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MATHEMATICAL AND NUMERICAL STUDY OF NONLINEAR PROBLEMS IN FLUID MECHANICS

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INTRODUCTION

The study of flow problems in their generality is very difficult since real flows are three-dimensional, nonstationary, viscous with large Reynolds numbers, rotational, turbulent, sometimes also more-fase and in regions with a complicated geometry. Therefore, we use simplified, usually two-dimensional and non-viscous models. (The effects of viscosity are taken into account additionally on the basis of the boundary layer theory.)

Here we give a surway of results obtained in the study of boundary value problems describing two-dimensional, non-viscous, stationary or quasistationary incompressible or subsonic compressible flows with the use of a stream function.

1. STREAM FUNCTION FORMULATION OF THE PROBLEM

On the basis of a detailed theoretical and numerical analysis of various types of flow fields (plane or axially symmetric channel flow, flow past an isolated profile, cascade flow etc.) a unified conception for the stream function-finite element solution of flow problems was worked out.

We start from the following assumptions:

1) The domain $\Omega \subset R_2$ filled by the fluid is bounded with a piecewise smooth, Lipschitz-continuous boundary $\partial\Omega$. (Usually Ω has the form of a curved channel with inserted profiles.)

2) $\partial\Omega = \Gamma_D \cup \Gamma_N \cup (\bigcup_{j=1}^{K_I} \Gamma_I^j) \cup (\bigcup_{j=1}^{K_T} \Gamma_T^j) \cup \Gamma_P^- \cup \Gamma_P^+$, where Γ_I^j, Γ_T^j

are arcs or simple closed curves, Γ_P^+, Γ_P^- are piecewise linear arcs, Γ_P is obtained by translating Γ_P in a given direction by a given distance. This translation is represented by a one-to-one mapping Z_P: $\Gamma_P^- \xrightarrow{\text{onto}} \Gamma_P^+$. Γ_D and Γ_N are formed by finite numbers of arcs. Of course, all these arcs and simple closed curves are mutually disjoint, except neighbouring arcs that have common initial or terminal points. We assume that $\Gamma_D \neq \emptyset$.

3) The differential equation has the form

$$\sum_{i=1}^{2} (b(x,u,(\nabla u)^2)u_{x_i})_{x_i} = f(x,u,(\nabla u)^2) \quad \text{in} \quad \Omega. \tag{1.1}$$

4) We admit the following boundary conditions:

$$u|\Gamma_D = u_D \quad \text{(Dirichlet condition)}, \tag{1.2}$$

$$b(.,u,(\nabla u)^2)\frac{\partial u}{\partial n}|\Gamma_N = -\varphi_N \quad \text{(Neumann condition)}, \tag{1.3}$$

$$u(Z_P(x)) = u(x) + Q, \tag{1.4,a}$$

$$(b(.,u,(\nabla u)^2)\frac{\partial u}{\partial n})(Z_P(x)) = -(b(.,u,(\nabla u)^2)\frac{\partial u}{\partial n})(x) \tag{1.4,b}$$

$\forall\ x \in \Gamma_P^-$ (periodicity conditions),

$$u|\Gamma_I^j = u_I^j + q_I^j, \quad q_I^j = \text{const}, \tag{1.5,a}$$

$$\int_{\Gamma_I^j} b(.,u,(\nabla u)^2)\frac{\partial u}{\partial n}dS = -\gamma_I^j \quad \text{(velocity circulation conditions)} \tag{1.5,b}$$

$j = 1,\dots,K_I$,

$$u|\Gamma_T^j = u_T^j + q_T^j, \quad q_T^j = \text{const}, \tag{1.6,a}$$

$$\frac{\partial u}{\partial n}(z_j) = 0, \quad z_j \in \Gamma_T^j, \quad \text{(Kutta-Joukowski trailing conditions)} \tag{1.6,b}$$

$j = 1,\dots,K_T$.

u_D, u_N, u_I^j, u_T^j are given functions, Q, γ_I^j - given constants, $z_j \in \Gamma_T^j$ are given trailing stagnation points. u is an unknown function and q_I^j, q_T^j are unknown constants.

The contact of some boundary conditions is prohibited e.g. (1.2) and (1.5,a-b). It is also necessary to consider the consistency of some types of these conditions as e.g. (1.2) and (1.4). For concrete examples see [7 - 10].

