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STRUCTURE OF GREEN'S OPERATORS AND ESTIMATES FOR THE CORRESPONDING EIGENVALUES.[1]

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It is well-known that one of the most difficult problems which mathematical physics poses to quantitative analysis is the *rigorous approximation* of the eigenvalues of certain boundary value problems which arise in applied mathematics. By *rigorous approximation* we mean giving lower and upper bounds for any particular eigenvalue such that these bounds approach this eigenvalue to any prescribed degree of accuracy. It is convenient to consider these problems in an abstract Hilbert space setting. To this end we consider a complex separable Hilbert space S and a linear operator T which maps S into itself. We suppose that T is a positive compact operator (PCO), i.e., T is such that $(Tu, u) > 0$ for $u \neq 0$ [$(.,.)$ is the scalar product in S] and T maps weakly convergent sequences onto strongly convergent sequences. It is well-known (and very easy to prove) that positiveness implies that T is hermitian [i.e., $(Tu, v) = (u, Tv)$ for any u and v in S] and compactness implies that T is bounded.

Let us consider the eigenvalue problem

$$Tu - \mu u = 0. \tag{1}$$

A fundamental theorem in Hilbert space theory states that the eigenvalues of problem (1) constitute a sequence of positive real numbers converging to zero if — as we shall suppose — S is infinite dimensional. Each eigenvalue has finite multiplicity, i.e., the kernel of the linear operator $T_\mu = T - \mu I$ has finite dimension (multiplicity of μ). Let

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k \geq \ldots$$

be the sequence of the eigenvalues of T, each repeated as many times as its multiplicity. From now on when we mention the sequence of eigenvalues of

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a PCO, it shall be understood that this sequence is ordered according to the criterion just specified; thus the statement: *μ is the k-th eigenvalue of a certain PCO is precise.*

The problem of the rigorous approximation of the eigenvalues of T is the following.

For any given k we want to construct two sequences $\{\mu_k^{(\nu)}\}$ and $\{\sigma_k^{(\nu)}\}$. $\nu = 1, 2, \ldots,$ *such that:*

(2) $\quad \mu_k^{(\nu)} \leq \mu_k^{(\nu+1)}, \qquad$ (3) $\quad \lim_{\nu \to \infty} \mu_k^{(\nu)} = \mu_k,$

(4) $\quad \sigma_k^{(\nu)} \geq \sigma_k^{(\nu+1)}, \qquad$ (5) $\quad \lim_{\nu \to \infty} \sigma_k^{(\nu)} = \mu_k.$

The sequence $\{\mu_k^{(\nu)}\}$ (called lower bounds) can be constructed in a rather simple way by means of the classical *Rayleigh-Ritz method.* Let $\{w_i\}$ be a sequence of linearly independent vectors complete in the space S. Let us denote by

$$\mu_1^{(\nu)} \geq \mu_2^{(\nu)} \geq \ldots \geq \mu_\nu^{(\nu)} \tag{6}$$

the roots of the determinantal equation

$$\det \{(Tw_i, w_j) - \mu(w_i, w_j)\} = 0, \qquad (i.\, j = 1, \ldots, \nu). \tag{7}$$

The following theorem, which goes back to Plancherel [11], states that $\{\mu_k^{(\nu)}\}$ is a sequence of the lower bounds.

Theorem 1. *The sequence $\{\mu_k^{(\nu)}\}$ obtained through the Rayleigh-Ritz method satisfies conditions* (2), (3).

It is of interest to remark, in view of applications to partial differential equations, that, instead of using Eq. (7), we may obtain the Rayleigh-Ritz approximations from the equation

$$\det \{(Tw_i, Tw_j) - \mu(Tw_i, w_j)\} = 0, \qquad (i, j = 1, \ldots, \nu). \tag{8}$$

Theorem I. still holds if we substitute Eq. (8) for Eq. (7).

The construction of the sequence $\{\sigma_k^{(\nu)}\}$ (called *upper bounds*) is a much more difficult problem. The first approach to this problem is due to A. Weinstein [16], who considered the eigenvalue problems connected with the classical boundary value problems of elastic plates. The Weinstein method, known now as the *method of intermediate problems*, was later reformulated in terms of a PCO in a Hilbert space and deeply investigated (and generalized) by Aronszajn (see [1], [2] and [10]). Further important results have been obtained by the Weinstein school, especially by Weinberger [15], Bazley [3] and Bazley—Fox [4].

It has been proved (see [6], [7]) that the Weinstein method can be included in the following formulation of the theory of intermediate problems, due to

Aronszajn. Let T_0 be a PCO such that $T_0 > T$. Assume that *the sequence of eigenvalues* $\{\mu_k^{(0)}\}$ *of* T_0 *and the corresponding sequence of eigenvectors are known.* T_0 is called a *base operator.* Then it is possible to construct a sequence $\{T_\nu\}$ of PCO's such that

i) $T_0 \geq T_\nu \geq T_{\nu+1} \geq T$;

ii) $T_0 - T_\nu$ is a degenerate operator. i.e., its range is finite dimensional;

iii) T_ν converges uniformly to T for $\nu \to \infty$, i.e. $\lim_{\nu\to\infty} ||T_\nu - T|| = 0$. If we let $\sigma_k^{(\nu)}$ be the k-th eigenvalue of T_ν, conditions i), iii) assure that $\sigma_k^{(\nu)}$ satisfies conditions (4), (5). Of course the problem of the actual computation of $\sigma_k^{(\nu)}$ must still be solved. To this end one takes advantage of condition ii) and, by using standard techniques of finite rank perturbation theory, it is possible to find eigenvalues of T_ν as zeros of certain meromorphic functions, which Weinstein introduced for the first time. One then uses two different procedures: one for the eigenvalues of T_ν which are not eigenvalues of the base operator; and another for the eigenvalues which are eigenvalues both for T_0 and T_ν. The main numerical difficulty occurs in finding the zeros of the above mentioned meromorphic functions. Some procedures have been given by Weinstein, Weinberger, Bazley and Fox in order to avoid this difficulty. As a matter of fact in many important applications the eigenvalues of T_ν can be found as zeros of very simple functions.

