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ON A GENERALIZATION OF FABRY'S AND SZÁSZ'S THEOREMS CONCERNING THE SINGULARITIES OF POWER SERIES

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The analytic function defined by the power series

$$f(z)=\sum_{n=1}^{\infty} a_n z^n,\ a_n=|a_n|e^{i\phi_n},\quad \overline{\lim_{n\to\infty}}\ |a_n|^{\frac{1}{n}}=1, \tag{1}$$

has one or more singularities on the circle of convergence. The number, position and quality of these singular points are functions of the infinite number of variables $a_1, a_2, a_3, \ldots$.

It has been shown by the author in some recently published memoirs,* that all the investigations concerning this subject can be considerably simplified and at the same time generalized by means of the so-called L series of coefficients

$$a_{n_1},\ a_{n_2},\ a_{n_3},\ \ldots$$

chosen from the set $a_1, a_2, a_3, \ldots$ in such a manner that

$$\lim_{q\to\infty} |a_{n_q}|^{\frac{1}{n_q}}=1.$$

Clearly every set $a_1, a_2, \ldots$ given by (1) includes an unlimited number of L series.

In the memoirs just cited the author has generalized the well known theorems of Vivanti-Dienes, Fatou-Polya and of Hadamard concerning singularities on the circle of convergence. The first and a special case of the third generalizations are reprinted as the second and third auxiliary theorems in part I of this memoir. The present communication is a further addition to this general theory based upon the systematic use of L series.

It seems quite probable that the results obtained by this method are an individual property of power series and cannot be extended to more general Dirichlet's series.

*(*a*) Rendiconti dei Lincei. XXXII (5), 1° sem. (1923) *Sur les singularités der séries entières*, p. 26-29. *Nouveaux théorèmes sur les singularités des séries entières*, p. 83-85. *Sur les singularites des séries entières*, p. 528-531.

(*b*) Rozpravy ces. akademie Praha. XXXII, tr. II, c. 35 (1923). *O singularitách rady mocninné lezících na krunici konvergencni*, p. 1-15. See also the extract in French: Bulletin internat. de l'Académie des Sciences de Bohême (1923), *Sur les singularités des séries entières situées sur le cercle de convergence*, p. 1-3.

I. Auxiliary Theorems.

We use in the following some auxiliary theorems. The *first* is Hadamard's* multiplication theorem:

First Auxiliary Theorem: If two analytic functions are defined by the convergent power series

$$\phi(x) = a_0 + a_1x + a_2x^2 + \ldots,$$

$$\psi(x) = b_0 + b_1x + b_2x^2 + \ldots,$$

the only singularities of the function

$$f(x) = a_0b_0 + a_1b_1x + a_2b_2x^2 + \ldots$$

will be points whose affixes γ_{ij} *are the product of affixes of the singular points* α_i *and* β_j *of the first two functions.*

For our purpose the proof of a special form of this theorem due to Pringsheim† is quite sufficient.

The *second* and *third* auxiliary theorems‡ are as follows, the *third* being a generalization of the well-known Hadamard's theorem§, concerning power series with an unlimited number of zero coefficients:

Second Auxiliary Theorem: If corresponding to some L series $a_{n_1}, a_{n_2}, \ldots, a_{n_q}, \ldots,$ *chosen from the coefficients of* (1), *a series of angles* $\psi_1, \psi_2, \ldots, \psi_q, \ldots$ *can be selected in such a manner, that*

$$\text{(a)} \qquad \lim_{q\to\infty} \left\{ \cos(\phi_{n_q} + \psi_q) \right\}^{\frac{1}{n_q}} = 1,$$

$$\text{(b)} \qquad \cos\{\phi_n + \psi_q\} \geqq 0,$$

for all indices n satisfying one of the inequalities

$$n_q(1-\mu) \leqq n \leqq n_q(1+\mu), \quad (q = 1, 2, 3, \ldots),$$

where μ *is a positive constant independent of q, then the point* $z = 1$ *is a singularity of the function* (1).

