely constant multiples of each other and it follows that all their zeros and all the zeros of their derivatives coincide. Hence the concept of conjugate numbers is inherent in the differential equation (q) itself and does not depend on the particular choice of the integrals u, v used in their definition.

3.2 Classification of differential equations (q) with respect to conjugate numbers

By consideration of conjugate numbers of the κ-th kind ($\kappa = 1, 2, 3, 4$) we can separate differential equations of the type (q) into two classes according as there exist, in the interval j, conjugate numbers of the κ-th kind or not. For example, taking the differential equation (q) with the carrier $q = -1$, consider an open interval j of length $|j|$. In the case $0 < |j| \leqslant \frac{1}{2}\pi$ there are no conjugate numbers, in the case $\frac{1}{2}\pi < |j| \leqslant \pi$ there are conjugate numbers only of the 3rd and 4th kinds, while for $\pi < |j|$ there are conjugate numbers of all kinds.

When we have differential equations of the type (q) for which κ-conjugate numbers exist, we call them *differential equations with conjugate numbers (points) of the κ-th kind*, or *differential equations with κ-conjugate numbers (points)*. We also express this fact by saying that the differential equation (q) admits of or possesses conjugate numbers (points) of the κ-th kind, or κ-conjugate numbers.

Differential equations of the type (q) for which no κ-conjugate numbers exist we call *differential equations without conjugate numbers (points) of the κ-th kind* or *differential equations without κ-conjugate numbers (points)*. In this case we also say that the differential equation (q) admits of or possesses no conjugate numbers (points) of the κ-th kind or no κ-conjugate numbers (points).

3.3 Properties of differential equations (q) with κ-conjugate numbers

In this and in §§ 3.4 to 3.9 we shall concern ourselves with differential equations (q_κ) with conjugate numbers of the κ-th kind ($\kappa = 1, 2, 3, 4$). In the cases $\kappa = 2, 3, 4$ it is convenient to suppose that $q_\kappa(t) < 0$ for $t \in j$, in order to simplify our study; in particular, this assumption makes it possible to apply the ordering theorems (§ 2.3) in the interval j.

Let (q_κ) be a differential equation with conjugate numbers of the κ-th kind. First we have the following theorems:

Given any number $x_1 \in j$, which possesses left conjugate numbers of the κ-th kind, there exist smaller numbers which also possess left conjugate numbers of the κ-th kind. All numbers greater than x_1 lying in the interval j possess left conjugate numbers of the κ-th kind. Given any number $x_1 \in j$ which possesses right conjugate numbers of the κ-th kind, there exist greater numbers which also possess right conjugate numbers of the κ-th kind. All numbers smaller than x_1 lying in the interval j possess right conjugate numbers of the κ-th kind.

Proof. We shall only give the proof of the first theorem. Let t_1 be the first left conjugate number with x_1 of the κ-th kind. We choose an arbitrary number $t_2 < t_1$, $t_2 \in j$. Then by the ordering theorems there is a number x_2 which is right conjugate with t_2 of the kind κ' ($\kappa' = \kappa$ or $\kappa' = \kappa \pm 1$) and $x_2 < x_1$. This number x_2 then possess the left

κ-conjugate number t_2, which proves the first part of our result. The second part follows immediately from the ordering theorems.

3.4 Fundamental numbers

We denote by R_κ, S_κ respectively the sets of numbers lying in the interval j which possess left (or right) conjugate numbers of the κ-th kind. Since, on our hypothesis, κ-conjugate numbers exist, in the cases $\kappa = 1, 2$ both sets R_κ, S_κ are non-empty, while in the cases $\kappa = 3, 4$ at least one of these is non-empty.

Let $R_\kappa \neq \varnothing$. If the set R_κ is bounded below, then it possesses a greater lower bound r_κ, which either belongs to the interval j, that is $a < r_\kappa$, or coincides with its end point a: $a = r_\kappa$. If the set R_κ is unbounded below, which is obviously only possible in the case $a = -\infty$, then we define the number r_κ as: $r_\kappa = a = -\infty$. The number r_κ we call the *left fundamental number of the κ-th kind* or the *left κ-fundamental number of the κ-th kind* or *the left κ-fundamental number* of the differential equation (q_κ). If $a < r_\kappa$, we call r_κ *proper*, and if $a = r_\kappa$ we call r_κ *improper*.