The method of intermediate problems has led to the solution of many interesting eigenvalue problems since Weinstein published his important paper [16] on 1937. Let us mention, among all these, the outstanding result obtained on 1961 by Bazley [3], who was able to give remarkable lower bounds for the first two eigenvalues of the helium atom.

However, one of the theoretical restriction in the method of the intermediate problems is the assumption that a base operator T_0 must be known. For example, if we consider the very simple and classical eigenvalue problem in the space $S \equiv L^2(0, 1)$, for the Fredholm operator

$$Tu = \int_0^1 K(x, y)\, u(y)\, dy,$$

($K(x, y)$ continuous and $K(x, y) = \overline{K(y, x)}$), we do not know, in general, how to construct a base operator.

Therefore in the last two years a different method has been developed by the author. This new method applies to a class of operators smaller than those considered in the theory of intermediate problems, but its application requires less (e.g., base operator) information. The resulting application of this method to eigenvalue problems for elliptic linear differential systems has led to new investigations in the theory of those systems, which, in the opinion of the author, are of interest on their own.

In the present paper we wish to expose the main results of this new method, together with some numerical applications which have been carried out at the Computing Center of the Faculty of Sciences at the University of Rome. For these numerical calculations the author wishes to express his sincere thanks to L. de Vito, director of the Computing Center, and to A. Fusciardi, F. Scarpini and M. Schaerf. A complete account of the theory of orthogonal invariants and their applications to partial differential equations can be found in [6] or in [7].

1. Method of orthogonal invariants.

Let T be the above considered PCO. Let us denote by $\Gamma^{(n)}(w_1, \ldots, w_s)$ the Gram determinant of s given vectors $w_1, \ldots, w_s$ in the space S, with respect to the scalar product $(T^n u, v)$; i.e., $\Gamma(w_1, \ldots, w_s) = \det\{(T^n w_i, w_j)\}$, $(i, j = 1, \ldots, s)$. Let $\{v_k\}$ $(k = 1, 2, \ldots)$ be an orthonormal complete system in the space S. We set

$$\mathscr{Y}_0^n(T) = 1$$

and, for $s > 0$,

$$\mathscr{Y}_s^n(T) = \frac{1}{s!}\sum_{k_1 \ldots k_s} \Gamma^{(n)}(v_{k_1}, \ldots, v_{k_s}). \tag{9}$$

The summation $\sum\limits_{k_1 \ldots k_s}$ must be understood to be over any set of s positive integers. Since the multiple series on the right hand side of (9) has non-negative terms, its sum — finite or not — is independent of the summation procedure. It is evident that

$$\mathscr{Y}_s^n(T^m) = \mathscr{Y}_s^{nm}(T).$$

The following theorems hold.

Theorem II. *$\mathscr{Y}_s^n(T)$ is independent of the particular orthonormal complete system used in its definition, i.e., $\mathscr{Y}_s^n(T)$ is an orthogonal invariant for the operator T.*

The index s will be called the *order* of the orthogonal invariant $\mathscr{Y}_s^n(T)$ and the index n the *degree* of this invariant.

Theorem III. *We have $\mathscr{Y}_s^n(T) < +\infty$ if and only if $\mathscr{Y}_1^n(T) < +\infty$.*

We denote by $\mathfrak{C}^n$ the class of all the PCO's such that $\mathscr{Y}_1^n(T) < +\infty$. We then have $\mathfrak{C}^m \subset \mathfrak{C}^n$ if $m < n$. There exists PCO's such that they do not belong to any $\mathfrak{C}^n$. However PCO's which are encountered in mathematical physics generally belong to some $\mathfrak{C}^n$ for n large enough.

Theorem IV. *The sequence* $\mathcal{Y}_s^n(T)$, $(s = 1, 2, \ldots)$ *is a complete system of invariants for unitary equivalence of two operators of the class* $\mathfrak{C}^n$.

This means that if $T, R \in \mathfrak{C}^n$, we have $T = U^{-1}RU$, with U an unitary operator, if and only if $\mathcal{Y}_s^n(T) = \mathfrak{J}_s^n(R)$, $s = 1, 2, \ldots$

Theorem V. *If* $T_2 \geq T_1$, $(T_i \in \mathfrak{C}^n,\ i = 1, 2)$, *then* $\mathcal{Y}_s^n(T_2) \geq \mathcal{Y}_s^n(T_1)$.

Theorem VI. *If* T_k *converges uniformly to* T, $(T_k, T \in \mathfrak{C}^n)$, *then* $\lim_{k\to\infty} \mathcal{Y}_s^n(T_k) = = \mathcal{Y}_s^n(T)$.

Let $\{w_i\}$ be the above introduced complete sequence of linearly independent vectors. Let W^ν be the ν-dimensional subspace spanned by the vectors $w_1, \ldots, \omega_\nu$ and P_ν the orthogonal projector which maps S onto W^ν. Let us consider the positive eigenvalues of the operator $P_\nu T P_\nu$, that is to say, the roots (6) of the equation (7). Let $\tilde{\omega}_k^{(\nu)}$ $(k \leq \nu)$ be an eigenvector corresponding to the eigenvalue $\mu_k^{(\nu)}$ of $P_\nu T P_\nu$. We denote by $\tilde{W}_k^\nu$ the one-dimensional subspace spanned by $\tilde{\omega}_k^{(\nu)}$. Let $P_k^{(\nu)}$ be the orthogonal projector of S onto $\tilde{W}^\nu \ominus W_k^\nu$. Let us remark that $P_\nu T P_\nu$ and $P_\nu^{(k)} T P_\nu^{(k)}$ are PCO's when considered in the spaces W^ν and $W^\nu \ominus \tilde{W}_k^\nu$ respectively.