2. THE PROBLEM WITHOUT TRAILING CONDITIONS

Since the problem (1.1) - (1.5) without trailing conditions has better properties from the mathematical point of view than the general problem (1.1) - (1.6), we shall treat these problems separately.

<u>2.1. Variational formulation of the problem (1.1) - (1.5).</u> We shall seek a weak solution in the well-known Sobolev space $H^1(\Omega) = W_2^1(\Omega)$. We define the set

$$\mathcal{V} = \{ v \in C^\infty(\bar{\Omega});\ v|\Gamma_D = 0,\ v(Z_P(x)) = v(x),\ x \in \Gamma_P^-,\ v|\Gamma_I^j = \text{const}\} \tag{2.1}$$

and the space

$$V = \{ v \in H^1(\Omega);\ v|\Gamma_D = 0,\ v(Z_P(x)) = v(x),\ x \in \Gamma_P^-,\ v|\Gamma_I^j = \text{const (in the sense of traces on } \partial\Omega)\}. \tag{2.2}$$

The validity of the following assertion is important:

The set $\mathcal{V}$ is dense in V, i.e.

$$\bar{\mathcal{V}}^{H^1(\Omega)} = V. \tag{2.3}$$

It is not easy to prove this. For a cascade flow problem see [10] .

Further, let $u^* \in H^1(\Omega)$ satisfy

$$\text{a) } u^*|\Gamma_D = u_D, \qquad \text{b) } u^*|\Gamma_I^j = u_I^j, \tag{2.4}$$
$$\text{c) } u^*(Z_P(x)) = u^*(x) + Q,\ x \in \Gamma_P^-.$$

Very often the existence of this u^* follows from the fact that u_D and u_I^j are indefinite integrals of functions from $L_2(\Gamma_D)$ and $L_2(\Gamma_I^j)$, respectively (cf. [20]).

Under the above notation the problem (1.1) - (1.5) is (formally) equivalent to the following variational formulation: Find u such that

$$\text{a) } u \in H^1(\Omega), \qquad \text{b) } u - u^* \in V, \tag{2.5}$$
$$\text{c) } a(u,v) = m(v) \quad \forall\, v \in V,$$

where

$$a(u,v) = \int_\Omega (b(.,u,(\nabla u)^2)\nabla u.\nabla v + f(.,u,(\nabla u)^2)v)dx \qquad \forall\, u,v \in H^1(\Omega), \tag{2.6,a}$$

$$m(v) = -\sum_{j=1}^{K_I} \gamma_I^j v|\Gamma_I^j - \int_{\Gamma_N} \varphi_N v dS, \qquad v \in V. \tag{2.6,b}$$

<u>2.2. Finite element discretization.</u> Let Ω be approximated by a polygonal domain Ω_h and let $\mathcal{T}_h$ be a triangulation of Ω_h with usual properties. We denote by $\sigma_h = \{P_1,\dots,P_N\}$ the set of all vertices of $\mathcal{T}_h$. Let the common points of Γ_D, Γ_N etc. and also the points of $\partial\Omega$, where the condition of smoothness of $\partial\Omega$ is not satisfied, belong to σ_h. Moreover, let $\sigma_h \cap \partial\Omega_h \subset \partial\Omega$, and $P_j \in \Gamma_P^- \cap \sigma_h \Leftrightarrow Z_P(P_j) \in \Gamma_P^+ \cap \sigma_h$. Hence, the sets Γ_D, Γ_N etc. are approximated by arcs or curves Γ_{Dh}, Γ_{Nh} etc. $\subset \partial\Omega_h$ in a natural way.

An approximate solution is sought in the space of linear conforming triangular elements

$$W_h = \{ v_h \in C(\bar{\Omega}_h);\ v_h|T \text{ is affine } \forall T \in T_h \} \qquad (2.7)$$