Third Auxiliary Theorem: If an L series $a_{n_1}, a_{n_2}, \ldots, a_{n_q}, \ldots,$ *derived from the coefficients of* (1) *exists, which has the property, that*

$$a_n = 0$$

for all indices n satisfying one of the inequalities

$$n_q(1-\mu) \leqq n \leqq n_q(1+\mu), \quad (q = 1, 2, 3, \ldots),$$

where μ *is a positive constant independent of q, with the sole exception of the central coefficients* a_{n_q}, *then the circle of convergence is the natural boundary of the function* (1).

*Acta Mathematica 22 (1898), p. 55.

†Münchener Berichte (1912), p. 11-92.

‡Consult for proof (*b*), p. 12 and 14, theorems (I) and (II) for $r_q = 0$ or Bulletin intern., p. 2-3 theorems (I) and (II).

§Jour. de Math. (4), vol. 8 (1892), p. 101-186.

It is obvious from this, that every coefficient of the L series used, *e.g*,, the coefficient $a_{n_{10}}$, as non-vanishing can be a member of only one group defined by

$$n_{10}(1-\mu) \leqq n \leqq n_{10}(1+\mu),$$

which fact involves the easily verified consequence

$$n_{q+1} \geqq n_q(1+\mu_1), \quad (q=1, 2, 3, \ldots),$$

where μ_1 is a positive constant dependent only on μ.

The *fourth* auxiliary theorem, which will be stated at the end of this section after we have proved a preliminary result, refers to a type of integral functions. Suppose that the positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_\mu, \ldots$ have the properties

$$\frac{\lambda_\nu}{\nu} \to \infty, \lambda_{\nu+1} - \lambda_\nu \geqq 1, \quad (\nu=1, 2, 3, \ldots).$$

Then the integral function

$$g(x) = \prod_{\nu=1}^{\infty} \left(1 - \frac{x^2}{\lambda_\nu^2}\right)$$

is of the "Minimaltypus der Ordnung eins"* and therefore

$$|g(x)| < e^{\delta|x|},$$

where δ is some positive constant independent of x, if only $|x|$ is greater than a suitably chosen $R(\delta)$. The consequence of this is

$$\text{(2)} \qquad \overline{\lim_{|x| \to \infty}} |g(x)|^{\frac{1}{|x|}} \leqq 1.$$

If now a series of positive numbers

$$a_1 < a_2 < a_3 < \ldots a_n < \ldots$$

s given in such a way, that†

$$a_n \to \infty, \left|a_n - \lambda_\nu\right| \geqq \kappa, \quad (n, \nu=1, 2, 3, \ldots),$$

then we can prove the following inequality‡:

$$|g(a_n)|^{\frac{1}{a_n}} > e^{-\delta},$$

where δ is as small a positive number as desired, if only n is sufficiently great.

To prove this preliminary theorem we first observe that for some given a_m only one index n can verify the relation

$$\lambda_{n-1} < a_m < \lambda_n.$$

*Cf. Pringsheim, *loc. cit.*, p. 85, 87-88.

†κ is a positive constant independent of n and ν.

‡This inequality represents a remarkable property of the function $g(x)$. In the special case $\frac{a_m}{\lg m} \to \infty$ the inequality has been proved by Szász: see Mathematische Annalen 85 (1922) s. 99-110, *Ueber Singularitäten von Potenzreihen und Dirichletschen Reihen e.s.o.*

