Alois Kufner

Boundary value problems in weighted spaces

In: Jaromír Vosmanský and Miloš Zlámal (eds.): Equadiff 6, Proceedings of the International Conference on Differential Equations and Their Applications held in Brno, Czechoslovakia, Aug. 26 - 30, 1985. J. E. Purkyně University, Department of Mathematics, Brno, 1986. pp. 35--48.

Persistent URL: http://dml.cz/dmlcz/700153

BOUNDARY VALUE PROBLEMS IN WEIGHTED SPACES

A. KUFNER
Mathematical Institute, Czechoslovak Academy of Sciences
115 67 Prague 1, Czechoslovakia

1. Introduction

Let us consider a linear differential operator of order $2k$ of the form

$$(Lu)(x) = \sum_{|\alpha|,|\beta|\leq k} (-1)^{|\alpha|} D^\alpha \left(a_{\alpha\beta}(x) D^\beta u(x)\right) , \quad x \in \Omega , \tag{1.1}$$

together with the associated bilinear form

$$a(u,v) = \sum_{|\alpha|,|\beta|\leq k} \int_\Omega a_{\alpha\beta}(x) D^\beta u(x) D^\alpha v(x)\, dx . \tag{1.2}$$

Here $a_{\alpha\beta}$ are given (real) functions defined on the domain $\Omega \in \mathbb{R}^N$.

The usual procedure for solving a boundary value problem for the operator L proceeds in the following fundamental steps:

(i) Choose an appropriate Banach space V such that the form $a(u,v)$ is *defined* and *continuous* on $V \times V$ and *elliptic* on V , i.e., that there exist constants $c_1 > 0$ and $c_0 > 0$ such that the following conditions are fulfilled:

$$|a(u,v)| \leq c_1 ||u||_V ||v||_V \quad \text{for every} \quad u, v \in V \tag{1.3}$$

(continuity of $a(u,v)$), and

$$a(u,u) \geq c_0 ||u||_V^2 \quad \text{for every} \quad u \in V \tag{1.4}$$

(ellipticity of $a(u,v)$).

(ii) Use the *Lax-Milgram Lemma* in order to obtain assertions about the existence and uniqueness of a (*weak*) solution in the space V .

If the coefficients $a_{\alpha\beta}$ of the operator L are *bounded*, i.e., if

$$a_{\alpha\beta} \in L^\infty(\Omega) \quad \text{for} \quad |\alpha|,\ |\beta| \leq k , \tag{1.5}$$

and if the operator L is (for simplicity) *uniformly elliptic*, i.e., if there exists a constant $c_0 > 0$ such that

$$\sum_{|\alpha|,|\beta|\leq k} a_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq c_0|\xi|^2 \tag{1.6}$$

for every $\xi \in \mathbb{R}^M$ ($\xi = \{\xi_\alpha, |\alpha| \leq k\}$), then the first step mentioned above can be realized if we choose for the space V the *Sobolev space* $W^{k,2}(\Omega)$ or one of its subspaces selected according to the type of the boundary conditions.

If one or both of the conditions (1.5) and (1.6) are violated, i.e., if operators with *singular coefficients* appear - condition (1.5) is not fulfilled - or if the operator becomes *degenerate* - the quadratic form on the left-hand side in (1.6) is only positive semidefinite - then the Sobolev spaces $W^{k,2}(\Omega)$ cannot be used in general. In these cases, appropriate *weighted Sobolev spaces* can be constructed which replace the classical spaces $W^{k,2}(\Omega)$. The *weight functions* appearing in these new spaces *are determined by the coefficients of the operator*, and the method of the proof of the corresponding existence and uniqueness theorem for weak solutions is the same as in the case of classical Sobolev spaces, the main tool being the Lax-Milgram Lemma.

On the other hand, there appear boundary value problems in which the operator L satisfies conditions (1.5) and (1.6) but the right-hand side in the equation $Lu = f$ or the right-hand sides in the boundary conditions $B_i u = g_i$, $i = 1,\dots,k$ (B_i being boundary operators) behave in such a way that the classical Sobolev spaces cannot be used: the function f is not an element of the dual space $(W^{k,2}(\Omega))^*$ or some of the functions g_i are not traces of functions from $W^{k,2}(\Omega)$ on the boundary $\partial\Omega$ of Ω. Also in such cases, weighted spaces can be sometimes used for obtaining assertions about existence and uniqueness of weak solutions. The bilinear form $a(u,v)$ is considered to be defined on a *weighted space* V or on a *product of two weighted spaces* $V_1 \times V_2$, and it is necessary to show for *which* such spaces conditions (1.3), (1.4) or their certain modifications are fulfilled.