From the theorems in Section 3 we conclude that if the left κ-fundamental number r_κ of the differential equation (q_κ) is proper, then it is the largest number $\in j$ for which there are no left κ-conjugate numbers. In this case the interval j separates into two sub-intervals $(a, r_\kappa]$, (r_κ, b), of which the first is composed of numbers which do not possess any left κ-conjugate numbers, while there are left κ-conjugate numbers for every member of the second sub-interval

Similarly, if $S_\kappa \neq \varnothing$ and S_κ is bounded above, it has a least upper bound s_κ, with $s_\kappa < b$ or $s_\kappa = b$. If S_κ is unbounded above, we define $s_\kappa = b = \infty$. This number s_κ is called the *right fundamental number of the κ-th kind* or the *right κ-fundamental number* of the differential equation (q_κ), and is *proper* if $s_\kappa < b$, *improper* if $s_\kappa = b$. If s_κ is proper the interval j is composed of the two sub-intervals (a, s_κ), $[s_\kappa, b)$; right κ-conjugate numbers are possessed by every number in (a, s_κ) and by none in $[s_\kappa, b)$.

3.5 Fundamental integrals and fundamental sequences

In this and the following paragraphs we are concerned principally with 1-conjugate and 2-conjugate numbers, and it is convenient to use λ in place of κ, allowing λ to take the values 1, 2 only. We consider a differential equation (q_λ) whose left λ-fundamental number r_λ is proper; $a < r_\lambda$.

Let u_λ be an integral of (q_λ) which vanishes or has its derivative vanishing at the point r_λ, according as $\lambda = 1$ or $\lambda = 2$, that is $u_1(r_1) = 0$, $u_2'(r_2) = 0$. Such an integral u_λ we shall call a *left fundamental integral* of the λ-th kind or a *left λ-fundamental integral*, of (q_λ). Obviously any integral of (q_λ) which is dependent on the integral u_λ is also a left λ-fundamental integral of (q_λ).

Now let $\nu = 1, 2, \ldots$. In the case $\lambda = 1$ let $r_{1,\nu}$ be the ν-th zero of the fundamental integral u_1 lying on the right of r_1. Moreover, if $q_1(t) < 0 \;\forall\; t \in j$, let $r_{4,\nu}$ be the ν-th zero of the derivative u_1' of u_1 on the right of r_4 (assuming, of course, that this zero exists.) The fundamental number r_4 definitely exists (§ 3.7). We set $r_1 = r_{1,0}$, $r_4 = r_{4,0}$.

In the case $\lambda = 2$ let $r_{2,\nu}$ be the ν-th zero of the derivative u_2' of u_2 on the right of r_2. Moreover, let $r_{3,\nu}$ be the ν-th zero of the fundamental integral u_2 on the right of r_3, (again assuming that these zeros exist.) The fundamental number r_3 does exist (§ 3.7). We set $r_2 = r_{2,0}$, $r_3 = r_{3,0}$.

The sequence (finite or infinite) of numbers $\in j$

$$r_{\kappa,0} < r_{\kappa,1} < r_{\kappa,2} < \cdots$$

is known as the *left fundamental sequence of the κ-th kind* or the *left κ-fundamental sequence* of the differential equation; $\kappa = 1, 2, 3, 4$. We denote this sequence by $\boldsymbol{R}_\kappa$. Obviously the fundamental sequence is an entity characteristic of the differential equation itself and does not depend on the choice of the particular fundamental integral used in its definition.