Theorem VII. *Let* $T \in \mathfrak{C}^n$. *Given* $s > 0$, *for* $\nu \geq s$ *let*

$$\sigma_k^{(\nu)} = \left\{ \frac{\mathcal{Y}_s^n(T) - \mathcal{Y}_s^n(P_\nu T P_\nu)}{\mathfrak{J}_{s-1}^n(P_\nu^{(k)} T P_\nu^{(k)})} + [\mu_k^{(\nu)}]^n \right\}^{\frac{1}{n}}. \tag{10}$$

Then the sequence $\{\sigma_k^{(\nu)}\}$ *satisfies conditions* (4), (5).[2]

The two orthogonal invariants $\mathcal{Y}_s^n(P_\nu T P_\nu)$ and $\mathcal{Y}_{s-1}^n(P_\nu^{(k)} T P_\nu^{(k)})$ must be considered as numerically known since they are expressed as follows through the Rayleigh-Ritz approximations

$$\mathcal{Y}_s^n(P_\nu T P_\nu) = \sum_{h_1 < \ldots < h_s}^{1,\ldots,\nu} [\mu_{h_1}^{(\nu)} \ldots \mu_{h_s}^{(\nu)}]^n,$$

$$\mathcal{Y}_{s-1}^n(P_\nu^{(k)} T P_\nu^{(k)}) = \sum_{h_1 < \ldots < h_{s-1}}^{1,\ldots,\nu(k)} [\mu_{h_1}^{(\nu)} \ldots \mu_{h_{s-1}}^{(\nu)}]^n.$$

The symbol $\sum_{h_1 < \ldots < h_{s-1}}^{1,\ldots,\nu(k)}$ means that the indices $h_1 \ldots h_{s-1}$ are always chosen among the integers $1, \ldots, k-1, k+1, \ldots, \nu$.

Theorem VII solves the problem of the upper approximation of μ_k, *provided that one of the orthogonal invariants* $\mathcal{Y}_s^n(T)$ *of* T *is known.*

Theoretically, because of the definition (9), $\mathcal{Y}_s^n(T)$ can be considered as known. However from the numerical point of view, we can only obtain a lower bound for $\mathcal{Y}_s^n(T)$ since it is expressed as a sum of a series with non-negative terms. On the other hand, formula (10) requires an upper bound for $\mathcal{Y}_s^n(T)$ if we wish $\sigma_k^{(\nu)}$ to be an upper bound for μ_k. In conclusion, we are allowed to use formula (10) for giving an upper bound for μ_k if we are able to estimate the remainder of the series which defines $\mathscr{Y}_s^n(T)$, or if we can compute this invariant by some different procedure. We shall see in the following how to overcome this difficulty.[2] We believe that theorem VII must be considered as a remarkable advance in the problem of obtaining upper approximations for the μ_k's, since *the problem of finding upper bounds for a sequence of numbers* $\mu_1, \mu_2, \ldots, \mu_k, \ldots$ *has been reduced to the problem of giving an upper bound to a single number*: *one of the orthogonal invariants of* T.

Let us now consider a measure space with a non-negative measure μ and let A be a measurable set in this space. Denote by $\mathrm{L}^2(A, \mu)$ the Hilbert space of complex valued functions $u(x)$ on A, with $|u(x)|^2$ summable on A with respect to the measure μ. The scalar product in $\mathrm{L}^2(A, \mu)$ is the following

$$(u, v) = \int_A u(x)\, \bar{v}(x)\, \mathrm{d}\mu_x.$$

Suppose that S is Hilbert-isomorphic to $\mathrm{L}^2(A, \mu)$. It is well-known that it is always possible to choose the measure-space, μ and A in such a way that this is true. For instance, we may take as A any bounded open set of an euclidean space and μ the classical Lebesgue measure.

Theorem VIII. *Let T belong to $\mathfrak{C}^n$. Then there exists a kernel $K^{(n)}(x, y)$ belonging to $\mathrm{L}^2(A \times A, \mu \times \mu)$ such that T^n admits the following representation in the space $\mathrm{L}^2(A, \mu)$*

$$T^n u = \int_A K^{(n)}(x, y)\, u(y)\, \mathrm{d}\mu_y. \tag{11}$$

From this theorem we can deduce the following one which provides an integral representation for the orthogonal invariants of $T \in \mathfrak{C}^n$.

Theorem IX. *Consider the function*

$$f(x_1, \ldots, x_s) = \begin{vmatrix} K^{(n)}(x_1, x_1) \;\ldots\ldots\; K^{(n)}(x_1, x_s) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ K^{(n)}(x_s, x_1) \;\ldots\ldots\; K^{(n)}(x_s, x_s) \end{vmatrix}.$$

It is summable on the cartesian product $\underset{1}{A} \times \underset{2}{A} \times \ldots \times \underset{s}{A}$ *with respect to the product-measure* $\mu_{x_1} \times \mu_{x_2} \times \ldots \times \mu_{x_s}$. *For* $T \in \mathfrak{C}^n$ *we have*

$$\mathscr{Y}_s^n(T) = \frac{1}{s!} \int_A \ldots \int_A f(x_1, \ldots, x_s)\, \mathrm{d}\mu_{x_1} \ldots \mathrm{d}\mu_{x_s}. \tag{12}$$

Representation (12) solves the problem of the computation of $\mathscr{Y}_s^n(T)$ when

[2] $\sigma_k^{(\nu)}$ obviously depends on s and n. However we don't want to indicate this dependence explicitly, since we assume s and n fixed and wish to avoid cumbersome notation.

the kernel $K^{(n)}(x, y)$ corresponding to the operator T^n is known. It follows that (10) furnishes a double sequence of formulas, each of them (for any fixed s and n) solving the problem of upper approximation of eigenvalues for Fredholm (hermitian) integral operators. We would like to observe that using the integral representation (12) of $\mathcal{Y}_s^n(T)$ and letting $n = 2$ and $s = 1$ in the general formula (10), we arrive at a particular formula already known to Trefftz [14].

