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MELLIN ANALYSIS OF SINGULAR AND NON–LINEAR PDE'S

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ABSTRACT. The article gives a review of the applications of the Mellin transformation to the geometric study of singular and non linear PDEs. Among the topics covered is the theory of second microlocalization and the radial regularity of solutions to Fuchsian type PDEs.

With the discovery of the notion of singular spectrum and of the related concept of a microfunction in the early 60-ties Miko Sato opened way to microlocal analysis and to the geometric study of singularities of solutions to linear PDEs. The essential idea of microlocal study of a function (distribution) u is to localize it first to a neighbourhood of a point $\mathring{x}$ by means of a bump function φu and then to study the behavior of the Fourier transform $\mathcal{F}(\varphi u)(\xi) = \varphi u[e^{ix\xi}]$ in a conical neighbourhood of a vector $\mathring{\xi}$. If $\mathcal{F}(\varphi u)$ is rapidly decreasing then we say that $(\mathring{x}, \mathring{\xi})$ does not belong to the wave front set of u (denoted $(\mathring{x}, \mathring{\xi}) \notin WFu$). The wave front set of a distribution u is an important invariant describing the local singularities of u. Geometrically it is a subset of the cotangent bundle $T^*(\mathbb{R}^n)$ and its behaviour under the action of linear PDEs can be estimated by means of the principal symbol of the operator: recall that if

$$P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha \quad \text{with} \quad D^\alpha = \frac{1}{i^{|\alpha|}} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

then the principal symbol $\sigma_P(x,\xi)$ is defined as

$$\sigma_P(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha .$$

The fundamental result reads (microelliptic regularity):

If $Pu = f$ then

$$WFu \subset WFf \cup \operatorname{Char} P \tag{$*$}$$

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where $\operatorname{Char} P = \{(x,\xi) \in T^*(\mathbb{R}^n) \colon \sigma_P(x,\xi) = 0\}$ is the characteristic set of P. In the case of strictly hyperbolic operators there is a refinement of $(*)$ stating that the set

$$WFu \setminus WFf \subset \operatorname{Char} P$$

is invariant under the Hamiltonian flow

$$H_P = \sum_{j=1}^{n} \frac{\partial \sigma_P}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial \sigma_P}{\partial x_j} \frac{\partial}{\partial \xi_j}.$$

This means that if $(\mathring{x}, \mathring{\xi}) \notin WF(u-f)$ then $\big(x(s), \xi(s)\big) \notin WF(u-f)$ for the bicharacteristic curve passing through $(\mathring{x}, \mathring{\xi})$.

The methods of microlocal analysis (based on the techniques of pseudodifferential and Fourier integral operators) were very useful in a whole range of problems concerning the existence and regularity of solutions to linear PDEs. However they break down almost completely in the case of non-linear and linear singular equations. The purpose of my talk is to address those two classes of equations which in a sense are closely related:

i) Singular operators:

An operator P is called *singular* if the coefficient $a_\alpha(x)$ for $|\alpha| = m$ vanish on a certain set (called the singular set of P). The Hamilton flow degenerates (or vanishes) on that set and we have no information on the way the solution approaches the singular set (also on the behaviour of solutions at infinity).

ii) Non-linear operators:

Non-linear singularities need not travel along bicharacteristic curves. In other words different non-linear waves may interact producing new singular waves. The most fundamental example here is the case of a semilinear wave operator in two space variables

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(t,x,y,u),$$

f smooth in a neighbourhood of $(t,x,y) = (0,0,0)$ ([B]).

