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ON THE CONVERGENCE OF DIFFERENCE SCHEMES FOR CLASSICAL AND WEAK SOLUTIONS OF THE DIRICHLET PROBLEM[1]

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I. Introduction

In the past much work has been done on convergence of sequences of solutions of difference analogs of the Dirichlet problem for second order uniformly elliptic equations and in particular Laplace's equation and Poisson's equation (c.f. FORSYTHE and WASOW [4], HUBBARD [5] and literature cited therein). Usually some rather restrictive conditions concerning smoothness of the solution of the continuous problem have been imposed in order to obtain the results. There have been, however, several studies of convergence properties under less stringent assumptions. Interesting results along these lines have been obtained for rectangular domains by WASOW [10], WALSH and YOUNG [9], and NITSCHE and NITSCHE [7] and for piecewise analytic boundaries with corners by LAASONEN [6]. Other important work has been done by CEA [3] who studied self-adjoint equations with bounded and measurable coefficients and obtained theorems on convergence of difference approximations to weak solutions in L_2.

In this paper some recent results of the author, the author and HUBBARD, and the author, HUBBARD and ZLÁMAL will be presented. Only indications of the proofs will be given since all of this work will be published elsewhere in complete detail. All the results share the common property that the smoothness conditions are much weaker than those classically assumed.

Although many of the results have been extended to equations with variable

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coefficients and various difference approximations, in order to minimize detail, I will consider only the Laplace operator and one of its simplest difference analogs.

II. Continuous and discrete problems.

Let R be a bounded region in N-dimensional Euclidean space with boundary δR. We shall, in the usual manner, consider the space as having been covered by hypercubes of side h and call the corners mesh points. Those mesh points in R shall be called R_h and the intersection of δR with the edges of the cubes will be called δR_h.

We shall denote the Laplace operator by Δ and the difference analog by Δ_h. The operator Δ_h will be defined for functions on $\bar{R}_h = R_h \cup \delta R_h$ as follows. When a point $x \in R_h$ has its $2N$ nearest neighbors also in R_h then Δ_h is the usual $2N + 1$ point approximation to Δ. We consider, at the remaining points of R_h, Δ_h to be defined as a locally $O(1)$ operator (bounded independent of h for smooth functions) and such that the matrix arrising from Δ_h operating on functions vanishing on δR_h is symmetric and of positive type. This is just one of the standard formulations which is globally second order for problems with smooth solutions.

We shall concern ourselves with approximating solutions to two problems. First, the solution u of the classical Dirichlet problem, which satisfies

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } R \\ u &= f \qquad \text{on } \delta R \end{aligned} \tag{2.1}$$

where f is a given continuous function on δR. That is, u is to be continuous on the closure, satisfy $\Delta u = 0$ in R and its restriction to δR should be equal to f. Conditions on δR in order for this problem to be solvable are, of course, well known.

The other problem to be discussed is a weak formulation of the problem for the inhomogeneous equation with homogeneous boundary values. More precisely we define the class

$$T = \{\varphi | \varphi \in C^{2+\alpha}(R) \cap C^0(\bar{R}) \colon \varphi(x) = 0,\ x \in \delta R;\ \Delta\varphi \in C_0^\alpha(R);\ \text{for some } \alpha\}.$$

In words, each member must have Hölder continuous second partial derivatives in R, be continuous on $\bar{R}$ (the closure of R), vanish on δR and its Laplacian must have compact support in R. Note that T contains the standard "test functions". We then want to consider the solution u, belonging to the real Banach space L_p. $1 \leq p < \frac{N}{N-2}$, of the equation

$$(2.2) \qquad \int_R u\, \Delta\varphi \, \mathrm{d}x = \int_R \varphi F \, \mathrm{d}x, \qquad \varphi \in T,$$

for a given $F \in L_1$. (If F and δR are sufficiently regular then the "weak solution" u will be the classical one, having zero boundary values.)

We shall consider the following approximating problems as analogs of (2.1) and (2.2) respectively:

$$(2.1\mathrm{h}) \qquad \begin{aligned} \Delta_h u_h(x) &= 0, & x &\in R_h \\ u_h(x) &= f(x). & x &\in \delta R_h \end{aligned}$$

and

$$(2.2\mathrm{h}) \qquad \begin{aligned} \Delta_h u_h(x) &= F_h(x), & x &\in R_h \\ u_h(x) &= 0, & x &\in \delta R_h. \end{aligned}$$

In (2.2h) F_h is defined as

$$F_h(x) = \frac{1}{h^N} \int_{S_h(x)} F(y) \, \mathrm{d}y$$

where $S_h(x)$ is the (normally oriented) hypercube of side h and center x, and F is extended to be zero outside R.

We shall in the sequel use the notation $\overleftarrow{V(x)}$ to mean the extension of a function $V(x)$ defined on R_h, as constant over $S_h(x) \cap R$ and zero outside $R \cap [\bigcup_{x \in R_h} S_h(x)]$.