Similarly, let us consider a differential equation (q_λ) $(\lambda = 1, 2)$ whose right λ-fundamental number s_λ is proper; $s_\lambda < b$. We define a *right fundamental integral of the λ-th kind* or a *right fundamental λ-integral* (of q_λ) as an integral v_λ such that $v_1(s_1) = 0$ or $v_2'(s_2) = 0$, according as $\lambda = 1, 2$. When $\lambda = 1$ we define $s_{1,\nu}$ as the ν-th zero of v_1 to the left of s_1 and if $q_1(t) < 0$ in j we define $s_{4,\nu}$ as the ν-th zero of v_1' to the left of s_4, (provided it exists). Further, when $\lambda = 2$ we define $s_{2,\nu}$ and $s_{3,\nu}$ correspondingly as the ν-th zeros of v_2' and v_2 to the left of s_2 and s_3; assuming again that these numbers exist (s_4 and s_3 always do). We write $s_2 = s_{2,0}$, $s_3 = s_{3,0}$.

We then have the *right fundamental sequence of the κ-th kind*, or the *right κ-fundamental sequence*, $\boldsymbol{S}_\kappa$

$$s_{\kappa,0} > s_{\kappa,1} > s_{\kappa,2} > \cdots,$$

The elements of $\boldsymbol{R}_\kappa$, $\boldsymbol{S}_\kappa$ in the interval j are called *singular numbers of the κ-th kind* of the differential equation.

Every two distinct terms ($\neq r_3$ or s_3) of the fundamental sequences $\boldsymbol{R}_1$, $\boldsymbol{R}_3$; $\boldsymbol{S}_1$, $\boldsymbol{S}_3$ are 1-conjugate with each other; every two distinct terms, ($\neq r_4$ or s_4) of each of the fundamental sequences $\boldsymbol{R}_2$, $\boldsymbol{R}_4$; $\boldsymbol{S}_2$, $\boldsymbol{S}_4$ are 2-conjugate with each other. The subintervals of j

$$i_{\kappa,\nu} = (r_{\kappa,\nu-1}, b) \qquad (j_{\kappa,\nu} = (a, s_{\kappa,\nu-1}))$$
$$(\kappa = 1, 2, 3, 4; \qquad \nu = 1, 2, \ldots),$$

consist precisely of those numbers $t \in j$ which possess the ν-th left or right κ-conjugate number respectively.

3.6 General and special equations of finite type

Following on from the above, let (q_λ) be a differential equation, for which both fundamental number r_λ and s_λ are proper $(\lambda = 1, 2)$. Then the differential equation (q_λ) is of finite type (m) (naturally with $m \geqslant 2$). For, if the fundamental numbers r_λ, s_λ are proper, then the left λ-fundamental integral u_λ has on the left of r_λ either no zero (if $\lambda = 1$) or at most one zero (if $\lambda = 2$), and v_λ has similar properties to the right of s_λ.

Generally, the fundamental numbers r_λ, s_λ are not λ-conjugate. In this case the differential equation (q_λ) is called *general*, or *non-special, of the λ-th kind*, more briefly *λ-general* or *λ-non-special.* In the case of a λ-general differential equation (q_λ), the two λ-fundamental integrals u_λ and v_λ are independent, and the two κ-fundamental sequences $\boldsymbol{R}_\kappa$, $\boldsymbol{S}_\kappa$, $\kappa = 1, 2, 3, 4$, have no common terms in the interval j.

If the fundamental numbers r_λ, s_λ are λ-conjugate, then the differential equation (q_λ) is called *special of the λ-th kind* or briefly *λ-special.* In the case of a λ-special differential equation (q_λ) the two λ-fundamental integrals u_λ and v_λ are dependent, and the two fundamental sequences $\boldsymbol{R}_\lambda$, $\boldsymbol{S}_\lambda$ coincide, forming the so-called *fundamental sequence of the λ-th kind*, or briefly the *λ-fundamental sequence* of the differential equation (q_λ).

3.7 Relations between conjugate numbers of different kinds

We now wish to consider relationships between conjugate numbers of different kinds of the same differential equation (q).