In what follows, we give a survey of results obtained in these two directions of application of weighted Sobolev spaces to the solution of boundary value problems.

<u>1.1. Definition of the weighted space</u>. Let $1 < p < \infty$ and let Ω be a domain in $\mathbb{R}^N$. For $k \in \mathbb{N}$ and for N-dimensional multiindices α such that $|\alpha| \leq k$ let $\sigma_\alpha = \sigma_\alpha(x)$ be *weight functions*, i.e. measurable and a.e. in Ω positive functions, and let us denote $S = \{\sigma_\alpha, |\alpha| \leq k\}$. The weighted Sobolev space

$$W^{k,p}(\Omega;S) \tag{1.7}$$

is defined as the set of all functions $u = u(x)$, $x \in \Omega$, such that

$$||u||^p_{k,p,S} = \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u(x)|^p \sigma_\alpha(x)\, dx < \infty , \tag{1.8}$$

the derivatives $D^\alpha u$ being considered in the sense of distributions. Further, let

$$W^{k,p}_0(\Omega;S) \tag{1.9}$$

be the closure (if it is meaningful) of the set $C^\infty_0(\Omega)$ with respect to the norm (1.8).

1.2. Theorem. *Let us suppose that*

$$\sigma_\alpha^{-1/(p-1)} \in L^1_{loc}(\Omega) \quad \text{for} \quad |\alpha| \leq k . \tag{1.10}$$

Then the linear set $W^{k,p}(\Omega;S)$ *is a Banach space with respect to the norm* $||\cdot||_{k,p,S}$ *defined by* (1.8). - *If, moreover,*

$$\sigma_\alpha \in L^1_{loc}(\Omega) \quad \text{for} \quad |\alpha| \leq k , \tag{1.11}$$

then the linear set $W^{k,p}_0(\Omega;S)$ *is a Banach space with respect to the same norm.*

1.3. Remark. Conditions (1.10), (1.11) are rather restrictive. In the paper A. KUFNER, B. OPIC [2] it is shown how to modify the definition of the weighted spaces if (1.10) and/or (1.11) are not fulfilled. - The most frequent type of weight functions σ_α are the so called *power type weights*

$$\sigma_\alpha(x) = [\mathrm{dist}(x,M)]^\varepsilon$$

with $M \subset \overline{\Omega}$ and $\varepsilon = \varepsilon(\alpha)$ real numbers. If M is a subset of the boundary $\partial\Omega$ of Ω , then conditions (1.10) and (1.11) are obviously satisfied.

2. Operators with singular or degenerating coefficients

Let us consider the operator L from (1.1) and let us suppose that its coefficients fulfil the following conditions:

$$a_{\alpha\alpha} > 0 \text{ a.e. in } \Omega ; \quad a_{\alpha\alpha},\ a^{-1}_{\alpha\alpha} \in L^1_{loc}(\Omega) , \quad |\alpha| \leq k ; \tag{2.1}$$

there exist constants $c_1 > 0$, $c_0 > 0$ such that

$$|a_{\alpha\beta}(x)| \leq c_1\sqrt{a_{\alpha\alpha}(x)a_{\beta\beta}(x)} \quad \text{a.e. in } \Omega \tag{2.2}$$

for $|\alpha|, |\beta| \leq k$, $\alpha \neq \beta$;

$$\sum_{|\alpha|,|\beta|\leq k} a_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq c_0 \sum_{|\alpha|\leq k} a_{\alpha\alpha}(x)\xi_\alpha^2 \quad \text{a.e. in } \Omega \tag{2.3}$$

for all $\xi \in \mathbb{R}^M$.

Conditions (2.1) indicate that the weighted spaces $W^{k,2}(\Omega;S)$ and $W_0^{k,2}(\Omega;S)$ with $\sigma_\alpha = a_{\alpha\alpha}$, $|\alpha| \leq k$, i.e. with

$$S = \{a_{\alpha\alpha} = a_{\alpha\alpha}(x) \ , \ |\alpha| \leq k\} \tag{2.4}$$

are Banach spaces. From the following theorem we see that these weighted spaces are just the right tool for solving boundary value problems.