We have now

$$|g(a_m)|^{-1}=\prod_{\nu=1}^{n-1}\frac{1}{\frac{a_m^2}{\lambda_\nu^2}-1}\cdot\prod_{\nu=n}^{\infty}\frac{1}{1-\frac{a_m^2}{\lambda_\nu^2}}$$

$$\leqq\prod_{\nu=1}^{n-1}\frac{\lambda_\nu}{a_m-\lambda_\nu}\cdot\prod_{\nu=n}^{\infty}\left(1+\frac{a_m^2}{(\lambda_\nu+a_m)(\lambda_\nu-a_m)}\right)$$

$$\leqq\frac{a_m^{n-1}}{\kappa(\kappa+1)\ldots(\kappa+n-2)}\cdot\prod_{\nu=n}^{\infty}\left(1+\frac{a_m^2}{\lambda_\nu(\kappa+\nu-n)}\right).$$

To simplify the proof* further we introduce the series of auxiliary numbers

$$\eta_m=\operatorname{Max}\frac{\nu}{\lambda_\nu},\quad \nu\geqq m,\quad (m=1,\,2,\,3,\,\ldots.).$$

Since

$$\frac{\nu}{\lambda_\nu}\to 0,$$

it is obvious that

$$\eta_m\to 0 \text{ if } m\to\infty.$$

It follows that

$$\frac{\nu}{\lambda_\nu}\leqq\eta_n \text{ if } \nu\geqq n,$$

and therefore

$$\frac{1}{\lambda_\nu}\leqq\frac{\eta_n}{\nu},$$

which gives

$$\prod_{\nu=n}^{\infty}\left(1+\frac{a_m^2}{\lambda_\nu(\kappa+\nu-n)}\right)\leqq\prod_{\nu=n}^{\infty}\left(1+\frac{(a_m\sqrt{\eta_n})^2}{\nu(\nu+\kappa-n)}\right).$$

If now the number n is so great, that

$$n\kappa>1,\ n+\kappa-2\geqq 0,$$

then putting $\nu=n+\mu$ we find

$$(n+\mu)\ (\kappa+\mu)>(\mu+1)^2$$

and therefore

$$\prod_{\nu=n}^{\infty}\left(1+\frac{(a_m\sqrt{\eta_n})^2}{\nu(\kappa+\nu-n)}\right)<\prod_{\mu=1}^{\infty}\left(1+\frac{(a_m\sqrt{\eta_n})^2}{\mu^2}\right)=\frac{\sin\pi i a_m\sqrt{\eta_n}}{\pi i a_m\sqrt{\eta_n}}.$$

see Göttinger Nachrichten Math. Phys. Klasse 1921; *Neuer Beweis und Verallgemeinerungen des*
*This follows the lines of the proof of a similar theorem used by F. Carlson and E. Landau: *Fabryschen Lückensatzes.*

Hence returning to the function $g(x)$, we have

$$|g(a_m)|^{-1} < \frac{a_m^{n-1}}{\kappa(\kappa+1)\dots(\kappa+n-2)} \cdot \frac{\sin \pi i a_m\sqrt{\eta_n}}{\pi i a_m\sqrt{\eta_n}}$$
$$< \frac{a_m^{n-1}}{(n-1)!}\frac{(n-1)}{\kappa} e^{\pi a_m\sqrt{\eta_n}}$$
$$< \frac{n-1}{\kappa}\frac{(ea_m)^{n-1}}{(n-1)^{n-1}} \cdot e^{\pi a_m\sqrt{\eta_n}}$$
$$< e^{ea_m\left\{\frac{\pi\sqrt{\eta_n}}{e} + \frac{n-1}{ea_m} lg \frac{ea_m}{n-1} + \frac{1}{ea_m} lg \frac{n-1}{\kappa}\right\}},$$

But the numbers n and m are conditioned by the inequalities

$$e\lambda_{n-1} < ea_m < e.\lambda_n,$$
$$e\frac{\lambda_{n-1}}{n-1} < \frac{ea_m}{n-1} < \frac{e.n}{n-1}\cdot\frac{\lambda_n}{n}.$$

