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1. Notations

The letter R denotes the set of all (finite) real numbers. The word function means a mapping to R. The domain of definition of a function f is denoted by Dom f.

For each $A \subset R$ let int A, cl A, $|A|$, and c_A denote the interior, the closure, the outer Lebesgue measure and the characteristic function of A, respectively. The symbols $f(a+)$ and $f(a-)$ stand for $\lim_{x\downarrow a} f(x)$ and $\lim_{x\uparrow a} f(x)$. Further, we set $N_0 = \{0, 1, ...\}$, $N = \{1, 2, ...\}$. Instead of lim sup a_n $(n \in N,\ n \to \infty)$ we write simply lim sup a_n; similarly for lim inf and lim. The meaning of $a_n \to a$ is obvious.

The symbols $[a, b]$, $[a, b)$ etc. $(a, b \in R,\ a \leqq b)$ have the usual meaning (in particular, $[a, a] = \{a\}$); $\int_a^b f$ or $\int_{[a,b]} f$ denotes the Lebesgue integral of f over $[a, b]$. (In this connection f will be always Riemann integrable.) The words almost and measurable refer to the Lebesgue measure.

2. $\mathfrak{D}$-and I$\mathfrak{D}$F-series

2.1. For each $n \in N_0$ let D_n be a finite set $\{d_{n,0}, d_{n,1}, ..., d_{n,r_n}\}$, where $0 = d_{n,0} < d_{n,1} < ... < d_{n,r_n} = 1$. Set $D = \bigcup_{n=0}^{\infty} D_n$. Assume that $D_0 \subset D_1 \subset ...$ and that D is dense in $[0, 1]$ (so that $\max\{d_{n,j} - d_{n,j-1};\ j = 1, ..., r_n\} \to 0$). For each $n \in N_0$ let $\mathscr{D}_n$ be the system of all intervals $[d_{n,j-1}, d_{n,j}]$ $(j = 1, ..., r_n)$. Let $\mathfrak{D}$ denote the sequence $D_0, D_1, ...$.

Let $n \in N_0$. For $x \in (0, 1]$ define $\alpha_n(x)$ and $\beta_n(x)$ by $\alpha_n(x) < x \leqq \beta_n(x)$ and $[\alpha_n(x), \beta_n(x)] \in \mathscr{D}_n$; for $x \in [0, 1)$ define $\alpha_n^*(x)$ and $\beta_n^*(x)$ by $\alpha_n^*(x) \leqq x < \beta_n^*(x)$ and $[\alpha_n^*(x),\ \beta_n^*(x)] \in \mathscr{D}_n$. Further set $\alpha_n(0) = \beta_n(0) = 0$, $\alpha_n^*(1) = \beta_n^*(1) = 1$ $(n \in N_0)$.

For each $x \in [0, 1]$ and each $n \in N_0$ set $J_n(x) = [\alpha_n(x), \beta_n(x)]$, $J_n^*(x) = [\alpha_n^*(x), \beta_n^*(x)]$ (thus $J_n(0) = \{0\}$, $J_n^*(1) = \{1\}$).

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Remark. If $x\in D_n$, then $\alpha_n^*(x)=\beta_n(x)=x$; if $x\in[0, 1]-D_n$, then $J_n(x)=$ $=J_n^*(x)$.

2.2. For each $n\in N_0$ let V_n be the system of all functions f on $[0, 1]$ with the following properties:

1) f is constant on int J for each $J\in\mathscr{D}_n$;

2) $f(x)=\frac{1}{2}(f(x+)+f(x-))$ for each $x\in(0, 1)$;

3) $f(0)=f(0+)$, $f(1)=f(1-)$.

Obviously $V_0\subset V_1\subset\ldots$. Set $V=\bigcup_{n=0}^{\infty} V_n$. Then V is a vector space and V_n is an r_n-dimensional subspace of V $(n\in N_0)$. It is easy to see that V becomes an inner product space, if we define the inner product of any elements f, g of V as $\int_0^1 fg$.

2.3. Let W be a system of functions on $[0, 1]$ and let I be a function on W with the following properties:

4) If $f_1, f_2\in W$, $a_1, a_2\in R$, then $a_1f_1+a_2f_2\in W$ and $I(a_1f_1+a_2f_2)=a_1I(f_1)+$ $+a_2I(f_2)$;

5) $V\subset W$ and $I(f)=\int_0^1 f$ for each $f\in V$;

6) if $f\in V$ and $g\in W$, then $fg\in W$;

7) if f is a function on $[0, 1]$ such that $f(x)=0$ for almost all $x\in[0, 1]$, then $f\in W$ and $I(f)=0$.

2.4. Let $f, g\in V$. Then $fg\in W$ and $\int_0^1 fg=I(fg)$.

Proof. There is an $h\in V$ such that $f(x)g(x)=h(x)$ for almost all $x\in[0, 1]$. Now we apply 5), 7) and 4).

2.5. Let T be a finite-dimensional subspace of V. Let $f\in W$. Then there is a unique $g\in T$ such that $I(f\varphi)=I(g\varphi)$ for each $\varphi\in T$. If functions $\varphi_1, \ldots, \varphi_m$ form an orthonormal basis of T, then $g=\sum_{j=1}^{m} I(f\varphi_j)\varphi_j$.

Proof. Easy.

2.6. (i) The element g of 2.5 (the orthogonal projection of f to T) will be denoted by o.p.(f, T).

(ii) Let $T_0=V_0$. For each $n\in N$ let T_n be the set of all elements f of V_n such that $I(fg)=0$ for each $g\in V_{n-1}$.

(iii) Any series $\sum_{n=0}^{\infty} f_n$ where $f_n\in T_n$, will be called a $\mathfrak{D}$-series.

(iv) Let $f \in W$. Then the series $\sum_{n=0}^{\infty} \text{o.p.}(f, T_n)$ will be called the $I\mathfrak{D}F$-series of f (F suggests Fourier).