<u>2.1. Theorem</u>. *Let the operator* L *from* (1.1) *fulfil conditions* (2.1) - (2.3). *Let* S *be given by* (2.4); *let* $f \in (W_0^{k,2}(\Omega;S))^*$ *and* $u_0 \in W^{k,2}(\Omega;S)$. *Then there exists one and only one weak solution* $u \in W^{k,2}(\Omega;S)$ *of the Dirichlet problem for the equation* $Lu = f$, *i.e., such a function* u *that*

$$u - u_0 \in W_0^{k,2}(\Omega) \tag{2.5}$$

and

$$a(u,v) = \langle f,v \rangle \quad \textit{for every} \quad v \in W_0^{k,2}(\Omega) \ . \tag{2.6}$$

Moreover, there is a constant $c > 0$ *such that*

$$||u||_{k,2,S} \leq c(||f||_* + ||u_0||_{k,2,S}) \ . \tag{2.7}$$

Idea of the p r o o f : Condition (2.2) implies the continuity of the bilinear form $a(u,v)$ from (1.2) on $V \times V$ with $V = W^{k,2}(\Omega;S)$ and conditions (2.3) imply its ellipticity while (2.1) guarantees that the space V and its subspace $W_0^{k,2}(\Omega;S)$ are well defined. A standard application of the Lax-Milgram Lemma then yields the existence and uniqueness of a weak solution $u \in W^{k,2}(\Omega;S)$ as well as the estimate (2.7) which expresses the continuous dependence of the solution on the data of the boundary value problem.

<u>2.2. Remarks.</u> (i) In the sequel, we shall give two examples of boundary value problems which *go beyond the frame of conditions* (2.1) - (2.3), but for which again existence and uniqueness of a weak solution can be proved. These examples indicate that conditions (2.1) - (2.3) can be substantially weakened and that the adequate weighted space can be constructed in a much more sophisticated way. A detailed description of

the (rather complicated) construction of these spaces can be found in A. KUFNER B. OPIC [1], [3].

(ii) Although Theorem 2.1 is a simplified version of an application of weighted Sobolev spaces to the solution of boundary value problems, some of its conditions can be weakened: E. g. condition (2.3) follows from (2.2) if the constant c_1 in (2.2) is sufficiently small, i. e. if $c_1 < 1/(M-1)$ where M is the number of multiindices α such that $|\alpha| \leqq k$.

(iii) The restriction to the Dirichlet problem in Theorem 2.1 is not substantial, either; other boundary value problems can be handled in the same manner.

2.3. Example. Let us consider the differential operator of order two, i.e. $k = 1$:

$$(Lu)(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)\,\frac{\partial u}{\partial x_i}\right) + a_0(x)u$$

where $a_i > 0$ for $i = 1,\dots,N$ but $a_0 < 0$, $a_0 = -\lambda b_0$ with $b_0 > 0$, $\lambda \geqq 0$. We suppose that $a_i, a_i^{-1} \in L^1_{loc}(\Omega)$ for $i = 0,1,\dots,N$. Here, one of the conditions in (2.1) is not fulfilled, but if we take $S = \{b_0, a_1, \dots, a_N\}$, then $W^{1,2}(\Omega;S)$ and $W_0^{1,2}(\Omega;S)$ are the adequate spaces which can be used for deriving existence and uniqueness theorems *provided the following inequality holds for all* $u \in C_0^\infty(\Omega)$:

$$\int_\Omega |u(x)|^2 b_0(x)\,dx \leqq c\sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^2 a_i(x)\,dx \tag{2.8}$$

with a constant c independent of u, and *provided the constant* λ *in* $a_0 = -\lambda b_0$ *is sufficiently small,* namely $\lambda < 1/c$.

2.4. Example. Let us consider the plane domain $\Omega = (0,\infty)\times(0,\infty)$ (i.e. $N = 2$) and the *fourth* order operator

$$(Lu)(x) = \frac{\partial^2}{\partial x_1 \partial x_2}\left(x_1^{\delta_1} x_2^{\delta_2}\,\frac{\partial^2 u}{\partial x_1 \partial x_2}\right) -$$

$$- \frac{\partial}{\partial x_1}\left(x_1^{\gamma_1} x_2^{\gamma_2}\,\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(x_1^{\beta_1} x_2^{\beta_2}\,\frac{\partial u}{\partial x_2}\right).$$

Here we have two possibilities:

(i) We can prove existence and uniqueness of a weak solution of the Dirichlet problem in the *anisotropic* space $W^{E,2}(\Omega;S)$ normed by

$$||u||^2 = \int_\Omega |u|^2 x_1^{\delta_1-2} x_2^{\delta_2-2}\,dx + \int_\Omega \left|\frac{\partial u}{\partial x_1}\right|^2 x_1^{\gamma_1} x_2^{\gamma_2}\,dx$$
$$+ \int_\Omega \left|\frac{\partial u}{\partial x_2}\right|^2 x_1^{\beta_1} x_2^{\beta_2}\,dx + \int_\Omega \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 x_1^{\delta_1} x_2^{\delta_2}\,dx \tag{2.9}$$

provided $\delta_1 \neq 1$, $\delta_2 \neq 1$ (these last conditions are caused by the fact that the ellipticity constant c_0 equals $16(\delta_1 - 1)^{-2}(\delta_2 - 1)^2 + 1$).