Now if $m\to\infty$ (we recall that $a_m\to\infty$) then also $n\to\infty$ and therefore*

$$\frac{\lambda_{n-1}}{n-1}\to\infty, \frac{\lambda_n}{n}\to\infty, \frac{ea_m}{n-1}\to\infty, \frac{n-1}{e\,.\,a_m} lg \frac{ea_m}{n-1}\to 0,\ \eta_n\to 0,\ \frac{1}{ea_m} lg\frac{n-1}{\kappa}\to 0.$$

We arrive at the following result: if one chooses some small positive number δ, then an integer $m(\delta)$ can be found such that for each $m > m(\delta)$, the inequality

(3) $$|g(a_m)|^{\frac{1}{a_m}} > e^{-\delta}$$

holds. Hence

(4) $$\underline{lim}_{m\to\infty} |g(a_m)|^{\frac{1}{a_m}} \geqq 1.$$

But from (2) we have,

$$\overline{\lim_{m\to\infty}} |g(a_m)|^{\frac{1}{a_m}} \leqq 1.$$

and from the two inequalities we finally obtain the *fourth auxiliary theorem* expressed by the following formula:

Fourth Auxiliary Theorem:

$$\lim_{m\to\infty} |g(a_m)|^{\frac{1}{a_m}} = 1.$$

*As to the last of these relations we have:

$$\frac{lg(n-1)}{a_m} < \frac{lg(n-1)}{\lambda_{n-1}} = \frac{lg(n-1)}{n-1}\frac{n-1}{\lambda_{n-1}} \to 0.$$

II. The Generalized Theorem of Fabry.

E. Fabry has proved the following well known theorem*:

If in the power series

$$f(z) = \sum_{p=0}^{\infty} a_p z^{m_p}$$

the conditions

$$m_1 < m_2 < m_3 < \ldots, \quad \lim_{p \to \infty} \frac{m_p}{p} = \infty,$$

are satisfied, then the circle of convergence is a natural boundary for $f(z)$.

Quite recently the validity of this theorem has been shown to hold for the general Dirichlet series

$$f(z) = \sum_{p=1}^{\infty} a_p\, e^{-\lambda_p s} \text{ if } \frac{\lambda_p}{p} \to \infty.$$

The author of this important generalization is O. Szász†, while a still more general form is due to F. Carlson and E. Landau‡.

In all these theorems the condition

$$\frac{\lambda_p}{p} \to \infty$$

must hold for *all* indices p, that is to say, for all members of the series.

The following pages contain a generalization of the theorem relating to power series *only*: there is, however, here no longer the necessity of imposing the condition $\frac{m_p}{p} \to \infty$ on all the members of the series.

Theorem A. Let $a_{n_1}, a_{n_2}, \ldots, a_{n_q}, \ldots$ be an L series derived from the coefficients of

$$f(z) = \sum_{n=1}^{\infty} a_n z^n,$$

where

$$n_{q+1} \geqq n_q(1+\mu_1)$$

and the positive number μ_1 is independent of q.

Let us arrange in an infinite series $a_{m_1}, a_{m_2}, \ldots, a_{m_p}, \ldots$, the non-vanishing

Annales de l'École Normale Supérieure (3) 13 (1896), p. 381-382, Acta Mathematica 22 (1899), p. 86. The theorem above is quoted in a form adapted to our notation.

†*loc. cit.* p. 106.

‡*loc. cit.*

coefficients of $f(z)$ which satisfy the conditions

$$n_q(1-\mu) \leqq n \leqq n_q(1+\mu), \; n \neq n_q, \quad (q=1, 2, 3, \ldots),$$

where μ is a suitably chosen positive constant, independent of q.*

If the condition

$$\frac{m_\nu}{\nu} \to \infty,$$

is satisfied, then the circle of convergence is a natural boundary for $f(z)$.

To prove this we first form the integral function

$$g(y) = \prod_{\nu=1}^{\infty} \left(1 - \frac{y^2}{m_\nu^2}\right),$$

which is of the form indicated in the fourth auxiliary theorem.