The proofs of the next three assertions are left to the reader.

2.7. Let $n \in N$, $f \in V_n$. Then the following three conditions are equivalent to each other:

(i) $\int_0^x f = 0$ for each $x \in D_{n-1}$;

(ii) $\int_J f = 0$ for each $J \in \mathscr{D}_{n-1}$;

(iii) $f \in T_n$.

Remark. If $J \in \mathscr{D}_n \cap \mathscr{D}_{n-1}$ and if $f \in T_n$, then $f = 0$ on int J.

2.8. Let $f \in W$, $g \in V$, $n \in N_0$. Then the following three conditions are equivalent to each other:

(i) $\int_0^x g = I(fc_{[0,x]})$ for each $x \in D_n$;

(ii) $g(x)|J| = I(fc_J)$ for each $J \in \mathscr{D}_n$ and each $x \in \text{int}\, J$;

(iii) $g = \text{o.p.}(f, V_n)$.

2.9. Let $f_k \in V$ $(k = 0, 1, \ldots)$, $f \in W$. Then $\sum_{k=0}^{\infty} f_k$ is the $I\mathfrak{D}F$-series of f iff $\sum_{k=0}^{n} f_k = \text{o.p.}(f, V_n)$ for each $n \in N_0$.

3. Auxiliary theorems

3.1. Throughout the paper, $\sum_{n=0}^{\infty} f_n$ is a $\mathfrak{D}$-series. We set

$$s_n = \sum_{n=0}^{n} f_k, \quad F_n(x) = \int_0^x f_n \quad (n \in N_0,\ x \in [0, 1]).$$

The sum of the series $\sum_{n=0}^{\infty} F_n(x)$ will be denoted by $F(x)$ at the points of its convergence.

We will often write $\alpha_n, \beta_n, \alpha_n^*, \beta_n^*, J_n, J_n^*$ instead of $\alpha_n(x), \ldots, J_n^*(x)$, respectively.

3.2. Let $n \in N_0$, $x \in D_n$. Then $F(x) = \int_0^x s_n$.

Proof. By 2.7 we have $F_k(x)=0$ for $k>n$. Thus $F(x)=\sum_{k=0}^{n} F_k(x)=\int_0^x s_n$.

3.3. Let $n\in N_0$, $x\in(0,1)-D_n$. Then $s_n(x)=(F(\beta_n)-F(\alpha_n))/(\beta_n-\alpha_n)$.

Proof. It follows from 3.2.

3.4. Let $0<q<1$. Suppose that $\mathfrak{D}$ has the following property: If $n\in N_0$, $J\in\mathscr{D}_n$, $K\in\mathscr{D}_{n+1}$, $K\subset J$, $K\neq J$, then $|K|\leqq q|J|$. Let $x\in[0,1]$ and let

$$\int_{J_n} f_n \to 0 \quad \Big[\int_{J_n^*} f_n \to 0\Big].$$

Then

$$\int_{J_n} s_n \to 0 \quad \Big[\int_{J_n^*} s_n \to 0\Big].$$

Proof. Let $\int_{J_n} f_n \to 0$. We will show that $\int_{J_n} s_n \to 0$. We may suppose that $x>0$. Set

$$b_n=\sup\Big\{\Big|\int_{J_k} f_k\Big|;\ k\geqq n\Big\},\quad B_n=b_0q^n+\ldots+b_{n-1}q+b_n\quad (n\in N_0),\quad B=\limsup B_n.$$

Since $B_n\leqq b_0/(1-q)$, we have $B<\infty$; since $B_{n+1}=qB_n+b_{n+1}$ and $b_n\to 0$, we have $B=qB$ so that $B=0$.

Let $P=\{n\in N;\ J_n\neq J_{n-1}\}$. We may write $P=\{p_1,p_2,\ldots\}$, where $p_1<p_2<\ldots$; further set $p_0=0$. Let $\varepsilon>0$. Since $B=0$, we can find an $m_0\in N$ such that $B_m<\varepsilon$ for each $m\geqq m_0$. Now let $n\in N$, $n\geqq p_{m_0}$. There is an $m\geqq m_0$ such that $p_m\leqq n<p_{m+1}$. Let $j\in N_0$, $j\leqq m$. Obviously $|J_{p_m}|\leqq q^{m-j}|J_{p_j}|$; since f_{p_j} is constant on int J_{p_j} and $p_j\geqq j$, we have

$$\Big|\int_{J_{p_m}} f_{p_j}\Big|\leqq q^{m-j}\Big|\int_{J_{p_j}} f_{p_j}\Big|\leqq q^{m-j}b_j.$$

If $k\in N-P$ and $k\leqq n$, then $J_{k-1}=J_k\supset J_n=J_{p_m}$ whence $\int_{J_{p_m}} f_k=0$. Thus

$$\Big|\int_{J_n} s_n\Big|\leqq\sum_{j=0}^{m}\Big|\int_{J_{p_m}} f_{p_j}\Big|\leqq\sum_{j=0}^{m} q^{m-j}b_j=B_m<\varepsilon.$$

Similarly can be proved that

$$\int_{J_n^*} s_n\to 0 \quad \text{if} \quad \int_{J_n^*} f_n\to 0.$$

3.5. (i) Let $\mu>0$. We say that $\mathfrak{D}$ fulfills (condition) $Q(\mu)$ iff it has the following property: If $n\in N_0$, $J\in\mathscr{D}_n$, $K\in\mathscr{D}_{n+1}$, $K\subset J$, then $|K|\geqq\mu|J|$. (In such a case, obviously, $\mu<1$.) We say that $\mathfrak{D}$ fulfills Q iff it fulfills $Q(\mu)$ for some $\mu>0$.