2. Structure of Green's operators.

If we wish to use representation (12) for eigenvalue problems for differential equations, we must face the main difficulty consisting in the actual knowledge of the Green's function for the associated boundary value problem. It is well-known that only in a very few cases — especially for partial differential equations — is the Green's function explicitly known.

We now want to show to overcome this difficulty for linear elliptic differential systems by means of a new approach to boundary value problems for these systems; the latter will lead us to an explicit construction of the Green's operator. This construction is particularly suitable for using formula (10) or the slight generalization of this formula, given by the following theorem.

Theorem X. *Let $\{T_\varrho\}$ be a decreasing sequence of PCO's uniformly converging to T. Let T_ϱ, T belong to $\mathfrak{C}^n$. Set*

$$\sigma_k^{(\varrho,\ )} = \left\{\frac{\mathcal{Y}_s^n(T_\varrho) - \mathcal{Y}_s^n(P_\nu T P_\nu)}{\mathcal{Y}_{s-1}^n(P_\nu^{(k)} T P_\nu^{(k)})} + [\mu_k^{(\nu)}]^n\right\}^{\frac{1}{n}}.$$

Then
$$\sigma_k^{(\varrho,\nu)} \geq \sigma_k^{(\tilde{\varrho},\tilde{\nu})} \qquad \text{for} \qquad \varrho \leq \tilde{\varrho}, \quad \nu \leq \tilde{\nu},$$
and

$$\lim_{\substack{\varrho\to\infty \\ \nu\to\infty}} \sigma_k^{(\varrho,\nu)} = \mu_k.$$

Let X^r be the r-dimensional real cartesian space; we denote by $x \equiv (x_1, \ldots, x_r)$ a variable point in X^r. If u and v are n-vectors with complex components, their scalar product $u_i \bar{v}_i$ will be denoted by uv. If $a \equiv \{a_{ij}\}$ is an $l \times l$ matrix with complex entries, the l-vector whose components are $a_{ij} u_j$ $(i = 1, \ldots, l)$, will be indicated by au, the adjoint matrix of a, i.e., the matrix $\{\alpha_{ij}\}$ with $\alpha_{ij} = \bar{a}_{ji}$ will be indicated by $\bar{a}$.

Let A be a bounded domain (connected open set) of X^r. We suppose that A is a properly regular domain(3). Let us consider the following linear differential matrix-operator of order $2m$

$$L(x, D) \equiv D^p a_{pq}(x)\, D^q, \qquad (0 \leq p \leq m, \ 0 \leq q \leq m),$$

where $a_{pq}(x)$ are $l \times l$ matrices, which — for simplicity — we assume to be of class C^∞ in the whole space X^r; if p is the multi-index $(p_1, \ldots, p_r)$, D^p denotes — as usual — the partial derivative

$$D^p = \frac{\partial^{|p|}}{\partial x_1^{p_1} \ldots \partial x_r^{p_r}}.$$

We make the following hypotheses.

1) $L(x, D)$ is elliptic for every $x \in X^r$, i. e., we have for any real non-zero n-vector ξ

$$\det a_{pq}(x)\, \xi^p \xi^q \neq 0 \qquad (|p| = |q| = m)$$
$$(\xi^p = \xi_1^{p_1} \ldots \xi_r^{p_r}, \qquad \xi_i^{p_i} = 1 \qquad \text{if } \xi_i = p_i = 0);$$

2) $L(x, D)$ is formally self-adjoint, i.e.,

$$a_{pq}(x) \equiv (-1)^{|p|+|q|} \bar{a}_{qp}(x);$$

3) The bilinear integro-differential form

$$B(u, v) = \sum_{pq} (-1)^{|p|} \int_A (a_{pq} D^q u)\, D^p v\, dx$$

is such that for any function[4] u of class C^∞ in X^r, we have

$$(-1)^m B(u, u) \geq c \sum_{|p|=m} \int_A |D^p u|^2\, dx,$$

where c is a positive constant independent of u.

4) There exists a linear operator R which enjoys the following properties: i) R is a bounded operator with domain $L^2(A')$ (A' is a domain such that $A' \supset \bar{A}$) and range in the Hilbert space $H_m(A')$ of functions with weak derivatives of order $\leq m$ belonging to $L^2(A')$; ii) R is hermitian on $L^2(A')$; iii) for any $f \in L^2(A')$ we have $LRf = f$.

Hypothesis 4) is satisfied when there exists a *fundamental solution* for the operator R. Hypotheses 1), 2), 3), 4) are satisfied by the classical differential operators encountered in eigenvalue theory.

Let us consider the space $C^\infty(\bar{A})$ of C^∞ functions in $\bar{A}$ and the finite dimensional manifold Γ of all the functions w such that $B(w, w) = 0$. Let us denote by $\mathscr{H}(A)$ the Hilbert space obtained through functional completion from the quotient space $C^\infty(\bar{A})/\Gamma$ by means of the norm introduced by the scalar product

$$((u, v)) = (-1)^m B(u, v).$$

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(3) For the precise definition of properly regular domain see [6] p. 21. Roughly speaking, a properly regular domain is a domain with a piece-wise regular boundary such that $\partial A = \partial \bar{A}$ and which satisfies a cone-hypothesis.

(4) The term "function" must be understood as "vector-valued function", since the values of the function are l-vectors with complex components.

Let $(.,.)_{A'}$ denote the scalar product in the space $L^2(A')$. Let R^* be the bounded linear operator with domain $\mathscr{H}(A)$ and range, in $L^2(A')$, defined by the equations

$$((Rf, g)) = (f, R^*g)_{A'}, \qquad [f \in L^2(A'),\ g \in \mathscr{H}(A)].$$

Let P be the orthogonal projector of $\mathscr{H}(A)$ onto its subspace $\Omega(A)$ determined by the solution of the homogeneous equations $Lu = 0$.