We take $q(t) < 0$ for $t \in j$, and let the symbols R_κ, S_κ, $\kappa = 1, 2, 3, 4$ have the same meaning as in § 3.4. Let $t \in R_1$; then every integral u of (q) which vanishes at t has a first zero x lying to the left of t, and between t, x there is precisely one zero x' of u'. The number x' is obviously left 3-conjugate with t, and t is right 4-conjugate with x'. Hence

$$R_1 \subset R_3, \qquad S_1 \subset S_3.$$

Moreover, from $R_1 \neq \varnothing$ or $S_1 \neq \varnothing$ it follows that $R_\kappa \neq \varnothing$, $S_\kappa \neq \varnothing$ for $\kappa = 1, 3, 4$. Similarly we obtain

$$R_2 \subset R_4, \qquad S_2 \subset S_4$$

and from $R_2 \neq \varnothing$ or $S_2 \neq \varnothing$ the conclusion that $R_\kappa \neq \varnothing$, $S_\kappa \neq \varnothing$ for $\kappa = 2, 3, 4$.

If, therefore, the differential equation (q) possesses conjugate numbers of the first or second kind then the fundamental numbers r_κ, s_κ exist for $\kappa = 1, 3, 4$ or $\kappa = 2, 3, 4$, respectively. We now assume that the differential equation (q) has conjugate numbers of all four kinds.

First, the above relations between the sets R_κ, S_κ give the following inequalities for the fundamental numbers:

$$r_3 \leqslant r_1, \qquad r_4 \leqslant r_2; \qquad s_1 \leqslant s_3, \qquad s_2 \leqslant s_4. \tag{3.1}$$

Obviously, if r_1 or r_2 is improper, then r_3 or r_4 is improper also, and if s_1 or s_2 is improper, so is s_3 or s_4. We have, moreover, the following theorem:

Theorem. Let one of the two fundamental numbers r_λ, $\lambda = 1, 2$ be improper, then the left fundamental numbers of all four kinds are improper. In this case the left end point a of the interval j is a cluster point of zeros of the individual integrals of (q).

If one of the two fundamental numbers r_λ, $\lambda = 1, 2$ is proper, so is the other. In this case the left end point a of j is not a cluster point of zeros of the individual integrals of (q).

The same is true with r_λ, "left" and a replaced by s_λ, "right" and b.

Proof. (a) For definiteness, let r_1 be improper; consider an integral u of (q). Assume a is not a cluster point of zeros of the integrals of (q); u has therefore a least zero $x \in j$. Then no number $t \in (a, x)$ possesses a left 1-conjugate number. We have therefore $a < x \leqslant r_1$, contrary to our assumption. Consequently, in every right neighbourhood of a there lie infinitely many zeros of u and consequently (§ 2.1) also of u', hence $r_2 = a$.

(b) The first part of the second statement follows from (a). Moreover if a is a cluster point of zeros of the individual integrals of (q), then in every right neighbourhood of a there are numbers which possess left conjugate numbers of the λ-th kind; hence $r_\lambda = a$, $\lambda = 1, 2$. Other cases are proved similarly.

3.8 Equations with proper fundamental numbers

Consider now a differential equation (q) with proper λ-fundamental numbers r_λ, s_λ $\lambda = 1, 2$. First we observe that every integral of (q) has therefore at least one zero. We recall also our assumption that $q(t) < 0 \ \forall\, t \in j$.

We now state and prove a theorem relating to left fundamental numbers and integrals. An exactly similar theorem holds for right fundamental numbers and integrals, which is obtained merely by replacing "left" by "right", u by v, a by b, r by s and $<$ by $>$, while the proof is entirely analogous. We shall not therefore write down this theorem or its proof explicitly.