Then the power series†

$$\sum_{\nu=1}^{\infty} g(\nu) z^\nu$$

defines an analytic function with the sole singular point $z=1$.

Next we form a new power series

$$F(z) = \sum_{\nu=1}^{\infty} g(\nu) a_\nu z^\nu,$$

all the singularities of which must (in accordance with the first auxiliary theorem) be such as are possible for the series

$$f(z) = \sum_{\nu=1}^{\infty} a_\nu z^\nu.$$

But of all the coefficients of $F(z)$ satisfying the inequality

$$n_q(1-\mu) \leqq n \leqq n_q(1+\mu)$$

only one, $a_{n_q} g(n_q)$, does not vanish. Furthermore

$$\lim |a_{n_q} g(n_q)|^{\frac{1}{n_q}} = 1,$$

as a consequence of the fourth auxiliary theorem and the condition

$$\lim |a_{n_q}|^{\frac{1}{n_q}} = 1.$$

Now it is obvious that the function $F(z)$ has all the qualities requisite for the application of the third auxiliary theorem. The circle of convergence is

*We choose the μ in such a manner that each a_{n_q} is a member of only one group defined by $n_q(1-\mu) \leqq n \leqq n_q(1+\mu)$. (Cf. the third auxiliary theorem).

†Consult for proof Pringsheim *loc. cit.*

therefore the natural boundary of $F(z)$ and since $f(z)$ has the same singularities, it is also the natural boundary of $f(z)$.

As an example we construct the following series:

$$f(z) = \sum_{n=1}^{\infty} a_n z^n.$$

Put the coefficients with indices n satisfying one of the inequalities

$$10^q - 10^{q-3} \leqq n \leqq 10^q + 10^{q-3}, \quad (q = 3, 4, 5, \ldots),$$

equal to zero with the following exception:

$$a_{10^q} = 1$$

and for coefficients with prime indices

$$1 = a_3 = a_5 = a_7 = a_{11} = \ldots.$$

On all the remaining coefficients with indices n such that

$$10^q + 10^{q-3} < n < 10^{q+1} - 10^{q-2}, \quad (q = 3, 4, 5, \ldots),$$

no limitations need be imposed except naturally

$$\overline{\lim} |a_n|^{\frac{1}{n}} = 1.$$

The choice of these latter coefficients cannot affect the singular character of the natural boundary $|z| = 1$.

It is obvious that the original Fabry's theorem is a special form of (A).

The theorem (A) remains a "Lückensatz," which means that the vanishing of some coefficients is essential. More explicitly stated: The replacing of the zero coefficients by a series $a_{r_1}, a_{r_2}, \ldots, a_{r_\nu}, \ldots$ with the properties

$$|a_{r_\nu}| > 0, \quad \overline{\lim} |a_{r_\nu}|^{\frac{1}{r_\nu}} = 1, \quad (\nu = 1, 2, 3, \ldots),$$

cannot be done without invalidating our proof. This remark is not superfluous because in the third auxiliary theorem, which represents a generalization of Hadamard's "Lückensatz," a suitable replacing of zero coefficients is permissible (cf. (b) second auxiliary theorem).

III. The Generalization of Szász's Theorem.

Szász (*loc. cit.*) has recently published a theorem concerning general *Dirichlet* series, which, restricted to power series, may be stated as follows:

Consider the power series with real coefficients

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad \overline{\lim} |a_n|^{\frac{1}{n}} = 1.$$

Let the series of coefficients a_1, a_2, ... have the following properties

$$a_1 \geqq 0,\ a_2 \geqq 0,\ \ldots,\ a_{q_1} \geqq 0;\ a_{q_1+1} < 0,\ a_{q_1+2} \leqq 0,\ \ldots,\ a_{q_2-1} \leqq 0,$$
$$a_{q_2} \leqq 0;\ a_{q_2+1} > 0,\ a_{q_2+2} \geqq 0,\ \ldots.$$

If now

$$\lim_{\nu \to \infty} \frac{q_\nu}{\nu} = \infty,$$

then $z=1$ is a singular point of $f(z)$.