Theorem XI. *Let $U(A)$ be the class of all functions belonging to $H_m(A) \cap \cap H_{2m}(A_0)$ for every domain A_0 such that $\bar{A}_0 \subset A$. Then for every $f \in L^2(A)$ there exists in the class $U(A)$ one and only one solution u of the boundary value problem*

$$(13) \qquad \begin{cases} Lu = (-1)^m f & \text{in } A \\ D^p u = 0 \qquad (0 \leq |p| \leq m-1) & \text{on } \partial A. \end{cases}$$

Set

$$G = R^*R - R^*PR.$$

Then the solution u of problem (13) *is given by $u = Gf$. Thus G is the Green operator for the boundary value problem* (13).

Let $\{\omega_k\}$ be a complete system in the space $\Omega(A)$ and $\Omega_\varrho(A)$ be the ϱ-dimensional manifold spanned by $\omega_1, \ldots, \omega_\varrho$. Let P_ϱ be the orthogonal projector of $\mathscr{H}(A)$ onto $\Omega_\varrho(A)$.

Theorem XII. *Set $G_\varrho = R^*R - R^*P_\varrho R$. Then both operators G and G_ϱ, as operators on the Hilbert space $L^2(A)$, belong to $\mathfrak{C}^n$ for any $n > r/2m$. Moreover $\lim_{\varrho \to \infty} ||G - G_\varrho|| = 0$ and $G_\varrho > G_{\varrho+1}$.*

The following eigenvalue problem, considered in the space $U(A)$

$$(14) \qquad \begin{cases} Lu - (-1)^m \lambda u = 0 & \text{in } A, \\ D^p u = 0 \qquad (0 \leq p \leq m-1) & \text{on } \partial A \end{cases}$$

has only positive eigenvalues. Letting $\lambda^{-1} = \mu$, problem (14) is equivalent to the following one in the space $L^2(A)$:

$$(15) \qquad Gu - \mu u = 0.$$

For the upper approximation of the eigenvalues of (15) [i.e., the lower approximation of the eigenvalues of (14)] we can apply theorem X with $T = G$ and $T_\varrho = G_\varrho$. This is possible by theorem XII. For the computation of $I_s^n(G_\varrho)$ we may use theorem IX if an integral representation of R is known; i.e., if a fundamental solution of L is available.

On some other cases the explicit representation of G_ϱ, which we have given, can be used in order to give upper bounds to the remainder of the series which defines $I_s^n(G_\varrho)$.

In the following sections we shall consider as examples some classical eigenvalue problems of mathematical physics.

3. Two or three-dimensional elasticity.

Let us consider the differential operator of classical elasticity, which we shall write as follows in the space X^r ($r = 2, 3$):

$$L_i u = u_{i/hh} + \alpha u_{h/ih}, \qquad (i = 1, \ldots, r);$$

α is a given real constant (depending on the elastic material) such that $\alpha > -1$.

From now on we shall consider only vector-valued functions with real components.

Let us consider the eigenvalue problem

$$(16) \qquad \begin{cases} Lu + \lambda u = 0 & \text{in } A \\ u = 0 & \text{on } \partial A. \end{cases}$$

We can use the following bilinear form

$$B(u, v) = -\int_A (u_{i/h} v_{i/h} + \alpha u_{i/i} v_{h/h})\, dx.$$

Set

$$\varphi(t) \begin{cases} = \log t^{-1} & \text{for } r = 2 \\ = t^{-1} & \text{for } r = 3 \end{cases}$$

$$F_{ij}(x-y) = \frac{\alpha}{8\pi(1+\alpha)} \frac{\partial^2 |x-y|^2 \varphi(|x-y|)}{\partial x_i\, \partial x_j} - \frac{\delta_{ij}}{2(r-1)\pi} \varphi(|x-y|);$$

$$\gamma_{ij}(x, y) = -\int_A \{F_{ik/h}(x-t) F_{jk/h}(t-y) + \alpha F_{ik/k}(x-t) F_{jh/h}(t-y)\}\, dt.$$

Let $\{\omega^s\}$ be a complete system of solutions of the homogeneous equations $Lu = 0$, such that $-B(\omega^s, \omega^l) = \delta_{sl}$(5). Set

$$\varrho_i^s(x) = \int_A \{F_{ik/h}(x-t)\, \omega_{k/h}^s(t) + \alpha F_{ik/k}(x-t)\, \omega_{h/h}^s(t)\}\, dt.$$

Let $\{w^i\}$ be any system of linearly independent functions such that $w^i = 0$ on ∂A and such that $\{Lw_i\}$ be complete in the space $L^2(A)$. Let $\mu_1^{(\nu)} \geq \geq \ldots \geq \mu_k^{(\nu)} \geq \ldots \geq \mu_\nu^{(\nu)}$ be the roots of the determinantal equation

$$\det \{\int_A w_h^i w_h^j\, dx + \mu \int_A w_h^i L_h w^j\, dx\} = 0 \qquad (i, j = 1, \ldots, \nu)$$

(Rayleigh-Ritz approximations). Set

(5) For the construction of a complete system of solutions for $Lu = 0$ see [8] chap. III. The orthonormality condition $-B(\omega^s, \omega^l) = \delta_{sl}$ is assumed here only for the sake of simplicity. It is not necessary in numerical applications.

$$\tau_k^{(\nu)} = \{\sum_{i,j}^{1,r}[\int_A\int_A |\gamma_{ij}(x,y)|^2\,dx\,dy + \sum_{s,l}^{1,\nu(k)}\int_A \varrho_i^s(x)\,\varrho_i^l(x)\,dx\int_A \varrho_j^s(x)\,\varrho_j^l(x)\,dx -$$

$$- 2\sum_s^{1,\nu}\int_A\int_A \gamma_{ij}(x,y)\,\varrho_i^s(x)\,\varrho_j^s(y)\,dx\,dy] - \sum_i^{1,\nu(k)}[\mu_i^{(\nu)}]^2\}^{-\frac{1}{2}},$$

with the usual meaning for the symbol $\sum\limits_i^{1,\nu}$. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \ldots$ be the eigenvalues of problem (16). Then we have

$$\tau_k^{(\nu)} \leq \lambda_k \leq \frac{1}{\mu_k^{(\nu)}}, \qquad (k \leq \nu), \tag{17}$$

and

$$\lim_{\nu\to\infty} \tau_k^{(\nu)} = \lim_{\nu\to\infty} \frac{1}{\mu_k^{(\nu)}} = \lambda_k. \tag{18}$$