To study those classes we need more information on the way the solution approaches the singular set (the point zero). This is acheaved by a further localization of solution called the second microlocalization. It was introduced in the 70-ties by M. Kashiwara to study systems of singular linear PDE the so called holonomic systems which can be regarded as a generalization of ordinary Fuchsian differential operators. In the case where the singular set reduces to the origin, second microlocalization consists (roughly speaking) in localizing u (in addition to microlocalizing it) to a conical neighbourhood (in the space x) of a vector $\delta \mathring{x}$ by means of a conical cut of function κ' with $\kappa'(\delta \mathring{x}) \neq 0$. The effect is then measured in terms of the scale of suitable weighted Sobolev spaces. For instance one may use the space $SP(s, s')$ defined as follows

DEFINITION. Let $u \in D'(\mathbb{R}^n \setminus \{0\})$, $u \equiv 0$ outside the unit ball in $\mathbb{R}^n$. Let $s, s' \in \mathbb{R}$ be such that $s+s'$ is a non-negative integer. We say that $u \in SP(s, s')$ if

$\|x\|^{-s+|\lambda|} D^\lambda u \in L^2(\mathbb{R}^n)$ for $0 \le |\lambda| \le s+s'$, $\lambda \in \mathbb{R}^n$, $|\lambda| = \lambda_1 + \cdots + \lambda_n$.

For remaining (s, s') the spaces $SP(s, s')$ are defined by duality and interpolation.

Fix $\delta \mathring{x} = (\delta \mathring{x}_1, \ldots, \delta \mathring{x}_n) \in \mathbb{R}^n_+ \stackrel{df}{=} (\mathbb{R}_+)^n$, $\mathring{\xi} = (\mathring{\xi}_1, \ldots, \mathring{\xi}_n) \in \mathbb{R}^n$ and let

1^o $\rho \in C^\infty(i(\mathbb{R}^n \setminus \{0\}))$ be a homogeneous function of order zero defined in a conical neighbourhood of $i\mathring{\beta} = (i\delta \mathring{x}_1 \mathring{\xi}_1, \ldots, i\delta \mathring{x}_n \mathring{\xi}_n)$, $\rho(i\mathring{\beta}) \neq 0$. The function ρ is extended to $\mathbb{R}^n + i(\mathbb{R}^n \setminus \{0\})$ by putting $\rho(z) = \rho(\operatorname{Im} z)$.

2^o $\kappa = \varphi \cdot \kappa'$ where φ is a C_0^∞ bump function at zero and $\kappa' \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is a cut-off function in a conical neighbourhood of $\delta \mathring{x}$, i.e., $\operatorname{supp} \kappa' \subset \mathbb{R}^n_+$, κ' is homogeneous of order zero and $\kappa'(\delta \mathring{x}) \neq 0$.

3^o $\chi \in C^\infty(i(\mathbb{R}^n \setminus \{0\}))$ be a function with $\operatorname{supp} \chi \subset \{i\tau \colon 1/4 \le \|\tau\|\}$ and $\chi(i\tau) \equiv 1$ for $\|\tau\| \ge \frac{1}{2}$. The function χ is extended to $\mathbb{R}^n + i(\mathbb{R}^n \setminus \{0\})$ by putting $\chi(z) = \chi(\operatorname{Im} z)$.

DEFINITION. Let $(\mathring{x}, \mathring{\xi}) \in T_0^*(\mathbb{R}^n) = \{(0, \xi) \colon \xi \in \mathbb{R}^n\}$, $\delta \mathring{x} \in \mathbb{R}^n_+$, $\delta \mathring{\xi} = 0$. Let $u \in SP(s, -\infty) = \bigcup_{\sigma'} SP(s, \sigma')$. We say that *$u$ belongs to $SP(s, s')$ 2-microlocally at the point* $(\mathring{x}, \mathring{\xi}, \delta \mathring{x}, \delta \mathring{\xi}) = (0, \mathring{\xi}, \delta \mathring{x}, 0)$ if there exist functions ρ, κ, χ satisfying conditions 1^o, 2^o, 3^o respectively such that

$$\widetilde{P}(x, D)u = \chi(xD)\rho(xD)\kappa(x)u \in SP(s, s') \qquad (xD = (x_1 D_1, \ldots, x_n D_n)).$$