(ii) For each $x\in(0, 1)$ and each $n\in N$ set

$$\delta_n(x) = |J_{n-1}|^{-1} \min\Big\{\Big|\int_{\alpha_{n-1}}^{\alpha_n} f_n\Big|, \Big|\int_{\beta_n}^{\beta_{n-1}} f_n\Big|\Big\}.$$

Further, we define

$$\delta(x) = \sup\{\delta_n(x)\colon n\in N\}, \quad S(x) = \sup\{|s_n(x)|\colon n\in N_0\}$$

for each $x\in(0, 1)$.

3.6. Let $\mathfrak{D}$ fulfill $Q(\mu)$. Let $x\in(0,1)-D$, $n\in N_0$. Set $\varepsilon=\sup\{\delta_k(x)\colon k>n\}$, $\eta=\sup\{|f_k(x)|\colon k>n\}$ and suppose that $\max\{\varepsilon, \eta\}<\infty$. Further, set $\theta=(\varepsilon+(1-\mu)\eta)/\mu^2$. Then $\sum_{k=0}^{\infty}|F_k(x)|<\infty$ (so that $F(x)$ has a meaning) and there is a $p\geqq n$ such that

$$|F(x)-F(\alpha_n)-(x-\alpha_n)s_p(x)| \leqq (x-\alpha_n)\theta, \tag{1}$$

$$|F(\beta_n)-F(x)-(\beta_n-x)s_p(x)| \leqq (\beta_n-x)\theta. \tag{2}$$

If, moreover, $S(x)<\infty$ and if λ is a number such that $\lambda\geqq(\delta(x)+2S(x))/\mu^2$, then

$$|F(x)-F(\alpha_n)| \leqq (x-\alpha_n)\lambda, \tag{3}$$

$$|F(\beta_n)-F(x)| \leqq (\beta_n-x)\lambda. \tag{4}$$

Proof. Let p be the greatest integer for which $\alpha_p=\alpha_n$. Since $[\alpha_p, \alpha_{p+1}]$ contains some element of $\mathscr{D}_{p+1}$, we have

$$x-\alpha_p > \alpha_{p+1}-\alpha_p \geqq \mu|J_p|. \tag{5}$$

Now choose a $k>p$. Let, e.g.,

$$\Big|\int_{\alpha_{k-1}}^{\alpha_k} f_k\Big| = \delta_k(x)|J_{k-1}|.$$

If $J_k\neq J_{k-1}$, then $\mu|J_{k-1}|\leqq|J_{k-1}-J_k|$; therefore (even if $J_k=J_{k-1}$)

$$\Big|\int_{\alpha_{k-1}}^{\alpha_k} f_k\Big| \leqq |J_{k-1}-J_k|\,\delta_k(x)/\mu.$$

Obviously

$$\Big|\int_{\alpha_k}^{x} f_k\Big| \leqq |f_k(x)|\,|J_k|, \quad F_k(x) = \int_{\alpha_{k-1}}^{\alpha_k} f_k + \int_{\alpha_k}^{x} f_k,$$

so that

$$|F_k(x)| \leqq |J_{k-1}-J_k|\,\delta_k(x)/\mu + |f_k(x)|\,|J_k|. \tag{6}$$

Let $P=\{k\in N;\ k>p,\ J_k\neq J_{k-1}\}$. We may write $P=\{p_1, p_2, \dots\}$, where $p<p_1<p_2<\dots$. If $k>p$, $k\notin P$, then $f_k(x)=0$; thus

$$\sum_{k=p+1}^{\infty} |f_k(x)|\,|J_k| \leqq \eta \sum_{r=1}^{\infty} |J_{p_r}|.$$

It is easy to see that

$$|J_{p_r}| \leqq (1-\mu)|J_{p_{r-1}}| \leqq \dots \leqq (1-\mu)^r |J_p|.$$

Now we get from (6) and (5)

$$\sum_{k=p+1}^{\infty} |F_k(x)| \leqq |J_p|\,\varepsilon/\mu + \eta |J_p|(1-\mu)/\mu \leqq (x-\alpha_p)\theta. \tag{7}$$

Since, by 3.2,

$$\sum_{k=0}^{p} F_k(x) - F(\alpha_p) = \int_{\alpha_p}^{x} s_p = (x-\alpha_p)s_p(x),$$

we have

$$F(x)-F(\alpha_p) = (x-\alpha_p)s_p(x) + \sum_{k=p+1}^{\infty} F_k(x).$$

This together with (7) proves (1).

If, finally, S and λ are as above, then

$$\eta \leqq 2S(x), \quad \theta + S(x) \leqq \big(\delta(x)+S(x)(2-2\mu+\mu^2)\big)/\mu^2 \leqq \lambda$$

and, by (1),

$$|F(x)-F(\alpha_n)| \leqq (x-\alpha_n)\big(\theta+S(x)\big).$$

This proves (3); (2) and (4) can be proved similarly.

3.7. Let $\mathfrak{D}$ fulfill $Q(\mu)$. Let g be a function such that $\operatorname{Dom} g \supset D$. Let $\varepsilon>0$, $r\in N_0$, $J\in\mathscr{D}_r$. Let $A\subset (J-D)\cap \operatorname{Dom} g$ and let

$$|g(x)-g(\alpha_k)| \leqq (x-\alpha_k)\varepsilon, \quad |g(\beta_k)-g(x)| \leqq (\beta_k - x)\varepsilon \quad (k = r, r+1, \dots)$$

for all $x\in A$.

Then

$$|g(y)-g(x)| \leqq |y-x|\,\varepsilon/\mu \quad \text{for all} \quad x, y\in A.$$

Proof. Let $x, y\in A$, $x<y$. Let n be the smallest integer for which $(x, y)\cap D_n \neq \emptyset$. Obviously $n>r$.