4. Vibrations of a clamped plate.

We assume $r = 2$. The eigenvalue problem is the following

$$\begin{cases} \Delta_2\Delta_2 u - \lambda u = 0 & \text{in } A, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial A. \end{cases}$$

($\Delta_2 \equiv$ Laplace operator, $\dfrac{\partial}{\partial n} \equiv$ differentiation along the normal); u is a real valued function. The $\mu_k^{(\nu)}$ are now the roots of the equations

$$\det\left\{\int_A w_i\,w_j\,dx - \mu\int_A \Delta_2\,w_i\,\Delta_2\,w_j\,dx\right\} = 0,\ (i, j = 1, \ldots\ldots, \nu),$$

where the sequence $\{w_i\}$ satiesfies the usual completeness condition and $w_i = \dfrac{\partial w_i}{\partial n} = 0$ on ∂A. Inequalities (17) and the limit relations (18) hold also in this case with

$$\tau_k^{(\nu)} = \left\{\frac{1}{4\pi^2}\int_A\int_A |\log|x-y||^2\,dx\,dy -\right.$$

$$\left. -\frac{1}{4\pi^2}\sum_i^{1,\nu}\int_A\left[\int_A \omega_i(t)\log|x-t|\,dt\right]^2 dx - \sum_i^{1,\nu(k)}\mu_i^{(\nu)}\right\}^{-1};$$

($\{\omega_i\}$ is a complete system of harmonic functions (harmonic polynomials if A is simply connected) orthonormalised in $L^2(A)$.

5. Buckling of a clamped plate.

The eigenvalue problem is now the following

$$\begin{cases} \Delta_2\Delta_2 u + \lambda\Delta_2 u = 0 & \text{in } A, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial A. \end{cases}$$

In this case the lower bounds for λ_k are given by

$$\tau_k^{(\nu)} = \left\{\frac{1}{4\pi^2}\left[\int_A\int_A |\log|x-y||^2\,dx\,dy - 2\sum_i^{1,\nu}\int_A\left[\int_A \omega_i(t)\log|x-t|\,dt\right]^2 dx + \right.\right.$$
$$\left.\left. + \sum_{h,j}^{1,\nu}\left(\int_A\int_A \log|x-y|\,\omega_h(x)\,\omega_j(y)\,dx\,dy\right)^2\right] - \sum_i^{1,\nu\,(k)}[\mu_i^{(\nu)}]^2\right\}^{-\frac{1}{2}}.$$

The upper bounds $[\mu_k^{(\nu)}]^{-1}$ are obtained from the equation

$$\det\left\{\int_A w_i\Delta_2 w_j\,dx + \mu\int_A \Delta_2 w_i\,\Delta_2 w_j\,dx\right\} = 0, \qquad (i, j = 1, \ldots, \nu).$$

The systems $\{\omega_k\}$ and $\{w_i\}$ are the same as in the preceding example.

6. Numerical examples.

We have included in this paper numerical results concerning eigenvalue problems for elastic plates. The upper bounds (i.e., the inverses of the lower bounds for the eigenvalues of the Green operator) have been obtained by the Rayleigh-Ritz method, wherein we have used systems of polynomials. The lower bounds have been obtained by the method of orthogonal invariants and the representation, of the Green operator described in the paper.

For numerical examples concerning ordinary differential equations see [9], [12], [13].

I) Square plate clamped along its boundary.

$$\Delta_2\Delta_2 u - \lambda u = 0 \qquad \text{in } A \equiv \left(-\frac{\pi}{2} < x_1 < \frac{\pi}{2},\ -\frac{\pi}{2} < x_2 < \frac{\pi}{2}\right)^{(6)},$$

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial A.$$

Let r_i be the x_i-axis ($i = 1, 2$). Let r_3 be the line $x_1 = x_2$. By $H^{(\alpha_1\alpha_2)}$, ($\alpha_1 = 0, 1$) we denote the subspace of $L^2(A)$ consisting of all functions which are symmetric with respect to r_i if $\alpha_i = 0$, anti-symmetric if $\alpha_i = 1$. By $H^{(\alpha_1\alpha_2\alpha_3)}$, ($\alpha_i = 0, 1$), we denote the subspace of $H^{(\alpha_1\alpha_2)}$ of all functions belonging to $H^{(\alpha_1\alpha_2)}$ which are symmetric (anti-symmetric) with respect to r_3 if $\alpha_3 = 0$

($\alpha_3 = 1$). The space $L^2(A)$ can then be decomposed into subspaces (which are invariant for the given problem) as follows:

$$L^2(A) = H^{(000)}(A) \oplus H^{(001)}(A) \oplus H^{(110)}(A) \oplus H^{(111)}(A) \oplus H^{(01)}(A) \oplus H^{(10)}(A).$$

	000 lower bound	000 upper bound	001 lower bound	001 upper bound
λ_1	13.29376	13.29378	177.7193	177.7401
λ_2	179.408	179.431	976.13	979.59
λ_3	496.55	497.03	1569	1584
λ_4	977.64	981.25	3158	3282
λ_5	1577	1593	4038	4306
λ_6	3120	3244	5865	6791
λ_7	3155	3284	6774	8330
λ_8	4037	4317	7555	9931
λ_9	5853	6817		
λ_{10}	6701	8276		

(6) For the analytical and numerical investigation of this problem see [5]. References concerning numerical work on the same problem can be found in [5].