Let u_λ be a left λ-fundamental integral of (q). Our theorem is:

1. *Let the fundamental number r_4 be proper, so that $a < r_4$, then the derivative u_1' of u_1 has precisely one zero to the left of r_1 and this latter zero is r_4. If the fundamental number r_4 is improper so that $a = r_4$, then the derivative u_1' of u_1 has no zero to the left of r_1. In this case $r_3 = r_1$ and the first zero of u_1' to the right of r_1 is the fundamental number r_2. Hence $r_1 < r_2$ and the two fundamental integrals u_1, u_2 are dependent.*

2. *Let the fundamental number r_3 be proper, so that $a < r_3$; then the fundamental integral u_2 has precisely one zero to the left of r_2, and this is r_3. If the fundamental number r_3 is improper, so that $a = r_3$, then the fundamental integral u_2 has no zero to the left of r_2. In this case $r_4 = r_2$ and the first zero of u_2 lying to the right of r_2 is the fundamental number r_1. Thus $r_2 < r_1$ and the two fundamental integrals u_1, u_2 are dependent.*

Proof. We shall restrict ourselves to the proof of the first statement.

(a) Let the fundamental number r_4 be proper. We consider a number $t_1 \in (a, r_4)$ and an integral u of (q) whose derivative vanishes at t_1; $u'(t_1) = 0$. On account of the fact that $t_1 < r_4$ the integral u has no zeros to the left of t_1. It therefore possesses zeros to the right of t_1; we consider the least of these, namely x_1. We must have $x_1 \leqslant r_1$, because otherwise x_1 would possess a left 1-conjugate number and consequently there would be a zero of u to the left of t_1. If $x_1 = r_1$, then the two integrals u, u_1 are dependent, and consequently their derivatives u', u_1' have the same zeros. In this case it follows that u_1' has at least one zero t_1 to the left of r_1. If $x_1 < r_1$, then we have the following situation: $t_1 < x_1$, $u'(t_1) = u(x_1) = 0$, $u(t) \neq 0$ for $t \in (t_1, x_1)$; $r_1 > x_1$, $u_1(r_1) = 0$. Then the third ordering theorem § 2.3 (3) shows that the function u_1 has a zero to the left of r_1.

The function u_1' has at most one zero to the left of r_1. For, if it vanishes more than once on the left of r_1, then on the left r_1 there lies at least one zero of u_1, which, however, conflicts with the definition of u_1. The function u_1' has therefore precisely one zero, t_2, to the left of r_1.

We now consider a number $t \in (a, r_1)$ and an integral u of (q), whose derivative u' vanishes at t: $u'(t) = 0$. By a similar argument to that used above, and applying the ordering theorem (3) of § 2.3, we conclude that; if $t > t_2$ then the integral u has one zero to the left of t but if $t < t_2$ then u has no zero to the left of t. It follows that $t_2 = r_4$.

(b) Let the fundamental number r_4 be improper. In this case every integral of the differential equation (q) has at least one zero to the left of a zero of its derivative. If, therefore, the derivative u_1' of the fundamental integral u_1 has a zero to the left of r_1, then u_1 has also such a zero. But the fundamental integral u_1 has no zero to the left o r_1, consequently the derivative u_1' of u_1 has no zero to the left of r_1.

We continue to employ the symbol R_κ ($\kappa = 1, 2, 3, 4$) with the meaning used above (see § 2.4).

Let $t \in R_3$, and consider an integral u of (q) which vanishes at t; $u(t) = 0$. The derivative u' of u has a zero, x, to the left of t. Because $a = r_4$ we have $x \in R_4$. The integral u has therefore a zero t_1 lying to the left of x, which is obviously left 1-conjugate with t. Hence $R_3 \subset R_1$ and $r_3 \geqslant r_1$, whence (3.1) gives $r_3 = r_1$.

Let x be the first zero of the derivative u_1' of u_1 lying on the right of r_1. We consider a number $t \in j$ and an integral u of (q) whose derivative vanishes at t; $u'(t) = 0$. Since $t \in R_4$, u has a greatest zero t_1 lying to the left of t. It is now necessary to distinguish two cases, according as $t > x$ or $t < x$. In the first case the ordering theorem (4) implies that $t_1 > r_1$ and since $r_3 = r_1$ it follows that $t_1 \in R_3$. The function u' has therefore at least one zero to the left of t_1, which obviously is left 2-conjugate with t. Hence $t \in R_2$.