(ii) We can prove existence and uniqueness in another anisotropic space $W^{\tilde{E},2}(\Omega;\tilde{S})$ normed by the expression obtained by omitting the second and third integrals in (2.9) provided $\gamma_1 = \delta_1$, $\gamma_2 = \delta_2 - 2$, $\beta_1 = \delta_1 - 2$, $\beta_2 = \delta_2$.

2.5. Remarks. (i) Example 2.4 shows that the structure of the operators as well as of the weighted spaces can be more general than that mentioned in formula (1.1) and in Definition 1.1; in particular, *anisotropic* operators and spaces can be treated by our method.

(ii) In Example 2.3, the estimate (2.8) played an important role. Estimates of such a type, which can be viewed as continuous imbeddings of $W^{1,2}(\Omega;S)$ into the weighted L^2-space $L^2(\Omega;b_0)$, are very useful tools both in the theory and in applications of weighted Sobolev spaces.

2.6. Nonlinear operators. Let us consider the *nonlinear* operator

$$(Lu)(x) = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^\alpha a_\alpha(x;\delta_k u(x)),\quad x \in \Omega, \tag{2.10}$$

where $\delta_k u = \{D^\beta u,\ |\beta| \leq k\}$. Using the *theory of monotone operators*, results concerning existence of weak solutions of boundary value problems for the equation $Lu = f$ in the weighted space $W^{k,p}(\Omega;S)$ with $1 < p < \infty$ and $S = \{\sigma_\alpha,\ |\alpha| \leq k\}$ can be derived provided the "coefficients" $a_\alpha(x;\xi)$ of the operator L satisfy the following three conditions:

(i) the *weighted growth condition*

$$|a_\alpha(x;\xi)| \leq \sigma_\alpha^{1/p}(x)\left[g_\alpha(x) + c_\alpha \sum_{|\beta| \leq k} |\xi_\beta|^{p-1} \sigma_\beta^{1/p}(x)\right], \tag{2.11}$$

$|\alpha| \leq k$, for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^M$ with given constants $c_\alpha \geq 0$ and functions $g_\alpha \in L^q(\Omega)$, $q = p/(p-1)$;

(ii) the usual *monotonicity condition*

$$\sum_{|\alpha|\leq k} [a_\alpha(x;\xi) - a_\alpha(x;\eta)][\xi_\alpha - \eta_\alpha] \geq 0 \tag{2.12}$$

for a.e. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^M$;

(iii) a *weighted coercivity condition*

$$\sum_{|\alpha|\leq k} a_\alpha(x;\xi)\xi_\alpha \geq c_0 \sum_{|\alpha|\leq k} |\xi_\alpha|^p \sigma_\alpha(x) \tag{2.13}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^M$ with a given constant $c_0 > 0$.

The following assertion is a nonlinear analogue of Theorem 2.1.

<u>2.7. Theorem.</u> *Let* $S = \{\sigma_\alpha, |\alpha| \leq k\}$ *be a given family of weight functions and* $W^{k,p}(\Omega;S)$ *the corresponding weighted space with* $p > 1$. *Let the operator* L *from* (2.10) *fulfil conditions* (2.11) - (2.13). *Let* $f \in (W_0^{k,p}(\Omega;S))^*$ *and* $u_0 \in W^{k,p}(\Omega;S)$ *be given. Then there exists at least one weak solution* $u \in W^{k,p}(\Omega;S)$ *of the Dirichlet problem for the equation* $Lu = f$, *i.e., such a function* u *that*

$$u - u_0 \in W_0^{k,p}(\Omega;S)$$

and

$$\sum_{|\alpha|\leq k} \int_\Omega a_\alpha(x;\delta_k u(x)) D^\alpha v(x)\, dx = \langle f,v\rangle \quad \textit{for every} \quad v \in W_0^{k,p}(\Omega) \ .$$

If the inequality in (2.12) *is strict, then the solution* u *is uniquely determined.*

Idea of the p r o o f : Let us consider the form

$$a(u,v) = \sum_{|\alpha|\leq k} \int_\Omega a_\alpha(x;\delta_k u(x)) D^\alpha v(x)\, dx \ .$$