1) Let $(x, y)\cap D_n$ contain only one point. Then $\alpha_n(y)=\beta_n(x)$ whence

$$|g(y)-g(x)| \leqq \big|g(y)-g(\alpha_n(y))\big| + \big|g(\beta_n(x))-g(x)\big| \leqq \varepsilon(y-x).$$

2) Let $(x, y)\cap D_n$ contain more points than one. Set $\alpha=\alpha_{n-1}(x)$, $\beta=\beta_{n-1}(x)$. Since $(x, y)\cap D_{n-1}=\emptyset$, we have $\alpha_{n-1}(y)=\alpha$ so that

$$|g(y)-g(x)| \leqq |g(y)-g(\alpha)|+|g(x)-g(\alpha)| \leqq (x+y-2\alpha)\varepsilon;$$

similarly

$$|g(y)-g(x)| \leqq (2\beta-x-y)\varepsilon.$$

If $x+y\leqq\alpha+\beta$, then $x+y-2\alpha\leqq\beta-\alpha$; if $x+y>\alpha+\beta$, then $2\beta-x-y<\beta-\alpha$. Since (x, y) contains some element of $\mathscr{D}_n$, we have $\mu(\beta-\alpha)\leqq y-x$ whence

$$|g(y)-g(x)| \leqq (\beta-\alpha)\varepsilon \leqq (y-x)\varepsilon/\mu.$$

3.8. For each $m>0$ set

$$E_m = \{x\in(0, 1)-D;\ \max\{\delta(x), S(x)\} \leqq m\}.$$

3.9. Let $m>0$. Then $\operatorname{cl} E_m-D=E_m$.

Proof. Obviously $E_m\subset\operatorname{cl} E_m-D$. Now let $x\in\operatorname{cl} E_m-D$ and let $n\in N_0$. There is a $y\in E_m\cap\operatorname{int} J_n(x)$. Thus $f_k(x)=f_k(y)$, $J_k(x)=J_k(y)$ for $k=0, \ldots, n$ so that $|s_n(x)|=|s_n(y)|\leqq m$ and, if $n>0$, also $\delta_n(x)=\delta_n(y)\leqq m$. Therefore $x\in E_m$.

Remark. It follows from 3.6 and 3.9 that if $\mathfrak{D}$ fulfills Q, $F(x)$ exists for each $x\in \bigcup_{m>0} \operatorname{cl} E_m$.

3.10. Let $\mathfrak{D}$ fulfill $Q(\mu)$. Let $m>\max\{|f_0(x)|:\ x\in[0, 1]\}$ and let $x, y\in\operatorname{cl} E_m$. Then

$$|F(y)-F(x)| \leqq |y-x|\, 3m/\mu^3. \tag{8}$$

Proof. Define $\lambda=3m/\mu^2$. Notice that the relations (3), (4) in 3.6 hold for each $x\in E_m$ and each $n\in N_0$.

(i) Let $x, y\in E_m$, $x<y$. Let p be the smallest integer for which $(x, y)\cap D_p\neq\emptyset$.

1) Suppose that $p=0$. Set $\beta=\beta_0(x)$, $\alpha=\alpha_0(y)$. Then $x<\beta\leqq\alpha<y$. By 3.6 we have

$$|F(\beta)-F(x)| \leqq (\beta-x)\lambda, \quad |F(y)-F(\alpha)| \leqq (y-\alpha)\lambda;$$

obviously

$$|F(\alpha)-F(\beta)| = \Big|\int_\alpha^\beta f_0\Big| \leqq (\beta-\alpha)m \leqq (\beta-\alpha)\lambda.$$

Therefore $|F(y)-F(x)|\leqq(y-x)\lambda$.

2) Suppose that $p>0$. Then we apply 3.7 with $g=F$, $\varepsilon=\lambda$, $r=p-1$, etc.

(ii) Let $x, y\in\operatorname{cl} E_m$. If $y\notin D$, then, by 3.9, $y\in E_m$ and we define $y_n=y$ for each $n\in N$. If $y\in D$, we fix a p such that $y\in D_p$ and proceed as follows: For each $n<p$ we choose an arbitrary element $y_n\in E_m$. For each $n\geqq p$ there is a $y_n\in E_m$ such

that either $y=\alpha_n(y_n)$ or $y=\beta_n(y_n)$; by 3.6,

$$|F(y_n)-F(y)| \leqq |y_n-y|\lambda.$$

Thus, in any case, $y_n \to y$ and $F(y_n) \to F(y)$. We find similarly points $x_n \in E_m$ such that $x_n \to x$ and $F(x_n) \to F(x)$. By (i) we have

$$|F(y_n)-F(x_n)| \leqq |y_n-x_n|\lambda/\mu \quad (n \in N);$$

this implies (8).

4. $\mathfrak{D}$-integral

4.1. (i) Let g be a function such that $\mathrm{Dom}\, g \supset D$ and let $x \in [0, 1]$. We say that g is $S\mathfrak{D}$-continuous at x iff $g(\beta_n^*)-g(\alpha_n) \to 0$ ($\beta_n^* = \beta_n^*(x)$ etc.). We set

$$S\mathfrak{D}\bar{g}(x) = \limsup\, (g(\beta_n^*)-g(\alpha_n))/(\beta_n^*-\alpha_n), \quad S\mathfrak{D}\underline{g}(x) = \liminf \ldots;$$

$S\mathfrak{D}g'(x)$ means $S\mathfrak{D}\bar{g}(x)$ provided that $S\mathfrak{D}\bar{g}(x) = S\mathfrak{D}\underline{g}(x) \in R$.

(ii) Let $x \in [0, 1]$ and let g be a function such that $\mathrm{Dom}\, g \supset D \cup \{x\}$. We say that g is $\mathfrak{D}$-continuous at x iff $\lim g(\alpha_n) = g(x) = \lim g(\beta_n^*)$.