In the second case the ordering theorem (4) gives $t_1 < r_1$, from which, taking into account the fact that $r_3 = r_1$ it follows that $t_1 \notin R_3$. The function u' has therefore no zero lying to the left of t_1. Hence $t \notin R_2$.

Grouping these results together: the set R_2 coincides with the interval (x, b). From this it follows that $x = r_2$ and the proof is complete.

3.9 Singular bases

The concept of fundamental integrals is closely connected with that of singular bases of a differential equation (q).

Let (q_λ) be a differential equation with λ-conjugate numbers, $\lambda = 1, 2$, and let (u, v) be a basis of (q_λ). We call the basis (u, v) a *left* (*right*) *principal basis of the λ-th kind*, more briefly a *left* (*right*) *λ-principal basis*, if the first term u is a left (right) λ-fundamental integral of (q_λ). Further, we call the basis (u, v) a *principal basis of the λ-th kind*, more briefly a *λ-principal basis* if the first term u is a left or right λ-fundamental integral of (q_λ) and also v is a right or left λ-fundamental integral, respectively, of (q_λ). In our study of principal bases it is therefore important to notice whether u is a left and v a right λ-fundamental integral or conversely. Principal bases, and left or right principal bases, will be called *singular bases* of the differential equation (q_λ). In the case of a λ-general differential equation (q_λ) of finite type with λ-conjugate numbers, there are left and right principal bases as well as principal bases of the λ-th kind. In the case of λ-special differential equations (q_λ) there occur left and right principal bases of the λ-th kind, and in fact every left or right principal basis of the λ-th kind is the same as a right or left principal basis of the λ-th kind. In left (right) oscillatory differential equations (q_λ) we only have right (left) principal bases of the λ-th kind. In oscillatory differential equations (q_λ) there are no singular bases).

One can write down without difficulty the system consisting of all the left (right) λ-principal bases or all the λ-principal bases of the differential equation (q_λ). Let u be a left (right) λ-fundamental integral of the differential equation (q_λ). Then the functions ρu, with arbitrary constant $\rho \neq 0$, constitute all the left (right) λ-fundamental integrals of the differential equation (q_λ). The set of all left (right) λ-principal bases is simply $(\rho u, \bar{v})$, in which $\bar{v}$ is any integral of (q_λ) independent of u. If the integral v of (q_λ) is chosen to be independent of u, then $\bar{v}$ can be represented in the form $\bar{v} = \sigma v + \bar{\sigma} u$, with appropriate constants σ, $\bar{\sigma}$ $(\sigma \neq 0)$. Plainly, therefore, the left (right) λ-principal bases of the differential equation (q_λ) form a two-parameter system: $(\rho u, \sigma v + \bar{\sigma} u)$; $\rho\sigma \neq 0$. If the differential equation (q_λ) is of finite type and λ-general, then one can choose v as a right (left) fundamental integral of (q_λ). The functions σv, constructed with arbitrary constant $\sigma \neq 0$, then constitute all the right (left) λ-fundamental integrals of (q_λ); the situation can therefore be summarized as: the set of all λ-principal bases of the differential equation (q_λ) are of the form $(\rho u, \sigma v)$, $\rho\sigma \neq 0$ and thus constitute two two-parameter systems, in one of which u is a left and v a right λ-fundamental integral of (q_λ), and in the other conversely.

3.10 Differential equations (q) with 1-conjugate numbers

In this section we give a survey of differential equations (q) with 1-conjugate numbers, based on the above results. In order to simplify our explanation, we shall now leave out the attribute "1", using the terms conjugate numbers, fundamental sequences, principal bases, etc. instead of 1-conjugate numbers, 1-fundamental sequences, 1-principal bases and so on. Differential equations (q) with conjugate numbers are either of finite type (m) with $m \geqslant 2$ or of infinite type.