If $u \in SP(s, -\infty)$ we define its *$SP(s, s')$-second wave front set* (denoted $2\,WF_{SP(s,s')}$) as a closed subset of the space

$$(T^*\mathbb{R}^n \setminus T_0^*(\mathbb{R}^n)) \cup N_{T_0^*(\mathbb{R}^n)} T(T^*\mathbb{R}^n)$$

$(N_{T_0^*(\mathbb{R}^n)} T(T^*\mathbb{R}^n) = T(T^*\mathbb{R}^n)/T(T_0^*(\mathbb{R}^n))$ — the space of normal vectors to $T_0^*(\mathbb{R}^n))$ consisting of the points $(x, \xi) \notin T_0^*(\mathbb{R}^n)$ such that $u \notin SP(s+s', 0)$ microlocally at (x, ξ) (observe that outside zero $SP(s, s') = SP(s+s', 0)$ coincides with the usual Sobolev space) and of the points $(0, \xi, \delta x, 0)$ such that $u \notin SP(s, s')$ 2 microlocally. (Note that in our case the space normal to $T_0^*(\mathbb{R}^n)$ at the point $(0, \mathring{\xi})$ can be identified with the set of vectors of the form $(\delta \mathring{x}, 0)$ with $\delta \mathring{x} \in \mathbb{R}^n$).

This definition however is not practical and one would like to have a definition (similar to that of the (first) wave front set) in term of growth conditions (rapid decay) of a suitable transformation. Clearly the Fourier transformation is not convenient here since the spaces $SP(s, s')$ are not isotropic (in his original paper [B] M. B o n y uses complicated techniques of Paley–Littlewood decompositions to overcome this difficulty). Instead, I suggest the approach based on the Mellin transformation which is adapted to the geometry of the singular set (point zero) [Z-K].

LEMMA. *Let $u \in SP(s,s')$, $\operatorname{supp} u \subset \Gamma$ (Γ a proper cone in $\mathbb{R}^n_+$) then*

$$\mathcal{M}u(z) = u[x^{-z-1}] \qquad (x^{-z-1} = x_1^{-z_1-1}\dots x_n^{-z_n-1})$$

is well defined and holomorphic for $z \in \mathbb{C}^n$ such that $\Sigma \operatorname{Re} z_j < s - \frac{n}{2}$.

$\mathcal{M}u(z)$ is called the *Mellin transform* of u (it is a distributional generalization of the classical formula $\int_0^\infty u(x)\,x^{-z-1}dx$ with the sign of z changed).

We have the following characterization of the space $SP_\Gamma(s,s')$ of distributions in $SP(s,s')$ supported by a proper cone $\Gamma \subset \mathbb{R}^n_+$: $u \in SP(s,s')$, $s+s' \in \mathbb{N}_0$ iff

i) $\mathcal{M}u \in \mathcal{O}(\Sigma \operatorname{Re} z_j < s - \frac{n}{2})$.
ii) $\mathcal{M}(\alpha + i\cdot) \in L^{2,s+s'}(\mathbb{R}^n) = L^2(\mathbb{R}^n, (1+\|\beta\|)^{s+s'})$ for α such that $\Sigma\alpha_j \le s - \frac{n}{2}$.

This leads to the following equivalent definition of the second wave front set.

DEFINITION. Let $(0,\mathring{\xi}) \in T_0^*(\mathbb{R}^n)$, $\delta\mathring{x} \in \mathbb{R}^n_+$, $u \in SP(s,-\infty)$. $u \in SP(s,s')$ *2 microlocally at the point* $(0,\mathring{\xi},\delta\mathring{x},0)$ if and only if there exist functions χ,κ,ρ satisfying the conditions $1^o, 2^o, 3^o$ such that

$$\chi(z)\rho(z)\mathcal{M}(\kappa u)(z)|_{z=\alpha+i\cdot} \in L^{2,s+s'}(\mathbb{R}^n) \qquad \text{for} \quad \alpha_1 + \dots + \alpha_n \le s - \frac{n}{2}.$$

COROLLARY. *The point $(0,\mathring{\xi},\delta\mathring{x},0)$ does not belong to $2\,WF_{SP(s,\infty)}$ iff for some χ,κ,ρ $\chi(z)\rho(z)\mathcal{M}(\kappa u)(z)$ as a function of $\operatorname{Im} z$ is rapidly decreasing for z with* $\operatorname{Re} z_1 + \dots + \operatorname{Re} z_n < s - \frac{n}{2}$.