(iii) Let $x \in (0, 1)$ and let g be a function such that $\mathrm{Dom}\, g \supset D \cup \{x\}$. By $\mathfrak{D}g'(x)$ we mean the common value of

$$\lim\, (g(\beta_n^*)-g(x))/(\beta_n^*-x) \quad \text{and} \quad \lim\, (g(x)-g(\alpha_n))/(x-\alpha_n)$$

provided that these limits are finite and equal.

4.2. Let ψ, Ψ be functions; set $Z = \mathrm{Dom}\, \Psi$. We say that Ψ is an indefinite $\mathfrak{D}$-integral of ψ iff the following conditions are fulfilled:

1) $\mathrm{Dom}\, \psi = [0, 1]$.

2) $D \subset Z \subset [0, 1]$ and $[0, 1]-Z$ is countable.

3) Ψ is $\mathfrak{D}$-continuous at each point of Z and $S\mathfrak{D}$-continuous at each point of $[0, 1]$.

4) There is a countable system $\mathfrak{A}$ of closed sets such that $Z = \bigcup \mathfrak{A}$ and that Ψ is absolutely continuous on A for each $A \in \mathfrak{A}$.

5) $S\mathfrak{D}\Psi'(x) = \psi(x)$ for almost all $x \in [0, 1]$.

4.3. Let Ψ be an indefinite $\mathfrak{D}$-integral of a function ψ such that $\psi(x)=0$ for almost all $x \in [0, 1]$. Then Ψ is constant.

Proof. Let M be the set of all points $x \in [0, 1]$ such that $\lim \Psi(\alpha_n(x))$ exists. For each $x \in M$ denote this limit by $\Phi(x)$. It follows from 4.2, 3) that Φ is an extension of Ψ. Let Z, $\mathfrak{A}$ be as in 4.2. Let G be the set of all points $x \in (0, 1)$ with the following property: There is an open interval $J \subset M$ such that $x \in J$ and that Φ is constant on J. Then G is open and Φ is constant on each component of G. Set $H = (0, 1)-G$.

Suppose that h is an isolated point of H. Then there are numbers v, w, λ, μ such that $v<h<w$, $\Phi=\lambda$ on (v, h) and $\Phi=\mu$ on (h, w). Obviously $h\in M$, $\Phi(h)=\lambda$. Since (see 4.2, 3)) Φ is $S\mathfrak{D}$-continuous at h, we have $\mu=\lambda$, $\Phi=\lambda$ on (v, w), $h\in G$, which is a contradiction. We see that H has no isolated point.

Suppose that $H\neq\emptyset$. Set $C=[0, 1]-Z$. Obviously,

$$H=(C\cap H)\cup\bigcup_{A\in\mathfrak{A}}(A\cap H).$$

The set C is countable and H is a G_δ-set. Since H has no isolated point, there is, by Baire's theorem, an open interval $J\subset(0, 1)$ and an $A\in\mathfrak{A}$ such that

$$\emptyset\neq J\cap H\subset A. \tag{9}$$

Let $U=(v, w)$ be a component of $J-H=J\cap G$. Then Φ is constant on U. If, e.g., $v\in J$, then $v\in H$, $v\in A\subset Z$ so that Φ is $\mathfrak{D}$-continuous at v. Thus Φ is constant on $J\cap \operatorname{cl} U$. This together with the absolute continuity of Φ on $J\cap H$ implies easily that Φ is absolutely continuous on J. Therefore $\Phi'(x)=S\mathfrak{D}\Phi'(x)=\psi(x)=0$ for almost all $x\in J$. We see that Φ is constant on J. It follows that $J\subset G$ which contradicts (9). Thus $H=\emptyset$, $G=(0, 1)$ so that Φ is constant on $(0, 1)$. Since Φ is $\mathfrak{D}$-continuous at 0 and 1, Φ is constant on $[0, 1]$ and Ψ is constant on Z.

4.4. Let Ψ be an indefinite $\mathfrak{D}$-integral of ψ and let $\gamma\in R$. Then $\gamma\Psi$ is an indefinite $\mathfrak{D}$-integral of $\gamma\psi$.

Proof. Easy.

4.5. Let Ψ_j be an indefinite $\mathfrak{D}$-integral of ψ_j $(j=1, 2)$. For any $x\in\operatorname{Dom}\Psi_1\cap\cap\operatorname{Dom}\Psi_2$ set $\Psi(x)=\Psi_1(x)+\Psi_2(x)$. Then Ψ is an indefinite $\mathfrak{D}$-integral of $\psi_1+\psi_2$.

Proof. Let Z_j, $\mathfrak{A}_j$ correspond to ψ_j, Ψ_j in the sense of 4.2. It is easy to see that $S\mathfrak{D}\Psi'(x)=\psi(x)$ for almost all $x\in[0, 1]$ and that the set $Z=Z_1\cap Z_2$ and the system $\mathfrak{A}$ of all sets $A_1\cap A_2$ $(A_j\in\mathfrak{A}_j)$ satisfy the requirements of 4.2 with respect to $\psi_1+\psi_2$ and Ψ.

4.6. Let Ψ_1, Ψ_2 be indefinite $\mathfrak{D}$-integrals of the same function. Then $\Psi_1(1)-\Psi_1(0)=\Psi_2(1)-\Psi_2(0)$.

Proof. It follows easily from 4.3—4.5.

4.7. A function which has an indefinite $\mathfrak{D}$-integral will be called $\mathfrak{D}$-integrable. Let ψ be such a function and let Ψ be its indefinite $\mathfrak{D}$-integral. According to 4.6, the number $\Psi(1)-\Psi(0)$ does not depend on the choice of Ψ; we call it the $\mathfrak{D}$-integral of ψ and denote it by $\mathfrak{D}\int\psi$.

4.8. Let $A \subset B \subset [0, 1]$. Let g be a function on B, $b \in B$. Let A be closed and let g be absolutely continuous on A. Set $g_1(x)=g(x)$ for $x \in B \cap [0, b]$, $g_1(x)=g(b)$ for $x \in B \cap (b, 1]$. Then g_1 is absolutely continuous on A.