The <u>discrete problem</u> is written down quite analogously as the continuous problem (2.5,a-c): Find u_h such that

$$\text{a) } u_h \in W_h, \qquad \text{b) } u_h - u_h^* \in V_h \qquad (2.8)$$

$$\text{c) } a_h(u_h, v_h) = m_h(v_h) \qquad \forall v_h \in V_h,$$

where

$$V_h = \{ v_h \in W_h;\ v_h|\Gamma_{Dh} = 0,\ v_h(Z_P(P_j)) = v_h(P_j),\ P_j \in \sigma_h \cap \Gamma_P^-,$$

$$v_h|\Gamma_{Ih}^j = \text{const} \} . \qquad (2.9)$$

The function $u_h^* \in W_h$ has the properties

$$\text{a) } u_h(P_i) = u_D(P_i),\ P_i \in \sigma_h \cap \Gamma_D,$$

$$\text{b) } u_h(P_i) = u_I^j(P_i),\ P_i \in \sigma_h \cap \Gamma_I^j, \qquad (2.10)$$

$$\text{c) } u_h(Z_P(P_i)) = u_h(P_i) + Q,\ P_i \in \sigma_h \cap \Gamma_P^- \quad \text{and}$$

plays the same role as u^* in the continuous problem. Further,

$$a_h(u_h, v_h) = \int_{\Omega_h} (b(.,u_h,(\nabla u_h)^2)\nabla u_h . \nabla v_h + f(.,u_h,(\nabla u_h)^2)v_h)dx \qquad (2.11,a)$$

$$m_h(v_h) = -\sum_{j=1}^{K_I} \gamma_I^j v_h|\Gamma_{Ih}^j - \int_{\Gamma_{Nh}} \varphi_h v_h dS. \qquad (2.11,b)$$

Usually, the integrals in (2.11,a-b) are evaluated by convenient <u>numerical quadratures</u>. Then, instead of u_h, a_h and m_h we have u_h^{int}, a_h^{int} and m_h^{int} in (2.8,a-c).

The problem (2.8,a-c) leads to a <u>system of algebraic equations</u>

$$A(\bar{u})\bar{u} = F(\bar{u}). \qquad (2.12)$$

Here $\bar{u} = (u_1,\dots,u_n)^T$ is a vector with components defining the approximate solution, $A(\bar{u})$ is an $n\times n$ $(n < N)$ symmetric positive definite matrix for all $\bar{u} \in R_n$ and $F: R_n \to R_n$.

Now let us introduce the <u>properties of the functions b and f:</u>

1) b and f depend on $x \in \bar{\Omega}$, $u \in R_1$, $\eta \geq 0$ ($\eta := (\nabla u)^2$).

2) b, f and their derivatives $\partial b/\partial x_i$, $\partial b/\partial u$, $\partial b/\partial \eta$, $\partial f/\partial x_i$ etc. are continuous and bounded.

3) $b \geq b_1 > 0$, $b_1 = \text{const}$, $\partial b/\partial \eta \geq 0$.

4) $\frac{\partial b}{\partial \eta}(x,u,s^2)s^2$, $\left|\frac{\partial b}{\partial u}(x,u,s^2)s\right| \leq \text{const} \quad \forall x \in \bar{\Omega}$, $\forall u, s \in R_1$.

5) $b(Z_P(x), u+Q, \eta) = b(x,u,\eta) \quad \forall x \in \Gamma_P^-$, $u \in R_1$, $\eta \geq 0$.

f satisfies the second inequality in 4) and the assumption 5).

<u>2.3. The solvability of the problem (2.5,a-c)</u> is a consequence

of the monotone and pseudomonotone operator theory ([19, 22]). If the flow is irrotational ($b = b(x, \eta)$, $f = 0$), then the form $a(u,v)$ satisfies the condition of strong monotony and the solution is unique. These results for various types of flows are contained in [1, 4, 5, 10, 15].

2.4. The study of the discrete problem. Its solvability easily follows from the Brower's fixed point theorem and the properties of b and f (cf. [19, 13]). Much more complex is the question on the convergence of the finite element method, since by Strang ([23]) we have commited three variational crimes (approximation of Ω by a polygonal domain; $W_h \not\subset H^1(\Omega)$, $V_h \not\subset V$; numerical integration), the problem is nonlinear and boundary conditions are nonhomogeneous and nonstandard.

We shall consider numerical quadratures of precision d=1 with nonnegative coefficients. Let φ_N and $\partial\Omega$ be piecewise of the class C^2.

Let us consider a regular system of triangulations $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ of Ω_h ($h_0 > 0$ is sufficiently small) and study the behaviour of u_h, if $h \to 0+$.