The operator $\hat{T}$, defined by the formula $a(u,v) = \langle \hat{T}u,v\rangle$, is, in view of condition (2.11), a bounded operator from $W^{k,p}(\Omega;S)$ into its dual. To find a solution u of the Dirichlet problem means to find a function $\hat{u} \in X = W_0^{k,p}(\Omega;S)$ such that $a(\hat{u} + u_0, v) = \langle f,v\rangle$ for every $v \in X$, i.e., that $\hat{T}(\hat{u} + u_0) = f$. If we denote by T the operator from X to X^* defined by $T\hat{u} = \hat{T}(\hat{u} + u_0)$, then our problem reduces to the solution of the equation $Tu = f$ in X with a given $f \in X^*$. Conditions (2.11) - (2.13) guarantee that the operator T is bounded, demicontinuous, monotone and coercive, and so, the existence of at least one solution of the Dirichlet problem follows by applying the method of monotone operators. Uniqueness follows by contradiction if we assume that the inequality in (2.12) is strict.

2.8 Example. As a typical example of a nonlinear operator closely connected with the weighted Sobolev space $W^{k,p}(\Omega;S)$, $S = \{\sigma_\alpha = \sigma_\alpha(x);\ |\alpha| \le k\}$, we can consider the operator

$$(Lu)(x) = \sum_{|\alpha|\le k} (-1)^{|\alpha|} D^\alpha \{ |D^\alpha u(x)|^{p-1} \operatorname{sgn} D^\alpha u(x)\, \sigma_\alpha(x) \} .$$

3. Elliptic operators with "bad" right-hand sides

Let us now suppose that the operator L from (1.1) satisfies condition (1.4) with a space $V \subset W^{k,2}(\Omega)$ (V is chosen in accordance with the type of the boundary conditions: for instance, $V = W_0^{k,2}(\Omega)$ for the Dirichlet problem and $V = W^{k,2}(\Omega)$ for the Neumann problem).

Further, let $S = \{\sigma_\alpha,\ |\alpha| \le k\}$ be a family of weight functions and let us denote by $1/S$ the family $\{1/\sigma_\alpha,\ |\alpha| \le k\}$. The functions $1/\sigma_\alpha$ are weight functions as well and consequently, we can consider the pair of weighted Sobolev spaces

$$H_1 = W^{k,2}(\Omega;S) \quad \text{and} \quad H_2 = W^{k,2}(\Omega;1/S) . \tag{3.1}$$

Rewriting the bilinear form $a(u,v)$ from (1.2) in the form

$$a(u,v) = \sum_{|\alpha|,\,|\beta|\le k} \int_\Omega a_{\alpha\beta}(x) D^\beta u(x) \sqrt{\sigma_\alpha(x)}\ D^\alpha v(x) \sqrt{\frac{1}{\sigma_\alpha(x)}}\, dx \tag{3.2}$$

we conclude immediately from Hölder's inequality that $a(u,v)$ *is a continuous bilinear form on* $H_1 \times H_2$.

This last property replaces the continuity condition (1.3). If we now replace the ellipticity condition (1.4) by the *pair of conditions*

$$\sup_{\|u_i\|_{H_i}\le 1} |a(u_1,u_2)| \ge c_j \|u_j\|_{H_j} , \quad i,\ j = 1,2 , \quad i \ne j , \tag{3.3}$$

– we say in this case that $a(u,v)$ *is* (H_1,H_2)*-elliptic* – we can again derive assertions about existence and uniqueness of weak solutions of the equation $Lu = f$ in H_1 (or its subspaces selected according to the type of the boundary conditions), $f \in H_2^*$. The main tool is a *modified version of the Lax-Milgram Lemma* due to J. NEČAS [1],[2], who also proposed the method roughly described above.

3.1. Problem. For what weight functions $\sigma_\alpha, |\alpha| \le k$, the conditions (3.3) are satisfied? More precisely: For what weight functions σ_α does the ellipticity condition (1.4) with V a subspace of the *classical* Sobolev space $W^{k,2}(\Omega)$ imply the *weighted* (H_1,H_2)-ellipticity?

3.2. Power type weights. Let us consider weight functions of the form

$$\sigma_\alpha(x) = [\mathrm{dist}(x,M)]^\varepsilon \quad \text{for all} \quad |\alpha| \leq k \tag{3.4}$$

where M is an m-dimensional manifold, $M \subset \partial\Omega$, and ε is a real number. The corresponding weighted space $W^{k,2}(\Omega;S)$ will be denoted by $W^{k,2}(\Omega;(\mathrm{dist})^\varepsilon)$, so that we have $W^{k,2}(\Omega;(\mathrm{dist})^{-\varepsilon})$ for $W^{k,2}(\Omega;1/S)$.