I. Let (q) be a differential equation of finite type (m), $m \geqslant 2$.

The differential equation (q) then possesses integrals with m zeros in the interval j but none with $m + 1$ zeros. In this case the ends a, b of j are obviously not cluster points of zeros of the individual integrals of (q); consequently both fundamental numbers r_1 and s_1 are proper, and the differential equation (q) possesses left and right fundamental integrals and also left and right fundamental sequences, each of which is composed of $m - 1$ terms. To simplify our notation we now denote these fundamental sequences by

$$a_1 < a_2 < \cdots < a_{m-1}; \qquad b_{-1} > b_{-2} > \cdots > b_{-m+1},$$

where $a_1 = r_1$; $b_{-1} = s_1$. In this, $a_1, a_2, \ldots, a_{m-1}$; $b_{-1}, b_{-2}, \ldots, b_{-m+1}$ are the singular numbers of (q).

Between the numbers $a_\mu, b_{-\mu}$ ($\mu = 0, 1, \ldots, m - 1, a_0 = a, b_0 = b$) there hold the following inequalities

$$a < b_{-m+1} \leqslant a_1 < b_{-m+2} \leqslant \cdots \leqslant a_r < b_{-m+r+1}$$
$$\leqslant a_{r+1} < \cdots \leqslant a_{m-2} < b_{-1} \leqslant a_{m-1} < b. \tag{3.2}$$

In these relations, either the strict inequality holds throughout, in which case the differential equation (q) is general, or equality holds everywhere, in which case the differential equation is special.

(a) Let the differential equation (q) be general.

The differential equation (q) then possesses left and right fundamental integrals, a left and a right fundamental integral are independent, and the differential equation (q) admits of principal bases.

Every integral of (q) which vanishes in one of the intervals $(a_\mu, b_{-m+\mu+1})$ has in every such interval one and only one zero; it follows that such an integral has precisely m zeros in the interval j. All other integrals of (q) have precisely $m - 1$ zeros in j.

(b) Let the differential equation (q) be special.

The differential equation (q) then possesses left and right fundamental integrals, and these coincide. The equation (q) admits of left and right principal bases; each left or right principal basis is the same as a right or left principal basis; each integral of (q) which is independent of a fundamental integral has precisely m zeros in j, while each fundamental integral has precisely $m - 1$ zeros.

II. Now let (q) be a differential equation of infinite type.

In this case the differential equation (q) is left or right oscillatory or oscillatory.

If the differential equation (q) is left (right) oscillatory then its left (right) fundamental number $r_1(s_1)$ is improper, while the right (left) is proper. In this situation the differential equation (q) possesses right (left) fundamental integrals, and the right (left) fundamental sequence

$$b_{-1} > b_{-2} > \cdots \qquad (a_1 < a_2 < \cdots), \tag{3.3}$$

which is always infinite; then $b_{-1}, b_{-2}, \ldots$ $(a_1, a_2, \ldots)$ are the singular numbers of the differential equation (q). The equation possesses right (left) principal bases.

Every integral of (q) independent of a right (left) fundamental integral has precisely one zero in every interval $(b_{-\mu}, b_{-\mu-1})$, $((a_\mu, a_{\mu+1}))$ $\mu = 0, 1, \ldots$; $b_0 = b$ $(a_0 = a)$.

If the differential equation (q) is oscillatory, then both fundamental numbers r_1, s_1 of (q) are improper; consequently in this case there are no fundamental integrals and naturally no fundamental sequences or singular bases.

Surveying these results, we see that among all types of differential equations (q), those which are general and of finite type (m) with $m \geqslant 2$ are the only ones for which there exist independent fundamental integrals (naturally, one left and one right) and consequently also principal bases.

3.11 Differential equations (q) with conjugate numbers of all four kinds

In this paragraph we collect together those properties of differential equations (q) with conjugate numbers of all four kinds, which are of importance for our further study.