Propagation of 2 microlocal singularities:

THEOREM 1. (Propagation of singularities along the incoming bicharacteristic) *Let $\delta\mathring{x} = (-1,0)$, $\mathring{\xi} = (0,\mathring{\xi}')$ and let $v \in SP(s,-\infty)$. Suppose $v \in H^{s+\sigma}$ microlocally at $(\mathring{x},\mathring{\xi})$ for every $\mathring{x} = (x_1,0)$, $x_1 < 0$ and $w = \frac{\partial}{\partial x_1}v \in SP(s-1,\sigma+1)$ 2 microlocally at $(0,\mathring{\xi},\delta\mathring{\xi},0)$. If $\sigma > -\frac{1}{2}$ then $v \in SP(s,\sigma)$ 2 microlocally at $(0,\mathring{\xi},\delta\mathring{x},0)$.*

THEOREM 2. (Propagation of singularities along second bicharacteristics) *Let Γ be a cone in $\mathbb{R}^n$ not tangent to x_1. Suppose $v \in SP(-\infty,-\infty)$ on Γ and let $\frac{\partial}{\partial x_1}v = w \in SP(s-1,\sigma+1)$ 2 microlocally at $(0,\mathring{\xi},\delta x,0)$ for $\delta x \in \Gamma$, $\mathring{\xi} = (0,\mathring{\xi}')$ for some fixed $\mathring{\xi}'$. If $v \in SP(s,\sigma)$ 2 microlocally at $(0,\mathring{\xi},\delta x,0)$ for $\delta x \in \Gamma'$ — a subcone of Γ then $v \in SP(s,\sigma)$ 2 microlocally at $(0,\mathring{\xi},\delta x,0)$ for $\delta x \in \Gamma$.*

THEOREM 3. (Propagation of singularities along the outgoing bicharacteristics) *Let $\delta\mathring{x} = (1,0)$, $\mathring{\xi} = (0,\mathring{\xi}')$ and let $v \in SP(s,-\infty)$ be such that $v \in SP(s,\sigma)$ 2-microlocally at $(0,\mathring{\xi},\delta x,0)$ where $\delta x \in \mathbb{R}^n_+$. Suppose $w = \frac{\partial}{\partial x_1}v$*

and $w \in SP(s-1,\sigma+1)$ *2 microlocally at* $(0,\overset{\circ}{\xi},\delta\overset{\circ}{x},0)$. *If* $\sigma < -\frac{1}{2}$ *then* $v \in SP(s,\sigma)$ *2 microlocally at* $(0,\overset{\circ}{\xi},\delta\overset{\circ}{x},0)$.

We have seen that the condition $u \in SP_\Gamma(s,-\infty)$ implies that $\mathcal{M}u$ is holomorphic for $\Sigma \operatorname{Re} z_j < s-\frac{n}{2}$. Further analysis of singularities of u is done by the study of analytic continuation of $\mathcal{M}u$ to $\mathbb{C}^n$. This is achieved by introducing subspaces $M(\Omega,\rho)$ of $SP(s,s')$ of distributions with continuous radial asymptotic defined as follows. Introduce variables

$$\zeta_1 = z_1 + \cdots + z_n\,,$$
$$\zeta_j = z_j \quad \text{for} \quad j = 2,\dots,n\,,$$

(this corresponds to the change of radial variables

$$x_1 = y_1, \quad x_j = y_1 y_j \quad \text{for} \quad j = 2,\dots,n)\,.$$

Let $\Omega = \Omega^1 \times \mathbb{C}^{n-1}$ where $\Omega^1 \subset \mathbb{C}$ is the complement of a number of half-lines $\overset{\circ}{\zeta} + \mathbb{R}_+,\dots,\zeta^k + \mathbb{R}_+$. Further introduce a "nondecreasing" function ρ: $\operatorname{Re}\Omega_1 \to \mathbb{R}$ (which will be used to measure the growth of the Mellin transforms in the imaginary direction, similarly to the spaces $L^{2,s}$).