Proof. Let $\varepsilon > 0$. Let us choose a $\delta > 0$ corresponding to ε and the absolute continuity of g on A. If $b \in A$, set $\delta_1 = \delta$; if $b \notin A$, choose an $\eta > 0$ such that $(b-\eta, b+\eta) \cap A = \emptyset$ and set $\delta_1 = \min\{\delta, \eta\}$. Now it is not difficult to prove that δ_1 fulfills the requirements corresponding to ε and the absolute continuity of g_1 on A.

4.9. Let Ψ be an indefinite $\mathfrak{D}$-integral of ψ and let $b \in \operatorname{Dom} \Psi$. Then

$$\mathfrak{D}\int \psi c_{[0,b]} = \Psi(b) - \Psi(0).$$

Proof. Let $\Psi_1(x) = \Psi(x)$ for $x \in [0, b] \cap \operatorname{Dom} \Psi$, $\Psi_1(x) = \Psi(b)$ for $x \in (b, 1] \cap \cap \operatorname{Dom} \Psi$. It is easy to prove (see 4.8) that Ψ_1 is an indefinite $\mathfrak{D}$-integral of $\psi c_{[0,b]}$. Obviously, $\Psi_1(1) - \Psi_1(0) = \Psi(b) - \Psi(0)$.

4.10. Let ψ be a function on [0, 1] whose Denjoy—Perron integral exists; let us denote it by P. Then $\mathfrak{D}\int \psi = P$.

Proof. Let Ψ be an indefinite (Denjoy—Perron) integral of ψ. Then Ψ is continuous on [0, 1], $\Psi'(x) = \psi(x)$ for almost all $x \in [0, 1]$ and there is a countable covering $\mathfrak{A}$ of [0, 1] such that Ψ is absolutely continuous on A for each $A \in \mathfrak{A}$. Since Ψ is continuous, we may suppose that each element of $\mathfrak{A}$ is closed. Therefore Ψ is an indefinite $\mathfrak{D}$-integral of ψ so that $\mathfrak{D}\int \psi = \Psi(1) - \Psi(0) = P$.

5. Recovery of terms of a $\mathfrak{D}$-series from its sum

5.1. Let $\mathfrak{D}$ fulfill Q. (See Section 3.5.) Let g be a function on D. Set

$$A = \{x \in [0, 1]: \text{ either } \; S\mathfrak{D}\bar{g}(x) < \infty \; \text{ or } \; S\mathfrak{D}\underline{g}(x) > -\infty\}.$$

Then $S\mathfrak{D}g'(x)$ exists for almost all $x \in A$.

This is Theorem 6 of Chapter 4 in [2]. The proof uses methods developed in [3, pp. 134—138].

5.2. Let W be the system of all $\mathfrak{D}$-integrable functions. For each $f \in W$ set $I(f) = \mathfrak{D}\int f$. Then W and I fulfill the assumptions of 2.3.

Proof. It follows from 4.2—4.10.

5.3. Let I, W be as in 5.2. Then, instead of $I\mathfrak{D}F$-series, we will say simply $\mathfrak{D}F$-series.

5.4. Theorem. *Let $\mathfrak{D}$ fulfill condition Q. Let* $\sum_{n=0}^{\infty} f_n$ *be a $\mathfrak{D}$-series such that*

$$\max\left\{\left|\int_{J_n(x)} f_n\right|, \left|\int_{J_n^*(x)} f_n\right|\right\} \to 0$$

for each $x \in [0, 1]$. *Let the set* $\{x \in (0, 1)\colon \max\{S(x), \delta(x)\} = \infty\}$ *be countable. Then there is a $\mathfrak{D}$-integrable function f such that* $f(x) = \sum_{n=0}^{\infty} f_n(x)$ *for almost all* $x \in [0, 1]$; $\sum_{n=0}^{\infty} f_n$ *is its $\mathfrak{D}F$-series.*

Proof. Let F have the usual meaning. Set $f(x) = S\mathfrak{D}F'(x)$ for each x for which $S\mathfrak{D}F'(x)$ exists, and $f(x) = 0$ for any other $x \in [0, 1]$. For each $x \in (0, 1) - D$ we have, by 3.3,

$$S\mathfrak{D}\overline{F}(x) = \limsup s_n(x), \quad S\mathfrak{D}\underline{F}(x) = \liminf s_n(x).$$

It follows easily from our assumptions and from 5.1 that

$$f(x) = S\mathfrak{D}F'(x) = \lim s_n(x) = \sum_{k=0}^{\infty} f_k(x) \quad \text{for almost all} \quad x \in [0, 1].$$

Let E_m be as in 3.8; set $Z = D \cup \bigcup_{m=1}^{\infty} E_m$. It is easy to see that $[0, 1] - Z$ is countable. Suppose that $\mathfrak{D}$ fulfills $Q(\mu)$. By 3.6 (with $\lambda = 3m/\mu^2$) F is $\mathfrak{D}$-continuous at each point of E_m for each $m \in N$. For every $x \in [0, 1]$ we have, by 3.2 and 3.4 (with $q = 1 - \mu$),

$$F(\beta_n) - F(\alpha_n) = \int_{J_n} s_n \to 0$$

and, similarly, $F(\beta_n^*) - F(\alpha_n^*) \to 0$. This shows that F is $\mathfrak{D}$-continuous at each point of D (thus at each point of Z) and $S\mathfrak{D}$-continuous at each point of $[0, 1]$.