	110 lower bound	110 upper bound	111 lower bound	111 upper bound
λ_1	120.2143	120.2143	601.488	601.983
λ_2	605.792	606.920	2133.1	2155.6
λ_3	1401.5	1415.7	3398	3491
λ_4	2111.8	2161.2	5429	5834
λ_5	3306	3506	6970	7894
λ_6	5037	5842	9366	12071
λ_7	5412	6451		
λ_8	6200	7931		

	lower bound	upper bound
λ_1	55.2982	55.2994
λ_2	279.35	279.50
λ_3	454.37	454.99
λ_4	896.8	901.6
λ_5	1180	1191
λ_6	1833	1875
λ_7	2171	2242
λ_8	2560	2677
λ_9	3371	3652
λ_{10}	4154	4716
λ_{11}	4556	5329
λ_{12}	4582	5372

II) Circular plate clamped along its boundary.

$$\Delta_2\Delta_2 u - \lambda u = 0 \qquad \text{in } A \equiv \{x_1^2 + x_2^2 < 1\}, \qquad u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial A.^{(7)}$$

The space $L^2(A)$ can be decomposed into the direct sum of a sequence of subspaces (which are invariant for the problem) as follows:

$$L^2(A) = H^{(0)} \oplus H^{(1)} \oplus \ldots \oplus H^{(k)} \oplus \ldots,$$

where $H^{(k)}$ is the subspace spanned by all functions of the type $f(\varrho)\cos k\vartheta$ and $g(\varrho)\sin k\vartheta$ where $x_1 = \varrho\cos\vartheta$, $x_2 = \varrho\sin\vartheta$, k is a non-negative integer and f and g are arbitrary functions. Each subspace $H^{(k)}$ (for $k > 0$) is itself decomposable into two invariant subspaces

$$H_1^{(k)} \equiv \{f(\varrho)\cos k\vartheta\}, \qquad H_2^{(k)} \equiv \{g(\varrho)\sin k\vartheta\}.$$

It is obvious that the eigenvalues in $H_1^{(k)}$ coincide with those of $H_2^{(k)}$. Therefore the eigenvalues included in the tables with index $k > 0$ must be considered as double eigenvalues.

(7) Application of the general method to this problem is due to M. Schaerf and will appear in a forthcoming paper. The numerical results exhibited in the present paper are due to this author.

$k = 0$ $k = 1$

	lower bound	upper bound	lower bound	upper bound
λ_1	104.36311051	104.3631056	452.00448	452.00452
λ_2	1581.742	1581.745	3700.11	370013
λ_3	7939.38	7939.55	144418.2	14419.1
λ_4	25017.2	25022.3	39606.2	39622.3
λ_5	60939.5	61012.2	88482.2	88661.1
λ_6	125786	126430	171901	173225
λ_7	230123	234133	300129	307340
λ_8	380355	399323	476778	507392
λ_9	569823	640349	689901	794004

$k = 2$ $k = 3$

	lower bound	upper bound	lower bound	upper bound
λ_1	1216.4072	1216.4076	2604.061	2604.065
λ_2	7154.14	7154.23	12325.4	12325.8
λ_3	23656.3	23659.1	36207.4	36215.6
λ_4	58870.7	58913.3	83526.1	83625.1
λ_5	123047	123437	165470	166244
λ_6	227594	230089	293711	298098
λ_7	381914	394063	476150	495553
λ_8	585981	632954	708346	777466
λ_9	822673	970669		

$k = 4$ $k = 5$

	lower bound	upper bound	lower bound	upper bound
λ_1	4853.31	4853.33	8233.49	8233.57
λ_2	19629.1	19630.3	29513.3	29516.3
λ_3	52658.5	52678.8	73627.7	73673.3
λ_4	114314	114523	152001	152404
λ_5	216597	218019	277274	279738
λ_6	371076	378366	460483	472040
λ_7	583460	613097	704421	748019
λ_8	844252	942444		

$k = 6$ | $k = 7$

	lower bound ($k=6$)	upper bound ($k=6$)	lower bound ($k=7$)	upper bound ($k=7$)
λ_1	13044.2	13044.5	19615.1	19615.8
λ_2	42457.8	42465.1	58973.7	58989.9
λ_3	99763.2	99857.0	131741	131922
λ_4	197374	198104	251235	252489
λ_5	348349	352407	430663	437070
λ_6	562700	580302	678459	704371
λ_7	839575	901677		

$k = 8$ | $k = 9$

	lower bound ($k=8$)	upper bound ($k=8$)	lower bound ($k=9$)	upper bound ($k=9$)
λ_1	28304.7	28306.3	39500.6	39504.1
λ_2	79602.4	79635.6	104914	104979
λ_3	170265	170590	216062	216620
λ_4	314402	316460	387704	390951
λ_5	525050	534802	632331	646714
λ_6	808467	845496		

$k = 10$ | $k = 11$

	lower bound ($k=10$)	upper bound ($k=10$)	lower bound ($k=11$)	upper bound ($k=11$)
λ_1	53618.9	53626.1	71103.5	71117.8
λ_2	135509	135626	172014	172215
λ_3	269882	270798	332495	333947
λ_4	471976	476928	568059	575391
λ_5	753313	773948	888788	917682

$k = 12$ | $k = 13$

	lower bound	upper bound	lower bound	upper bound
λ_1	92426.2	92452.6	118085	118133
λ_2	215080	215414	265387	265921
λ_3	404689	406917	487270	490593
λ_4	676795	687371	799027	813933

$k = 14$ | $k = 15$

	lower bound	upper bound	lower bound	upper bound
λ_1	148607	148687	184542	184674
λ_2	323636	324465	390553	391804
λ_3	581056	585887	686877	693747
λ_4	935594	956172		

$k = 16$ | $k = 17$

	lower bound	upper bound	lower bound	upper bound
λ_1	226468	226678	274986	275311
λ_2	466883	468724	553390	556044
λ_3	805574	815148	937997	951097

$k = 18$ | $k = 19$

	lower bound	upper bound	lower bound	upper bound
λ_1	330725	331214	394333	395054
λ_2	650861	654609	760097	765295

$k = 20$

	lower bound	upper bound
λ_1	466485	467526
λ_2	881916	889004

Several text-books exhibit the following numerical table due to H. Carrington (London—Edinburgh Phil. Mag., vol. 55 pp. 1261—64, 1925), for $\mu = \sqrt[4]{\lambda}$. It was obtained by computing the zeros of a well-known transcendental function expressed by means of Bessel functions.