III. Some results on convergence.

We call a domain R which has no "unstable" boundary points a regular domain (c.f. Brelot [1]). [This condition admits quite general domains and in particular problem (2.1) is always solvable for such regions.]

Theorem 1. *Let R be a regular domain and u the solution of* (2.1). *Then if u_h is the solution of* (2.1h), *$\tilde{u}_h \to u$ uniformly on R as $h \to 0$.*

Although there are several theorems on convergence of difference approximations in the literature, it is not clear what the most general known result is for the classical Dirichlet problem. In any case this theorem gives a quite general result. The proof is quite simple and relies on an approximation theorem of the type studied by Walsh [8]. The appropriate theorem is given in Brelot [1]. To extend this theorem to more general second order operators an approximation theorem of Browder [2] is used. More restrictions must be placed on the domain in this case (he calls the resulting domains "firmly regular") but the result is still quite general.

The following existence and uniqueness theorem is easily proved.

Theorem 2. *Let R be a regular domain and let $F \in L_1$. Then there exists a unique $u \in L_p$, $1 \leq p < \frac{N}{N-2}$, such that* (2.2) *holds.*

Such a theorem can also be proved for operators with variable coefficients for "firmly regular" domains provided the coefficients and those of the formal adjoint satisfy some smoothness conditions.

From our point of view here, an interesting method of proof makes use of the difference approximations. We obtain the following convergence theorem as a byproduct.

Theorem 3. *Let R be a regular domain and $u \in L_p$, $1 \leq p < \frac{N}{N-2}$, be the solution of* (2.2). *If u_h is the solution of* (2.2h) *then $\tilde{u}_h \to u$, strongly in L_p, $1 \leq p < \frac{N}{N-2}$, as $h \to 0$.*

The proof involves showing that the functions $\tilde{u}_h$ are uniformly bounded in L_p for all $1 \leq p < \frac{N}{N-2}$. By the weak compactness of bounded sets in L_p, $1 < p < \infty$, we obtain a weak limit point which is then shown to satisfy (2.2). The uniqueness tells us that $\tilde{u}_h \to u$, weakly in L_p as $h \to 0$. An additional argument can then be employed to show the strong convergence. The extension of this theorem to operators with variable coefficients, though true, is not a triviality.

It is interesting to note here that even in two dimensions there are problems of the form (2.2) whose solutions are not continuous. This is true only for $F \in L_1$, and $F \notin L_p$, $p > 1$. If $F \in L_p$, $p > 1$ and $N = 2$ then u will be continuous.

IV. Some results on rates of convergence.

In this section we shall consider regions R whose boundaries are no worse than of class C^2 (or piecewise C^2). We have the following:

Theorem 4. *Let $\delta R \in C^2$ and suppose that the solution u of* (2.1) *is of class $C^{m+\lambda}(\bar{R})$, $m = 0, 1, \ldots, 0 \leq \lambda \leq 1$. Then if u_h is the solution of* (2.1h) *it follows that*

$$(4.1) \qquad \max_{x \in R_h} |u_h(x) - u(x)| \leq K(\varepsilon) \begin{cases} h^{m+\lambda-\varepsilon} + h^{2-\varepsilon}; & m = 0, 1, 2 \\ h^2; & m \geq 3 \end{cases}$$

where ε is an arbitrary positive number and $K(\varepsilon)$ depends on ε and u but not on h.

The proof of this theorem is based on some delicate estimates of the behavior of the discrete Green's function.

It should be pointed out here that an order h^2 estimate is essentially achieved when $u \in C^{2+0}(\bar{R})$. Previous results required that $u \in C^{4+0}(\bar{R})$ in order to obtain a second order error estimate. The present theorem yields a great deal more information than other theorems on this subject. The author has subsequently become aware of a paper of Bahualov (Vestnik Moskov. Univ. Meh. Astronom. Fiz. Chem. (1959) pp. 171—195) which essentially contains this result.

In the important case $N = 2$ the results are better, in that piecewise C^2 boundaries are treated. We have

Theorem 5. *Let* $N = 2$ *and* $\delta R \in C^2$ *piecewise with no reentrant cusps, i.e,* R *is composed of a finite number of* C^2 *arcs meeting at (interior) angles* π/α_i, $i = 1, \ldots, k$, $\alpha_i > 1/2$. *Then* (4.1) *holds.*

We now consider the case of problem (2.2). It is possible to obtain rate of convergence estimates even assuming no more than that $F \in L_1$. In this case we obtain only interior L_p estimates.