By $\|\cdot\|_{1,\Omega_h}$ we denote the usual norm in $H^1(\Omega_h)$ and put

$$|v|_{1,\Omega_h} = \left(\int_{\Omega_h} (\nabla v)^2 dx\right)^{1/2} . \tag{2.13}$$

It is important that

$$\|v_h\|_{1,\Omega_h} \le c\, |v_h|_{1,\Omega_h} \qquad \forall v_h \in V_h \quad \forall h \in (0,h_0) \tag{2.14}$$

with a constant $c > 0$ independent of v_h and h (see [13] or [24]).

By [21], the solution u of the continuous problem and the function u^* posses the Calderon extensions from Ω to a domain $\tilde{\Omega}$ such that $\Omega, \Omega_h \subset \tilde{\Omega}$ $\forall h \in (0,h_0)$ and $u, u^* \in H^1(\tilde{\Omega})$.

Further, let us assume that

$$\|u^* - u_h^*\|_{1,\Omega_h} \to 0, \text{ if } h \to 0+. \tag{2.15}$$

In some cases (cf. e.g. [10]) $u^* \in W_2^{1+\varepsilon}(\tilde{\Omega})$, $\varepsilon > 0$, and we can put $u_h^* = r_h u^*$ (= the Lagrange interpolation of u^*). Then, since

$$\|u^* - r_h u^*\|_{1,\Omega_h} \le c h^{\varepsilon} \|u^*\|_{W_2^{1+\varepsilon}(\tilde{\Omega})} \tag{2.16}$$

(with c independent of u^* and h) we have (2.15).

First let us consider an irrotational flow. The study of the convergence $u_h \to u$, if $h \to 0+$, is based on the following results.

2.4.1. Theorem. There exist α, $K > 0$ such that

$$a_h(u_1,u_1-u_2) - a_h(u_2,u_1-u_2) > \alpha\, |u_1-u_2|^2_{1,\Omega_h}, \tag{2.17}$$

$$|a_h(u_1,v) - a_h(u_2,v)| \le K \|u_1-u_2\|_{1,\Omega_h} \|v\|_{1,\Omega_h} \tag{2.18}$$

$$\forall u_1,u_2,v \in H^1(\Omega_h), \quad \forall h \in (0,h_0)$$

and

$$a_h^{int}(u_1,u_1-u_2) - a_h^{int}(u_2,u_1-u_2) \geq \alpha |u_1-u_2|^2_{1,\Omega_h} \qquad (2.17^*)$$

$$|a_h^{int}(u_1,v) - a_h^{int}(u_2,v)| \leq K \| u_1-u_2\|_{1,\Omega_h} \| v \|_{1,\Omega_h} \qquad (2.18^*)$$

$$\forall u_1,u_2,v \in W_h , \quad \forall h \in (0,h_0) .$$

Proof follows easily from the properties of the functions b and f, the Mean Value Theorem and [3, Theorem 4.1.5] .

Now let us introduce abstract error estimates.

2.4.2. Theorem. There exist constants A_1, A_2, A_3 independent of h such that

$$\|u-u_h\|_{1,\Omega_h} \leq A_1 \inf_{w_h \in u_h^*+V_h} \|u-w_h\|_{1,\Omega_h} + \qquad (2.19)$$

$$+ A_2 \sup_{v_h \in V_h} (|a_h(u,v_h)-m_h(v_h)| / \|v_h\|_{1,\Omega_h})$$

and

$$\|u_h-u_h^{int}\|_{1,\Omega_h} \leq \qquad (2.20)$$

$$A_3 \sup_{v_h \in V_h} (| a_h(u_h,v_h)-a_h^{int}(u_h,v_h)|+|m_h(v_h)-m_h^{int}(v_h)|)/\|v_h\|_{1,\Omega_h} .$$

Proof is a consequence of Theorem 2.4.1.

2.4.3. Theorem. Let $u,u^* \in H^2(\tilde{\Omega})$. Then $\|u-u_h\|_{1,\Omega_h} = O(h)$.

Proof. We apply the technique common in linear problems (cf. [3]) based on estimates (2.19) and (2.16) with $\varepsilon = 1$ and a similar estimate for u. This, the use of Green's theorem and the fact that $meas((\Omega - \Omega_h) \cup (\Omega_h - \Omega)) \leq ch^2$ give the result. (Another approach avoiding the use of Green's theorem is used in [18].)