In the case of the *Dirichlet problem* the solution of Problem 3.1 is given by the following statement.

3.3. Theorem. *There exists an interval* J *containing* 0 *such that for* $\varepsilon \in J$ *the* (H_1,H_2)*-ellipticity conditions* (3.3) *are satisfied with* $H_1 = W_0^{k,2}(\Omega;(\mathrm{dist})^\varepsilon)$, $H_2 = W_0^{k,2}(\Omega;(\mathrm{dist})^{-\varepsilon})$.

The p r o o f is based on the imbedding

$$W_0^{1,2}(\Omega;(\mathrm{dist})^\varepsilon) \hookrightarrow L^2(\Omega;(\mathrm{dist})^{\varepsilon-2}) \tag{3.5}$$

which holds for $\varepsilon \neq 2 + m - N$ $(m = \dim M)$. Using the ellipticity condition (1.4) and repeatedly the imbedding (3.5) (also for higher derivatives and for both ε and $-\varepsilon$) we obtain the lower estimates (3.3) with constants c_j which depend on the coefficients of the operator L (i.e. on the L^∞-norm of $a_{\alpha\beta}$, on the ellipticity constant c_0), on geometrical properties of Ω (especially on $m = \dim M$ and on the smoothness of M) and on the norm of the imbedding operator from (3.5) (i.e. mainly on the value of the parameter ε). The interval J is determined by the requirement of the positivity of the constants c_j. A detailed derivation can be found in A. KUFNER [1] who extended to arbitrary $M \subset \partial\Omega$ the ideas developed by J. NEČAS [1] for the case $M = \partial\Omega$.

3.4. The size of J. Theorem 3.3 states the *existence of an interval* J; consequently, the existence and uniqueness of a weak solution of the Dirichlet problem for the equation $Lu = f$ in $W^{k,2}(\Omega;(\mathrm{dist})^\varepsilon)$ is guaranteed provided $\varepsilon \in J$. For applications it is necessary to know the exact *size* of the interval J of admissible powers ε in the weight function (3.4). It depends on L, Ω and M, but the estimates derived in the proof of Theorem 3.3 are very rough, and therefore, it is necessary to evaluate the interval J in every particular case separately. For example, for the operator $L = -\Delta$ it can be shown that Theorem 3.3 holds for $|\varepsilon| < 1$ so that we have $J = (-1,1)$ but in the case of $M = \{x_0\}$ with x_0 $\partial\Omega$ (i.e., the case of the weight function $|x - x_0|^{\varepsilon}$) where Ω is a plane domain with the *outer cone property* at the point x_0 (the cone being characterized by the angle ω) we have a better estimate $|\varepsilon| < 2\pi/(2\pi - \omega)$. In this connection, let us mention the recent result of J. VOLDŘICH [1] who has shown that for any given $\varepsilon \neq 0$, $|\varepsilon|$ arbitrary small, a second order elliptic differential operator L can be constructed (depending on ε) such that *the Dirichlet problem for* L *has no solution in the space* $W^{1,2}(\Omega;(\mathrm{dist})^{\varepsilon})$.

In the case of other than power type weights, certain results concerning weight functions of the type

$$\sigma_\alpha(x) = s(\mathrm{dist}(x,M)) ,$$

$s = s(t)$ a positive function on $(0,\infty)$, have been derived by B. OPIC. These results as well as other examples concerning the Dirichlet problem can be found in the book A. KUFNER [1].

3.5. Other boundary value problems. For non-Dirichlet problems, Problem 3.1 has been investigated only for power type weights, and results similar to Theorem 3.3 have been established. The fundamental difference as compared to the Dirichlet problem consists in the fact that *serious restrictive conditions appear*. E. g., in the case of the Neumann problem, where one has to work with the space $W^{k,2}(\Omega;S)$ instead of $W_0^{k,2}(\Omega;S)$, the following analogue of Theorem 3.3 holds.