This is a generalization of a well known theorem of Vivanti*. We now propose to prove a further generalization:

Theorem B. Let $a_{n_1}, a_{n_2}, \ldots, a_{n_q}, \ldots$ be some L series derived from the coefficients of

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad \overline{\lim} |a_n|^{\frac{1}{n}} = 1,$$

and $\psi_1, \psi_2, \ldots, \psi_q, \ldots$ a corresponding series of angles selected in such a manner that†

1° $$\lim_{q \to \infty} \{\cos(\phi_{n_q} + \psi_q)\}^{\frac{1}{n_q}} = 1.$$

Let μ be a positive constant independent of q. The first of the indices n defined by

$$n_q(1-\mu) \leqq n \leqq n_q(1+\mu),$$

which satisfies the inequality

2° $$\cos(\phi_n + \psi_q) < 0,$$

we denote by n_{q_1}; the next index which satisfies the inequality

3° $$\cos(\phi_n + \psi_q) > 0$$

we denote by $n_{q_2}+1$, so that

$$\cos(\phi_{n_{q2}} + \psi_q) \leqq 0$$

and so forth.

For simplicity of presentation denote the series of positive integers

$$n_{1,1} < n_{1,2} < n_{1,3} < \ldots < n_{2,1} < n_{2,2} < \ldots < n_{q1} < n_{q2} < \ldots$$

by

$$n_{1,1} = p_1,\ n_{1,2} = p_2, \ldots$$

*Riv. di Matem. 3 (1893), p. 111-114.

†Obviously the equation 1° implies the condition $\cos(\phi_{n_q} + \psi_q) > 0$ for sufficiently large values of q.

If this series has the property

$$\lim_{\nu \to \infty} \frac{p_\nu}{\nu} = \infty,$$

then $z=1$ *is a singular point of* $f(z)$.

To prove this we start with the function

$$g(y) = \prod_{\nu=1}^{\infty} \left(1 - \frac{y^2}{p_\nu^2}\right)$$

and make use of the first auxiliary theorem.

Clearly $g(y)$ is negative for every value of y, which lies within one of the intervals

$$(p_1, p_2), \ (p_3, p_4), \ \ldots$$

and is positive for every other positive value of y; in particular it is positive and different from zero, for every value

$$y = n_1, n_2, \ldots, n_q, \ldots$$

because these numbers are all *outside* the intervals. Therefore the values of $g(y)$ attached to these arguments satisfy the equation

$$\lim \left|g(n_q)\right|^{\frac{1}{n_q}} = 1.$$

This follows from the fourth auxiliary theorem.

Consider now the power series

$$F(z) = \sum_{\nu=1}^{\infty} g(\nu) a_\nu z^\nu$$

which is derived by applying Hadamard's multiplicative process to the following two functions

$$f(z) = \sum_{\nu=1}^{\infty} a_\nu z^\nu,$$

$$h(z) = \sum_{\nu=1}^{\infty} g(\nu) z^\nu.$$

The function $h(z)$ has the sole singular point $z=1$. Hence it follows from the first auxiliary theorem, that $F(z)$ has no singular points other than those which are possible for $f(z)$. The series $F(z)$, however, obviously satisfies the conditions (a) and (b) of the *second* auxiliary theorem, because the real part of $g(\nu) a_\nu e^{i\psi_\nu}$ cannot be negative for indices ν defined by

$$n_q(1-\mu) \leqq \nu \leqq n_q(1+\mu), \quad (q=1, 2, 3, \ldots).$$

Hence the function $F(z)$ and consequently also $f(z)$ has the singularity $z=1$.

Clearly the original Szász's theorem can be obtained as a specialization of B.