2.4.4. Theorem. Let $u \in H^1(\tilde{\Omega})$ and let (2.15) be satisfied. Then

$$\lim_{h\to 0+} \|u-u_h\|_{1,\Omega_h} = 0.$$

Proof. From (2.3) and (2.15) we get

$$\lim_{h \to 0+} \inf_{w_h \in u_h^*+V_h} \|u-w_h\|_{1,\Omega_h} = 0.$$

The convergence of the second term in (2.19) to zero is proved by introducing convenient modifications $\hat{v}_h \in V$ of $v_h \in V_h$. Hence,

$$a(u,\hat{v}_h) = m(\hat{v}_h).$$

This, the estimates of $v_h-\hat{v}_h$ derived in [24] and the estimates of $a(u,\hat{v}_h)-a_h(u,v_h)$ and $m_h(v_h)-m(\hat{v}_h)$ imply the desired result.

2.4.5. Theorem. If we use numerical integration of precision d=1 with nonnegative coefficients and $\partial\Omega$, φ_N are piecewise of class C^2, then $\|u_h-u_h^{int}\|_{1,\Omega_h} = O(h)$.

Proof follows from the estimate (2.20), [3, Theorem 4.1.5] and the boundedness of $\{u_h\}_{h\in(0,h_0)}$.

2.4.6. Remark. The convergence of the finite element solution with numerical integration applied to a nonlinear elliptic problem was proved in [13]. A more complex analysis will be given in [18].

For general rotational flows, instead of strong monotony, we have pseudomonotony only. Then by the application of methods from the pseudomonotone operator theory ([19, 22]) we get the following result:

2.4.7. Theorem. Let Ω be a polygonal domain and let the conditions (2.3) and (2.15) be satisfied. Let the forms a and m are evaluated by means of numerical quadratures of precision d=1 with nonnegative coefficients. Hence, a and m are approximated by $a_h := a_h^{int}$ and $m_h := m_h^{int}$, respectively. Then it holds:
1) To each $h \in (0,h_0)$ at least one solution u_h of (2.8,a-c) exists.
2) There exists $c > 0$ such that $\|u_h\|_{1,\Omega} \le c$ for all $h \in (0,h_0)$.
3) If $\{u_{h_n}\}_{n=1}^{\infty}$ is a subsequence of the system $\{u_h\}_{h\in(0,h_0)}$, $h_n \to 0$ and $u_{h_n} \rightharpoonup u$ weakly in $H^1(\Omega)$ for $n \to \infty$, then u is a solution of the continuous problem (2.5,a-c) and $u_{h_n} \to u$ strongly in $H^1(\Omega)$.

Proof. Let $A: H^1(\Omega) \to (H^1(\Omega))^*$ be the operator defined by the relation

$$\langle A(u), v\rangle = a(u,v), \quad u, v \in H^1(\Omega) . \tag{2.21}$$

From the properties of b and f it follows:
a) A is Lipschitz-continuous and bounded.
b) A satisfies the generalized property (S), i.e. it holds:

$$z_n, z \in V,\ z_n \rightharpoonup z \text{ weakly},\ u_n^* \to u^* \text{ strongly},\ \langle A(u_n^*+z_n)-A(u^*+z), z_n-z\rangle \to 0 \quad \Rightarrow \quad u_n^*=u_n+z_n \to u=u^*+z \text{ strongly}.$$

The proof of the assertion a) follows from the properties of the functions b and f. Let us show that also b) is valid. We assume that $z_n, z \in V$, $z_n \rightharpoonup z$ weakly, $u_n \to u$ strongly, $u=u^*+z$, $u_n=u_n^*+z_n$ and

$$J_n = \langle A(u_n) - A(u), z_n - z\rangle \to 0.$$

If we put

$$I_n = a(u_n,u_n-u) - a(u,u_n-u),$$

then

$$J_n = I_n + a(u,u_n^*-u^*) - a(u_n^*,u_n-u^*).$$

Since $u_n^* \to u^*$ we find out that

$$a(u,u_n^*-u^*) - a(u_n,u_n^*-u^*) \to 0.$$

Hence, $I_n \to 0$.