3.6. Theorem. *Let for* $\Omega \subset \mathbb{R}^N$ *and* $M \subset \partial\Omega$ *with* $m = \dim M$ *the following condition hold:*

$$N - m \geqq 2k + 1 . \tag{3.6}$$

Then there exists an interval J *containing* 0 *such that for* $\varepsilon \in J$ *the* (H_1,H_2)*-ellipticity conditions* (3.3) *are satisfied with* $H_1 = W^{k,2}(\Omega;(\mathrm{dist})^{\varepsilon})$, $H_2 = W^{k,2}(\Omega;(\mathrm{dist})^{-\varepsilon})$ (and consequently, existence and uniqueness of a weak solution $u \in H_1$ of the Neumann problem for an elliptic equation of order $2k$ is guaranteed).

The p r o o f uses the same ideas as the proof of Theorem 3.3, but it is based on the imbedding

$$W^{1,2}(\Omega;(\mathrm{dist})^{\varepsilon}) \subsetneq L^2(\Omega;(\mathrm{dist})^{\varepsilon-2})$$

which, in contrary to the imbedding (3.5), holds only for $\varepsilon > 2 + m -$

N . This difference leads to the unpleasant restriction (3.6).

3.7. Remark. For *second order* equations, i.e. for $k = 1$, condition (3.6) has the form

$$N - m \geq 3$$

and excludes many important and interesting special cases of M as points (vertices - $m = 0$) or lines (edges - $m = 1$) on the boundaries of domains Ω of dimension $N = 2$ or $N = 3$, respectively. Nonetheless, for some *special domains* (cubes) and *special operators* ($-\Delta$), J. VOLDŘICH derived results analogous to Theorem 3.3, even in the case if (3.6) is violated. For details see A. KUFNER, J. VOLDŘICH [1].

3.8. Another approach. The method described above is a little more complicated than the usual method mentioned in Introduction: It needs a *pair* of Banach spaces and *two* "ellipticity" conditions (3.3) instead of one simpler condition (1.4) and involves the Lax-Milgram-Nečas Lemma mentioned in the beginning of this section. In the paper of A. KUFNER, J. RÁKOSNÍK [1], another method is proposed which uses only *one* (weighted) space and requires the classical version of the Lax-Milgram Lemma.

Let us describe the method for the *Dirichlet problem*. We introduce a *new* bilinear form b by the formula

$$b(u,v) = a(u,\sigma v) \tag{3.7}$$

where σ is a (sufficiently smooth) weight function, and consider the weighted space $W^{k,2}(\Omega;S)$ with the family $S = \{\sigma_\alpha(x) = \sigma(x)$ for all $|\alpha| \leq k\}$ as well as the corresponding space $W_0^{k,2}(\Omega;S)$. For a given functional $f \in (W_0^{k,2}(\Omega;S))^*$ and a given function $u_0 \in W^{k,2}(\Omega;S)$, we say that the function $u \in W^{k,2}(\Omega;S)$ is a *σ-weak solution of the Dirichlet problem for the operator* L if

$$u - u_0 \in W_0^{k,2}(\Omega;S)$$

and

$$b(u,v) = \langle f,v \rangle \quad \text{for every} \quad v \in W_0^{k,2}(\Omega;S)$$

(provided $b(u,v)$ is meaningful for $u, v \in W^{k,2}(\Omega;S)$).

Let us further consider a weight function σ which satisfies the following conditions: There exist a weight σ_0 and constants c_1 , c_2 such that

$$\int_\Omega |u(x)|^2 \sigma_0(x)\, dx \leq c_1 \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^2 \sigma(x)\, dx \tag{3.8}$$

for every $u \in C_0^\infty(\Omega)$ and

$$|\nabla\sigma(x)|^2/\sigma(x) \leq c_2\sigma_0(x) \quad \text{a.e. in } \Omega . \tag{3.9}$$

Then it can be shown that the form $b(u,v)$ is bounded on $W^{1,2}(\Omega;S) \times W^{1,2}(\Omega;S)$. Further, it can be shown that if the constant c_1c_2 is sufficiently small then the form $b(u,v)$ satisfies the ellipticity condition $b(u,u) \geq c_0 \| u\|^2_{1,2;S}$, so that the existence and uniqueness of a σ-weak solution follows by a standard application of the Lax-Milgram Lemma.

3.9. Remarks. (i) The last result was derived for $k = 1$, i.e. for the *second order* operators only. For $k > 1$, we have to consider weights σ which fulfil conditions (3.8), (3.9) repeatedly (i.e. for σ there must exist the corresponding σ_0 , for σ_0 the corresponding $(\sigma_0)_0$ etc. k-times).

(ii) The pair of conditions (3.8), (3.9) on σ can be replaced by the single condition

$$|\nabla\sigma(x)| \leq c_2\sigma(x) \quad \text{a.e. in } \Omega \tag{3.10}$$

(in the case $k = 1$). For such weights we again deduce that $b(u,v)$ is continuous and, moreover, for $c_2 > 0$ sufficiently small also elliptic, so that existence and uniqueness of a σ-weak solution in $W^{1,2}(\Omega;S)$ follows in a standard way.