It follows from 3.9 that $Z = D \cup \bigcup_{m=1}^{\infty} \operatorname{cl} E_m$. Since $E_1 \subset E_2 \subset \ldots$, we infer from 3.10 that F is absolutely continuous on $\operatorname{cl} E_m$ for each $m \in N$. This shows that the restriction of F to Z is an indefinite $\mathfrak{D}$-integral of f. By 3.2 and 4.9 we have

$$\int_0^x s_n = F(x) = F(x) - F(0) = \mathfrak{D}\int f c_{[0,x]} \quad \text{for each} \quad x \in D_n \quad (n \in N_0).$$

Now we apply 2.8 (with $g = s_n$) and 2.9.

Remark 1. Let us keep the assumptions and the notations of 5.4. Then (see 2.8 (ii))

$$f_0(x) = |J_0(x)|^{-1}\mathfrak{D}\int f c_{J_0(x)} \quad \text{for each} \quad x \in (0, 1) - D_0,$$
$$f_n(x) = |J_n(x)|^{-1}\mathfrak{D}\int f c_{J_n(x)} - |J_{n-1}(x)|^{-1}\mathfrak{D}\int f c_{J_{n-1}(x)}$$
$$\text{for each} \quad x \in (0, 1) - D_n \quad (n \in N).$$

Remark 2. If $D_n \cap \operatorname{int} J$ has at most one element for each $J \in \mathscr{D}_{n-1}$ and each $n \in N$, then, obviously, $\delta(x)=0$ for each $x \in (0, 1)$ (which enables us to simplify the assumptions of 5.4).

5.5. In sections 5.5 and 5.6 we suppose that D_n has $n+2$ points; we write $D_n = \{0, 1, c_1, \ldots, c_n\}$ $(n \in N_0)$. Set $\varphi_0(x)=1$ $(x \in [0, 1])$. For each $n \in N$ let φ_n be a function in T_n such that

$$\varphi_n(x) = \left(\frac{|J_n^*(c_n)|}{|J_n(c_n)|\,|J_{n-1}(c_n)|}\right)^{1/2} \quad \text{for} \quad x \in \operatorname{int} J_n(c_n),$$

$$\varphi_n(x) = -\left(\frac{|J_n(c_n)|}{|J_n^*(c_n)|\,|J_{n-1}(c_n)|}\right)^{1/2} \quad \text{for} \quad x \in \operatorname{int} J_n^*(c_n).$$

It is easy to see that the function φ_n forms an orthonormal basis for T_n $(n \in N_0)$. Thus, a series $\sum_{n=0}^{\infty} f_n$ is a $\mathfrak{D}$-series iff there are numbers a_n such that $f_n = a_n \varphi_n$; such a series is the $\mathfrak{D}F$-series of a function f iff $a_n = \mathfrak{D}\int f\varphi_n$ $(n \in N_0)$.

5.6. Theorem. *Let $\mathfrak{D}$ fulfill condition Q. For each $x \in [0, 1]$ define $M(x) = \{n \in N\colon \varphi_n(x) \neq 0\}$. Let $a_0, a_1, \ldots$ be numbers such that*

(10) $$a_n/\varphi_n(x) \to 0 \quad (n \in M(x),\ n \to \infty) \quad \textit{for each} \quad x \in [0, 1]$$

and that the set

$$\left\{x \in [0, 1]\colon \limsup \left|\sum_{k=0}^{n} a_k \varphi_k(x)\right| = \infty\right\}$$

is countable. Then there is a $\mathfrak{D}$-integrable function f such that $f(x) = \sum_{k=0}^{\infty} a_k \varphi_k(x)$ for almost all $x \in [0, 1]$; we have $a_k = \mathfrak{D}\int f\varphi_k$ $(k \in N_0)$.

Proof. Let $x \in [0, 1]$, $n \in M(x)$, $x \neq c_n$. Obviously $x \in [\alpha_n(c_n), \beta_n^*(c_n)]$. Suppose first that $\alpha_n(c_n) \leqq x < c_n$. Then $0 < \varphi_n(x) \leqq \varphi_n(y)$ for each $y \in \operatorname{int} J_n^*(x)$ so that

$$\varphi_n(x) \int_{J_n^*(x)} \varphi_n < \int_0^1 \varphi_n^2 = 1.$$

If $x = \alpha_n(c_n)$, then $\int_{J_n(x)} \varphi_n = 0$; if $\alpha_n(c_n) < x < c_n$, then $J_n(x) = J_n^*(x)$. Thus

(11) $$\max\left\{\left|\int_{J_n(x)} \varphi_n\right|, \left|\int_{J_n^*(x)} \varphi_n\right|\right\} < 1/|\varphi_n(x)|.$$

In a similar way we can prove (11) for $c_n < x \leqq \beta_n^*(c_n)$. If $n \notin M(x)$, $x \neq c_n$, then

$$\int_{J_n(x)} \varphi_n = \int_{J_n^*(x)} \varphi_n = 0.$$

This shows that

$$\max\left\{\left|\int_{J_n(x)} a_n\varphi_n\right|, \left|\int_{J_n^*(x)} a_n\varphi_n\right|\right\} \to 0. \tag{12}$$

Obviously $\delta(x)=0$ for each $x\in(0,1)$. Now we apply 5.4.

Remark. It is not difficult to prove that (under the assumptions of 5.6) conditions (10) and (12) are equivalent for each $x\in[0,1]$.

6. Additional remarks

6.1. Let ψ, Ψ be functions. We say that Ψ is an indefinite $\mathfrak{D}_{as}$-integral of ψ iff the following holds:

(i) The requirements 1)—4) of 4.2 (with $Z=\operatorname{Dom}\Psi$) are fulfilled;

(ii) $\Psi'_{as}(x)=\psi(x)$ for almost all $x\in[0,1]$.

The reader can easily formulate the analogues of sections 4.3—4.7, 4.9, 4.10, 5.2 and 5.3 for the $\mathfrak{D}_{as}$-integral. In the analogue of 4.10 we may even replace the Denjoy—Perron integral by the Denjoy—Khintchine integral. The modification of the proofs is trivial.