Let (q) be a differential equation with conjugate numbers of all four kinds. We assume that $q(t) < 0$ for $t \in j$. For each kind κ (= 1, 2, 3, 4) an open interval $i_{\kappa,\nu}$ or $j_{\kappa,\nu}$, $\nu = 1, 2, \ldots$, is formed by those numbers $t \in j$ for which the ν-th left or right κ-conjugate number exists. If the zeros of integrals of the differential equation (q) cluster towards the left or right end point a or b of the interval j, so that (q) is left or right oscillatory, or oscillatory, then $i_{\kappa,\nu} = j$ or $j_{\kappa,\nu} = j$ for $\kappa = 1, 2, 3, 4$ and for all $\nu = 1, 2, \ldots$. If on the other hand the zeros of integrals of the differential equation (q) do not cluster about the left end point a of j, so that the differential equation (q) is of finite type or right oscillatory, then for every $\kappa = 1, 2, 3, 4$ we have

$$i_{\kappa,\nu} = (r_{\kappa,\nu-1}, b); \qquad a \leqslant r_{\kappa,0} < r_{\kappa,1} < \cdots < b$$

in which the sequence $\{r_{\kappa,\nu-1}\}$, $\nu = 1, 2, \ldots$, is finite if the differential equation (q) is of finite type, while it is infinite in the case of a right oscillatory differential equation (q).

Similarly, if the zeros of integrals of (q) do not cluster about b, then (q) is of finite type or left oscillatory, so $j_{\kappa,\nu} = (a, s_{\kappa,\nu-1})$; $a < \cdots < s_{\kappa,1} < s_{\kappa,0} \leqslant b$ and the sequence $\{s_{\kappa,\nu-1}\}$ is finite or infinite according as (q) is finite or left oscillatory. If the differential equation (q) is oscillatory, then corresponding to every number $t \in j$ there is a left and a right κ-conjugate ν-th number for all $\kappa = 1, 2, 3, 4$ and all $\nu = 1, 2, \ldots$.

3.12 Bilinear relations between integrals of the differential equation (q)

In the transformation theory to be considered later certain bilinear relations between integrals of a differential equation (q) play an important role. We consider a differential equation (q).

The fundamental theorem is the following:

Theorem. For an arbitrary basis (u, v) *of the differential equation* (q) *the following relations hold at two different points* $t, x \in j$

$$\begin{aligned} &1.\ u(t)v(x) - u(x)v(t) = 0,\\ &2.\ u'(t)v'(x) - u'(x)v'(t) = 0,\\ &3.\ u(t)v'(x) - u'(x)v(t) = 0 \end{aligned}$$

if and only if the numbers t, x *are connected, respectively, in the following ways:*

1. *t, x are 1-conjugate,*
2. *t, x are 2-conjugate,*
3. *x is 3-conjugate with t, and consequently t is 4-conjugate with x.*

Proof. We shall confine ourselves to the proof of 1.

(a) Let the bilinear relation 1 hold between two given distinct numbers $t, x \in j$. Then the linear equations

$$c_1u(t) + c_2v(t) = 0, \qquad c_1u(x) + c_2v(x) = 0$$

are satisfied for appropriate values c_1, c_2 with $c_1^2 + c_2^2 \neq 0$ and consequently the numbers t, x are zeros of the integral $y = c_1u + c_2v$ of (q); hence the numbers t, x are 1-conjugate.

(b) Let the numbers t, x be 1-conjugate. Then $t \neq x$, and there is an integral $y = c_1u + c_2v$ of (q) which vanishes at t and x, with $c_1^2 + c_2^2 \neq 0$. From this follows the bilinear relationship 1.

This theorem can obviously be formulated briefly as: conjugate numbers of the various kinds represent zeros of basis functions of the differential equation (q).

The ratio u/v of the integrals u, v of a basis of the differential equation (q), or the ratio u'/v' of their derivatives, respectively, take the same value at two different points $t, x \in j$ (i.e.

$$\frac{u(t)}{v(t)} = \frac{u(x)}{v(x)} \quad \text{or} \quad \frac{u'(t)}{v'(t)} = \frac{u'(x)}{v'(x)}$$

respectively) if and only if the numbers t and x are 1-conjugate or 2-conjugate respectively. Moreover, the relationship

$$\frac{u(t)}{v(t)} = \frac{u'(x)}{v'(x)}$$

holds if and only if x is 3-conjugate with t and t is consequently 4-conjugate with x.