3.10. Examples. (i) For $\sigma(x) = [\mathrm{dist}(x,M)]^\varepsilon$, condition (3.9) is satisfied with $\sigma_0(x) = [\mathrm{dist}(x,M)]^{\varepsilon-2}$ and $c_2 = \varepsilon^2$ and condition (3.8) is satisfied with $c_1 \approx |\varepsilon - 1|^{-\frac{1}{2}}$ if $\varepsilon \neq 1$ and with $c_1 \approx |\varepsilon + N - m - 2|^{-\frac{1}{2}}$ if $\varepsilon \neq m + 2 - N$ ($m = \dim M$, $N = \dim \Omega$). Consequently, we obtain an assertion about the existence and uniqueness of a $(\mathrm{dist})^\varepsilon$-weak solution u of the Dirichlet problem in the space $W^{1,2}(\Omega;(\mathrm{dist})^\varepsilon)$ *provided* $|\varepsilon|$ *is sufficiently small*, i.e. $\varepsilon \in J^*$ where J^* is an interval containing the origin. Thus, we have obtained a result similar to **Theorem** 3.3, and a comparison of the interval J from **Theorem** 3.3 with the interval J^* shows that (at least in some special cases) $J^* \supset J$ so that our second approach improves the set of admissible powers.

(ii) The weight $\sigma(x) = \exp(\varepsilon\, \mathrm{dist}(x,M))$ satisfies condition (3.10) with the constant $c_2 = |\varepsilon|$. Weights of such a type are suitable for *unbounded* domains Ω and the existence and uniqueness of a σ-weak solution is guaranteed for $|\varepsilon|$ small.

3.11. Other boundary value problems can be dealt with in the same manner, and similar difficulties arise as in the first approach. E. g., if we consider the Neumann problem in the space $W^{1,2}(\Omega;S)$ with $\sigma(x) = [\mathrm{dist}(x,M)]^{\varepsilon}$, we obtain a result about the existence and uniqueness for $\varepsilon \in J^*$ which is the same interval as in the case of the Dirichlet problem, but under the restrictive condition $N - m \geq 3$. Further, one can show that for $N - m = 1$ our method cannot be used while for $N - m = 2$ we find that admissible values are positive ε's from J^*. - On the other hand, the mixed boundary value problem admits existence for $\varepsilon \in J^*$ without restriction on the dimension of M.

3.12. Remark. Since the form $a(u,v)$ was derived from the operator L by using Green's formula for the integral $\int_\Omega Lu\, v\, dx$, $v \in C_0^\infty(\Omega)$, the form $b(u,v) = a(u,\sigma v)$ can be derived in the same way from the integral $\int_\Omega Lu(\sigma v)\, dx = \int_\Omega \sigma Lu\, v\, dx$. Consequently, we can treat our σ-weak solution as the solution of a boundary value problem for the operator σLu. Since $\sigma(x) > 0$ a.e. in Ω, the difference between a weak and a σ-weak solution is more or less formal.

References

KUFNER, A.:

[1] *Weighted Sobolev spaces.* J. Wiley & Sons, Chichester-New York -Brisbane-Toronto-Singapore 1985

KUFNER, A.; OPIC, B.:

[1] *The Dirichlet problem and weighted spaces I.* Časopis Pěst. Mat. 108(1983), 381-408

[2] *How to define reasonably weighted Sobolev spaces.* Comment.Math. Univ. Carolinae 25(3)(1984), 537-554

[3] *The Dirichlet problem and weighted spaces II.* To appear in Časopis Pěst. Mat.

KUFNER, A.; VOLDŘICH, J.:

[1] *The Neumann problem in weighted Sobolev spaces.* Math. Rep. Roy. Soc. Canada

KUFNER, A.; RÁKOSNÍK, J.:

[1] *Linear elliptic boundary value problems and weighted Sobolev spaces: A modified approach.* Math. Slovaca 34(1984), No.2 185-197

NEČAS, J.:

[1] *Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle.* Ann. Scuola Norm. Sup. Pisa 16(1962), 305-326

[2] *Les méthodes directes en théorie des équations elliptiques.* Academia, Prague & Masson et C^{ie}, Paris 1967

VOLDŘICH, J.:

[1] *A remark on the solvability of boundary value problems in weighted spaces.* To appear in Comment Math. Univ. Carolinae