6.2. Let $\mathfrak{D}$ fulfill $Q(\mu)$. Let B be a measurable set, $D\subset B\subset(0,1)$. Let h be a function measurable on B such that $\mathfrak{D}h'(x)$ exists for all $x\in B$. Then $h'_{as}(x)$ exists and equals $\mathfrak{D}h'(x)$ for almost all $x\in B$.

Proof. We may suppose that $B\cap D=\emptyset$. Let r_n and $d_{n,j}$ be as in 2.1. Let Ω be a countable dense subset of R. Choose an $\varepsilon>0$. For $n\in N_0$, $j=1,\dots,r_n$ and $\omega\in\Omega$ let $B(n,j,\omega)$ be the set of all points $x\in B\cap(d_{n,j-1},d_{n,j})$ such that

$$\left|\frac{h(x)-h(\alpha_k)}{x-\alpha_k}-\omega\right|<\varepsilon,\quad \left|\frac{h(\beta_k)-h(x)}{\beta_k-x}-\omega\right|<\varepsilon\quad (k=n,n+1,\dots).$$

It is easy to see that the sets $B(n,j,\omega)$ cover B. Now choose n,j,ω as above and set $A=B(n,j,\omega)$. According to 3.7 with $g(x)=h(x)-\omega x$, $r=n$ etc, we have

$$\left|\frac{h(y)-h(x)}{y-x}-\omega\right|\leqq\varepsilon/\mu\quad (x,y\in A,\ x\neq y).$$

We see, first of all, that h is absolutely continuous on A. It follows that $h'_{as}(x)$ exists and that $|h'_{as}(x)-\omega|\leqq\varepsilon/\mu$ for almost all $x\in A$. Obviously $|\mathfrak{D}h'(x)-\omega|\leqq\varepsilon$ for all $x\in A$. Therefore $|h'_{as}(x)-\mathfrak{D}h'(x)|<2\varepsilon/\mu$ for almost all $x\in A$, $|h'_{as}(x)-\mathfrak{D}h'(x)|<$ $<2\varepsilon/\mu$ for almost all $x\in B$ and, finally, $h'_{as}(x)=\mathfrak{D}h'(x)$ for almost all $x\in B$.

6.3. Theorem. *Let all the assumptions of* 5.4 *be fulfilled. Suppose, moreover, that* $\delta_n(x)\to 0$ *for almost all* $x\in(0,1)$. *Let f be as in* 5.4. *Then f is* $\mathfrak{D}_{as}$*-integrable and* $\sum_{n=0}^{\infty} f_n$ *is its* $\mathfrak{D}_{as}F$*-series.*

Proof. If $x\in(0,1)-D$, $\sum_{n=0}^{\infty} f_n(x)=f(x)$ and if $\delta_n(x)\to 0$, then, according to 3.6 (see (1) and (2)) we have $\mathfrak{D}F'(x)=f(x)$. It follows from 6.2 that $F'_{as}(x)=f(x)$ for almost all $x\in[0,1]$. Further, we proceed as in the proof of 5.4.

6.4. Theorem. *Let all the assumptions of* 5.6 *be fulfilled. Let f be as in* 5.6. *Then f is* $\mathfrak{D}_{as}$*-integrable and* $a_k=\mathfrak{D}_{as}\int f\varphi_k$ $(k\in N_0)$.

The proof is left to the reader.

6.5. Let $c_1=\frac{1}{2}$, $c_2=\frac{1}{4}$, $c_3=\frac{3}{4}$, $c_4=\frac{1}{8}$, $c_5=\frac{3}{8}$, $c_6=\frac{5}{8}$, $c_7=\frac{7}{8}$, $c_8=\frac{1}{16}$, ..., $c_{15}=\frac{15}{16}$, ..., $c_{2^n}=1/2^{n+1}$, ...; set $D_n=\{0,1,c_1,\dots,c_n\}$ $(n\in N_0)$. Let $\varphi_0,\varphi_1,\dots$ be as in 5.5. Then $\mathfrak{D}$ fulfills condition $Q\left(\frac{1}{2}\right)$, φ_n are the Haar functions, a $\mathfrak{D}$-series is a Haar series and the $\mathfrak{D}_{as}$-integral is the HD-integral defined in [4]. We see that our assertion 6.4 is a generalization of Theorem 2 in [5] (which, in turn, is a generalization of Theorem 4 in [4]).

6.6. Let $D_n=\{k/2^n;\ k=0,1,\dots,2^n\}$ $(n\in N_0)$ and let f be a Perron integrable function on [0, 1]. Let $\sum a_n\chi_n$ and $\sum b_n\psi_n$ be the Haar- and Walsh-Fourier series of f, respectively. Let $n\in N_0$ and let $m=2^n$. As $\chi_0,\dots,\chi_{m-1}$ is an orthonormal basis of V_n and as the same is true for $\psi_0,\dots,\psi_{m-1}$, we have

$$\sum_{k=0}^{m-1} a_k\chi_k = \text{o.p.}(f, V_n) = \sum_{k=0}^{m-1} b_k\psi_k$$

(see [6]).

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Об одном классе ортогональных рядов

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Предположим, что задана такая последовательность разбиений отрезка [0, 1], что $(n+1)$ разбиение всегда мельче n-го. Такая последовательность естественным образом порождает последовательность попарно ортогональных пространств кусочно постоянных функций. Некоторые свойства соответствующих ортогональных рядов изучались в работе [2]. Цель настоящей работы — найти при некоторых дополнительных предположениях члены такого ряда исходя из его суммы (см. 5.4 и 5.6). Некоторая модификация этих результатов приводит в разд. 6.4 к обобщению теоремы 2 из работы [5]. В наших доказательствах часто используются соображения, разработанные в [4]. Некоторые близкие вопросы исследовались, например, в [6] и [7]. Основные результаты работы без доказательства были сформулированы в [1].

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