From the definition of the form a it follows that

$$I_n = \int_\Omega \{ \sum_{i=1}^{2} (b(.,u_n,(\nabla u_n)^2) \frac{\partial u_n}{\partial x_i} - b(.,u,(\nabla u)^2)\frac{\partial u}{\partial x_i}) \frac{\partial (u_n-u)}{\partial x_i} +$$

$$+ (f(.,u_n,(\nabla u_n)^2) - f(.,u,(\nabla u)^2))(u_n-u)\} dx .$$

As $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$, $u_n \to u$ strongly in $L_2(\Omega)$. The properties of f and b imply the relations

$$I_n \geq \alpha |u_n-u|^2_{1,\Omega} + c_n, \qquad \alpha > 0,$$

$$c_n = \int_\Omega \{(b(.,u_n,(\nabla u_n)^2) - b(.,u,(\nabla u)^2))\nabla u \,.\, \nabla(u_n - u) +$$
$$+ (f(.,u_n,(\nabla u_n)^2) - f(.,u,(\nabla u)^2))(u_n - u)\} dx,$$

$$c_n = (b(.,(\nabla u_n)^2) - b(.,u,(\nabla u)^2))\nabla u \,.\, \nabla(u_n - u) +$$
$$+ (f(.,u_n,(\nabla u_n)^2) - f(.,u,(\nabla u)^2))(u_n - u)dx,$$

(The sequence $\{\|u_n-u\|_{1,\Omega}\}_{n=1}^{\infty}$ is bounded.) From this and equivalence of the norms $\|.\|_{1,\Omega}$ and $|.|_{1,\Omega}$ in the space V we already conclude that $u_n \to u$ strongly.

Since $\{u_h\}_{h\in(0,h_0)}$ is a bounded set and A is a bounded operator we can assume that we have a sequence $u_n:=u_{h_n}$ such that

$$h_n \to 0,\ u_n \rightharpoonup u \text{ weakly in } H^1(\Omega) , \tag{2.22}$$
$$A(u_n) \rightharpoonup X \text{ weakly in } (H^1(\Omega))^* .$$

In view of (2.15), it is evident that $u=u^*+z$, $z \in V$ and $z_n=z_{n_h} \rightharpoonup z$ weakly.

Similarly as in [13] or [18] we derive the estimates

$$|a(u_h,v_h) - a_h(u_h,v_h)| \leq ch\|v_h\|_{1,\Omega} \quad \forall\, v_h \in V_h \quad \forall\, h \in (0,h_0) \tag{2.23}$$

$$|m(v_h) - m_h(v_h)| \leq ch\|v_h\|_{1,\Omega} \quad \forall v_h \in V_h \quad \forall h \in (0,h_0) \tag{2.24}$$

with c independent of v_h and h.

Let $v \in V$. By (2.16) $v_n:=r_{h_n} v \to v$ in $H^1(\Omega)$, $v_n \in V_{h_n}$. We have

$$\langle A(u_n),v_n \rangle = m(v_n) + (a(u_n,v_n) - a_{h_n}(u_n,v_n)) +$$
$$+ (m_{h_n}(v_n) - m(v_n)).$$

From this, (2.22)-(2.24) and (2.3) we derive the relation

$$\langle X,v \rangle = m(v) \qquad \forall v \in V. \tag{2.25}$$

Further, by (2.21)-(2.25),

$$\langle A(u_n) - A(u),\ z_n - z \rangle \to 0.$$

Now, if we use the generalized property (S) of the operator A, we find out that $z_n \to z$ and thus, $u_n \to u$ (strongly). As the operator A is continuous, $A(u) = \lim_{n\to\infty} A(u_n) = X$. By (2.25),

$$\langle A(u), v \rangle = m(v) \qquad \forall v \in V,$$

which we wanted to prove.

2.4.8. Remark. Instead of Lipschitz-continuity of the operator A it is sufficient to use its demicontinuity: "$u_n \to u$ strongly $\Rightarrow$ $\Rightarrow A(u_n) \rightharpoonup A(u)$ weakly. The proof of the convergence of the approximate solution obtained without numerical integration is similar (and of course more simple). The case of the problem in a nonpolygonal domain Ω remains open.

3. ON THE GENERAL PROBLEM (1.1) - (1.6)

In practice the complete problem (1.1) - (1.6) is very important, but its mathematical study is unfortunately much more difficult because of the discrete trailing conditions (1.6,b). Therefore, the results are not so complete as in the case of the problem (1.1) - (1.5) and we present here only a brief surway.