We say that a *(Mellin) distribution* u *supported by a proper cone* Γ *belongs to* $M(\Omega,\rho)$ if

i) $\mathcal{M}u(\zeta) \in \mathcal{O}(\Omega)$,
ii) $|\mathcal{M}u(a+ib)| \le C(1+\|b\|)^{\rho(a_1)}$
locally uniformly in a_1 and a' (outside the set $\Lambda = \mathbb{C}\backslash\Omega^1$).

The spaces $M(\Omega,\rho)$ allow us to generalize the concept of the second wave front set (by studying the growth in the imaginary direction of the analytic continuation of the Mellin transformation). Moreover new information is contained in the singular set $\Lambda = \mathbb{C}^n\backslash\Omega$ of the Mellin transform. The set Λ controls the asymptotic expansion of u at the origin with respect to the radial variable.

Since the set Λ^1 is a finite number of half lines we may compute the difference of the boundary values $T_1,\dots,T_k$ of $\mathcal{M}u$ across these line. $T_1,\dots,T_k$ usually are distributions and they turn out to be a generalization of the classical Borel transformation. This leads to the following Borel resumation of u

$$u(y_1,y') \simeq T_1(y')[y_1^\theta] + \cdots + T_k(y')[y_1^\theta]$$

which gives the asymptotic expansion of u with respect to y_1.

For instance in the case of the so called corner operator (i.e., operators of the form

$$R = \Sigma a_\alpha(x)\widetilde{D}^\alpha \quad \text{with} \quad \widetilde{D}^\alpha = \Big(x_1\frac{\partial}{\partial x_1}\Big)^{\alpha_1}\cdots\Big(x_n\frac{\partial}{\partial x_n}\Big)^{\alpha_n}$$

we have the following "elliptic regularity result":

THEOREM. *Let* $w \in M(\Omega, s)$ $(\rho \equiv s - \text{const})$ *2 locally at* $(0, \delta \mathring{x})$. *Then for every* $\overset{*}{\alpha} \in \mathbb{R}^n$ *with* $\{\operatorname{Re} \zeta_1 < \overset{*}{\alpha}_1\} \subset \Omega^1$ *there exists* $u_{\overset{*}{\alpha}} \in M\big(\Omega \setminus \bigcup_{j=0}^{\infty} (\operatorname{char}_{\overset{*}{\alpha}} R + j), s - m\big)$ *2 locally at* $(0, \delta \mathring{x})$ *such that* $R u_{\overset{*}{\alpha}} = w$ *2 locally at* $(0, \delta \mathring{x})$.

Here $\operatorname{char}_{\overset{*}{\alpha}} R$ is the so called radial characteristic set of R (see [S-Z] and [2]) defined by means of the principal Mellin symbol of R

$$\big(R = P(\widetilde{D}) + \Sigma x_j Q(x, \widetilde{D})\big)$$

which generalizes the notion of the characteristic roots for ordinary Fuchsian differential operators.

Still deeper analysis of solutions to singular operators will be achieved if instead of radial we study complete asymptotic (we do not 2–localize). This is done by applying a transformation adapted to the geometry of the singular set. In the case of a corner operator this is the multiple Mellin transformation and the analysis leads to study of holomorphic continuation of functions of several complex variables and computing their boundary values by applying the techniques of the hyperfunction theory.

The program outlined here can be regarded as a multidimensional version of the study of linear and non-linear singular ordinary differential equations conducted by means of the resumation techniques by Malgrange, Ramis, Shibuya, Ecalle. In particular the resurgence effects of J. Ecalle have their multidimensional counterparts (characteristic resurgence) which allows one to expect that the program may be successful in the study of local singularities of solutions to non-linear PDEs.

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