Theorem 6. *Let* $\delta R \in C^2$ *and* u *and* u_h *be solutions of* (2.2) *and* (2.2h) *for a given* $F \in L_1$. *Then if* $\Psi \in C_0^\infty(R)$ *the following estimate holds for* $N = 2$.

$$(4.2) \qquad ||(\tilde{u}_h - u)\,\Psi||_{L_p} \leqq K(p, \Psi)\,||F||_L \begin{cases} h; & 1 \leq p < \dfrac{N}{N-1} \\ h|\ln h|; & p = \dfrac{N}{N-1} \\ h^{\frac{2}{p}}; \; 2 < p < \infty \end{cases}$$

where $K(p, \Psi)$ *is a constant depending on* p *and* Ψ *but not on* h. *The notation* $||\cdot||_{L_p}$ *is just the usual* L_p*-norm,* $1 \leq p < \infty$.

This result is obtained from a careful estimation of the difference between the discrete and continuous Green's functions. Theorem 4 is used in the derivation of this estimate. Since the analysis is based on the knowledge of the discrete and continuous fundamental solutions, the result only has been obtained for the Laplace operator. A similar result should be true in the more general case.

Other results of this type have been obtained. For example, when F is Hölder continuous with exponent α the estimates go up to $h^{1+\alpha}$ on compact subsets. Also if F is smooth on an open subset Ω of R, local maximum norm estimates can be obtained on compact subsets of Ω. This type of result shows that the local properties of elliptic operators are carried over to local convergence properties of corresponding difference approximations.

Finally, we consider the case where more precise knowledge of F is given. In particular we suppose that F is smooth, except at the origin O, (an arbitrary point of $\bar{R}$) and for simplicity that δR is smooth. For convenience we suppose that O lies at the center of a mesh hypercube for every h. We also prefer here to state the hypotheses on the solution u itself, rather than as conditions on F.

Theorem 7. *Let u be the solution of* (2.2) *and F be such that*

$$u \in C^{4+0}(\bar{R} - 0)$$

$$|D^k u(x)| \leq K \begin{cases} 1; & k \leq m \\ |x|^{m+\lambda-k}; & m+1 \leq k \leq 4, \end{cases} \tag{4.3}$$

$k = 0, 1, \ldots, 4$, *where $|x|$ is the distance from x to 0 and D^k stands for an arbitrary partial derivative of order k. In* (4.3) *m is an integer (not necessarily positive) less than or equal to 3, $0 < \lambda \leq 1$ and $m + \lambda > 2 - N$. Then if u_h is the solution of* (2.2h) *we have the estimates, for $x \in R_h$,*

$$|u_h(x) - u(x)| \leq K(\varepsilon) \begin{cases} h^{m+\lambda+N-2-\varepsilon}|x|^{\varepsilon+2-N} & 2 - N < m + \lambda \leq 4 - N \\ h^2|x|^{m+\lambda-2}, & 4 - N < m + \lambda < 2 \\ h^2, & 2 < m + \lambda \end{cases} \tag{4.4}$$

where ε is an arbitrary positive number and $K(\varepsilon)$ depends on ε but not on h. If $N > 3$ then the last inequality is valid for $2 \leq m + \lambda$.

The proof of this result is again based on the Green's function method. It involves the construction of certain majorants and the development of some new discrete inequalities suggested by known continuous ones.

Again it should be pointed out that this type of result displays the local effect of singularities on the convergence rate of difference analogs of elliptic problems. Note that we still get convergence away from the origin for any function whose singularity is not as bad as that of the fundamental solution and quadratic convergence even allowing bad behavior at the origin.

BIBLIOGRAPHY

[1] BRELOT, M.: *Sur l'approximation et la convergence dans la theorie des fonctions harmoniques ou holomorphes. Bull. Soc. Math. France 73, 55—70 (1945).*

[2] BROWDER, F. E.: *Approximation by solutions of partial differential equations. Am. J. Math. 84, 134—160 (1962).*

[3] CÉA, J.: *Sur l'approximation des problemes aux limites elliptiques. Compte rendus 254, 1729—1731 (1962).*

[4] FORSYTHE, G. AND WASOW, W.: *Finite difference methods for partial differential equations. Wiley, N. Y. (1960).*

[5] HUBBARD, B. E.: *Remarks on the order of convergence in the discrete Dirichlet problem. Numerical Solution of Partial Differential Equations (edited by J. H. Bramble) Academic Press, New York and London, 21–23 (1966).*

[6] LAASONEN, P.: *On the degree of convergence of discrete approximations for the solutions of the Dirichlet problem. Ann. Acad. Sci. Fenn. Ser. A. 246, 1–19 (1957).*

[7] NITSCHE, J. AND NITSCHE, J. C. C.: *Error estimates for the numerical solution of elliptic differential equations. Arch. Rat. Mech. Anal. 5, 293–306 (1960).*

[8] WALSH, J. L.: *The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions. Bull. Am. Math. Soc. 35, 499–514 (1929).*

[9] WALSH, J. L. AND YOUNG, D.: *On the degree of convergence of solutions of difference equations to the solution of the Dirichlet problem. J. Math. Phys. 33, 80–93 (1954).*

[10] WASOW, W.: *On the truncation error in the solution of Laplace's equation by finite differences. J. Res. Nat. Bur. Standards, 48, 345–348 (1952).*