3.1. The solvability of the continuous problem has to be studied in classes of classical solutions. The main tool for proving the solvability are appriori estimates of solutions to linear and nonlinear elliptic equations and the strong maximum principle. The study was successful for incompressible irrotational and rotational flows ([6, 8]) and for irrotational compressible flows ([9]). The solvability of the general rotational compressible flow problem remains open.

3.2. Finite element discretization. Let us consider a triangulation $\mathcal{T}_h$ of the domain Ω_h with the properties from 2.2. Moreover, we assume that to each trailing point $z_j \in \Gamma_T^j$ there exists a triangle $T_j \in \mathcal{T}_h$ with vertices $\tilde{P}_j = z_j$ and $P_j^* \in \Omega_h$ such that the side $S_j = \tilde{P}_j P_j^*$ is normal to Γ_T^j. Then, if we discretize the condition (1.6,b) by its finite-difference analogue and consider (1.6,a), we derive the conditions (for simplicity we assume that $u_T^j = 0$)

$$u_h(P_k) = q_T^j = u_h(P_j^*) \qquad \forall P_k \in \sigma_h \cap \Gamma_T^j \tag{3.1}$$

Now the discrete problem to (1.1) - (1.6) is written down in the following way: Find u_h such that

$$\text{a) } u_h \in W_h, \qquad \text{b) } u_h - u_h^* \in \tilde{V}_h, \tag{3.2}$$

$$\text{c) } a_h(u_h, v_h) = m_h(v_h) \qquad \forall v_h \in V_h.$$

Here,

$$V_h = \{ v_h \in W_h;\ v_h|\Gamma_{Dh} = 0,\ v_h(Z_P(P_i)) = v_h(P_i), \tag{3.3}$$
$$P_i \in \sigma_h \cap \Gamma_P^-,\ v_h|\Gamma_{Ih}^j = \text{const},\ v_h|\Gamma_{Th}^j = 0 \},$$

$$\tilde{V}_h = \{ v_h \in W_h;\ v_h|\Gamma_{Dh} = 0,\ v_h(Z_P(P_i)) = v_h(P_i), \tag{3.4}$$
$$P_i \in \sigma_h \cap \Gamma_P^-,\ v_h|\Gamma_{Ih}^j = \text{const},\ v_h|\Gamma_{Th}^j \cup S_j = \text{const}\},$$

a) $u_h^* \in W_h,\ u_h^*(P_i) = u_D(P_i),\ P_i \in \sigma_h \cap \Gamma_D$, (3.5)

b) $u_h^*(P_i) = u_I^j(P_i),\ P_i \in \sigma_h \cap \Gamma_I^j$,

c) $u_h^*(Z_P(P_i)) = u_h^*(P_i) + Q,\ P_i \in \sigma_h \cap \Gamma_P^-$,

d) $u_h^*|\Gamma_{Th}^j \cup S_j = 0$.

a_h and m_h are again defined by (2.11,a-b).

The problem (3.2,a-c) is equivalent to a system (2.12). Since $V_h \neq \tilde{V}_h$, the matrix $A(\bar{u})$ is not more symmetric. However, if all angles of all $T \in \mathcal{T}_h$ are less then or equal to 90^o, then $A(\bar{u})$ is an irreducibly diagonally dominant matrix and the system (2.12) has a solution. Under the same assumption, with the use of the discrete maximum principle, we can prove the convergence of the method: if $u \in C^2(\bar{\Omega})$ and the problem is linear, then $\|u-u_h\|_{L_\infty(\Omega_h)} \leq ch$ for all $h \in (0,h_o)$. For details see [14].

4. ITERATIVE SOLUTION OF THE DISCRETE PROBLEM

It is convenient to distinguish several cases:

4.1. Irrotational incompressible flow ($b = b(x)$, $f = 0$): The system (2.12) is linear and we use the SOR method.

4.2. Irrotational compressible flow ($b = b(x,\eta)$, $f = 0$): Among the methods we have tested the following iterative process occurs as an effective one:

a) $\bar{u}^0 \in R_n$ (a convenient initial approximation) (4.1)

b) $B\bar{u}^{k+1} = B\bar{u}^k - \omega(A(\bar{u}^k)\bar{u}^k - F(\bar{u}^k)),\ k \geq 0,\ \omega > 0.$

The speed of the convergence depends on the choice of ω (its estimate can be obtained on the basis of the behaviour of the function b) and of a preconditioning positive definite matrix B.

4.3. Rotational incompressible flow ($b = b(x)$, $f = f(x,u)$): Similarly as in [7] we can apply a Newton relaxation method. If the vorticity is too strong, it is better to proceed as in the following case.

4.4. Rotational compressible flow: As a sufficiently robust the method of least squares and conjugate gradients by Glowinski et al. appears (see [2]). The details will be the subject matter of an intended paper.

5. EXAMPLES

As a simple test problem we introduce a flow through a plane channel. On the inlet (left side of the boundary - see Fig. 1) and outlet (right side of the boudary) we consider the Neumann condition $\partial u/\partial n = 0$. On the lower wall we put $u = 0$ and on the upper wall $u = 25$. We consider a rotational flow described by the equation

$$\Delta u = -200 \operatorname{arctg} u.$$

The uniqueness of this boundary value problem is not sure.

This problem was successfully solved by the method of least squares and conjugate gradients starting from the solution of the corresponding linear irrotational flow ($\Delta u = 0$). In Fig. 1 we see the triangulation used. The iterative process was stopped after 6 conjugate gradient iterations, when the resulting value of the cost functional was 10^{-5}. For one-dimensional minimization the golden-section method was applied. In Fig. 2 the calculated velocity field is plotted. It is interesting with backward flows caused by a strong vorticity.

The second example represents an industrial application of the presented theory and numerical methods - a result of a cascade flow calculation (cf. [10 - 14, 16, 17]). In Fig. 3 we show velocity vectors plotted in the domain representing one period of a cascade of profiles.

For other examples see [11, 12, 17].

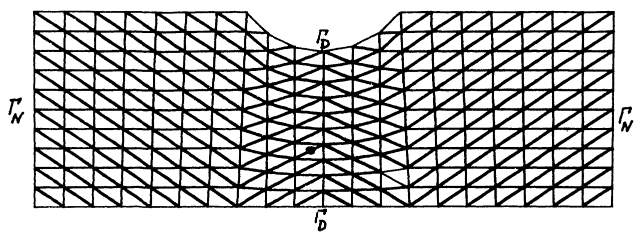

Fig. 1

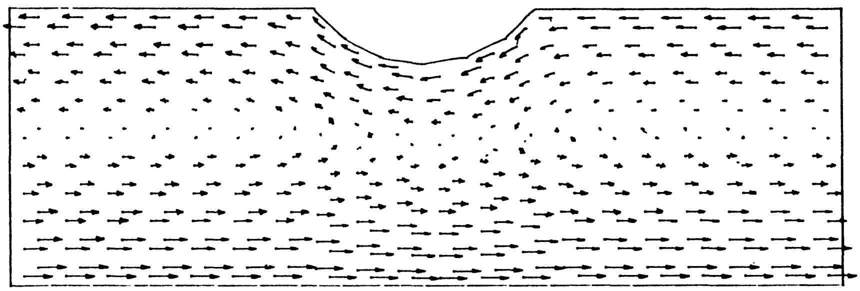

Fig. 2

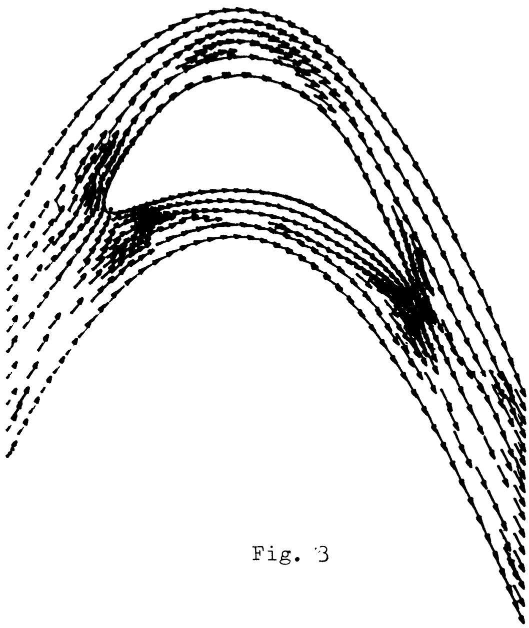

Fig. 3

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