	$k = 0$	$k = 1$	$k = 2$	$k = 3$
μ_1	3.1961	4.6110	5.9056	7.1433
μ_2	6.3064	7.7993	9.1967	10.537
μ_3	9.4395	10.958	12.402	13.795
μ_4	12.577	14.108	15.579	
μ_5	15.716			

Dr. Schaerf gets the following results.

	$k = 0$		$k = 1$	
	lower bound	upper bound	lower bound	upper bound
μ_1	3.19622	3.19623	4.61089	4.61090
μ_2	6.30643	6.30644	7.79926	7.79928
μ_3	9.43945	9.43950	10.9579	10.9581
μ_4	12.5764	12.5772	14.1072	14.1087
μ_5	15.7117	15.7165	17.2470	17.2558

It is interesting to observe that the numerical application of the methods described in this paper proves that some of the classical numerical results are incorrect in the fifth digit. On the other hand the numerical application of our method is simpler than the numerical solution of the classical above mentioned, transcendental equation.

	$k = 2$		$k = 3$	
	lower bound	**upper bound**	**lower bound**	**upper bound**
μ_1	5.90567	5.90568	7.14352	7.14354
μ_2	9.19685	9.19689	10.5366	10.5367
μ_3	12.4018	12.4023	13.7942	13.7951
μ_4	15.5766	15.5795	17.0002	17.0053

7. The problem of estimating eigenvalues when estimates for invariant subspaces are known.

Let us consider the linear operator L with domain the linear variety $\mathscr{D}_L$ of the Hilbert space S. Let V be a linear subvariety of $\mathscr{D}_L$. The following hypothesis be satisfied:

There exists a PCO G of the space S such that: i) *the range $G(S)$ of G is contained in V*; ii) $GL = LG = I$.

Let us consider the eigenvalue problem

$$Lv - \lambda v = 0, \qquad v \in V. \tag{19}$$

This problem is equivalent to the following one

$$Gu - \mu u = 0, \qquad u \in S \tag{20}$$

where $\mu = \lambda^{-1}$. It follows that all the eigenvalues of (19) constitute a non-decreasing sequence tending to $+\infty$

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$$

Each eigenvalue appears — as usual — in the above sequence as many times as its multiplicity.

Let us suppose that we can decompose the space S as direct sum of a finite or a countable set of mutually orthogonal subspaces, each of them being an invariant subspace for G.

$$S = H_1 \oplus H_2 \oplus \ldots \oplus H_s \oplus \ldots$$

Problem (20) is equivalent to the following system of eigenvalue problems:

$$Gu - \mu^{(s)} u = 0, \qquad u \in H_s \qquad (s = 1, 2, \ldots) \tag{20$_s$}$$

Set $V_s = G(H_s)$. It is easy to prove that

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_s \oplus \ldots$$

and that problem (19) is equivalent to the following system of eigenvalue problems:

$$(19_s) \qquad Lv - \lambda^{(s)} v = 0, \qquad v \in V_s \qquad (s = 1, 2, \ldots)$$

Let $\lambda_1^{(s)} \leq \lambda_2^{(s)} \leq \ldots \leq \lambda_k^{(s)} \leq \ldots$ be the sequence of the eigenvalues of problem (19_s). Suppose we have obtained for the first $p_s \geq 1$ eigenvalues of problem (19_s) the following *table of estimates* (t_s)

$$(t_s) \qquad \begin{array}{c} \delta_1^{(s)} \leq \lambda_1^{(s)} \leq \varepsilon_1^{(s)} \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ \delta_{p_s}^{(s)} \leq \lambda_{p_s}^{(s)} \leq \varepsilon_{p_s}^{(s)} \end{array}$$

We suppose that the upper bounds $\varepsilon_k^{(s)}$ have been computed by the Rayleigh-Ritz method. That means that the $\varepsilon_k^{(s)}$ are the roots of the determinantal equation

$$\det \{(Lw_h^{(s)}, w_k^{(s)}) - \lambda(w_h^{(s)}, w_k^{(s)})\} = 0 \qquad (h, k = 1, \ldots, p_s),$$

where $w_1^{(s)}, \ldots, w_s^{(s)}$ are p_s linearly independent vectors of V_s.

The problem now arises. *From estimates of the tables* (t_s) *is it possible to deduce estimates for the k-th eigenvalue* λ_k *of problem* (19)?

In solving this problem we shall not make any assumption on the method used for computing the lower bounds $\delta_k^{(s)}$. We only assume — without any loss — that $\delta_1^{(s)} \leq \delta_2^{(s)} \leq \ldots \leq \delta_{p_s}^{(s)}$.

In order to consider a more concrete situation we shall assume that the tables (t_s) are given only for $s = 1, \ldots, q$ $(q \geq 1)$. Assuming that is necessary if the spaces H_s are infinitely many. For $s > q$ we shall only suppose that we know a positive real number c_s such that for any eigenvalue of (19_s) with $s > q$ we have $\lambda_k^{(s)} > c_s$. Moreover $\lim_{s \to \infty} c_s = +\infty$, if the H_s are infinitely many.

For instance, in the case of the example II, considered in section 6 (circular clamped plate), it is possible to show that we may assume

$$c_s = 16(s + 1)(s + 2)(s + 3).$$

Let us consider the two sequences $\{\delta_k^{(s)}\}$, $\{\varepsilon_k^{(s)}\}$ $(s = 1, \ldots, q;\ k = 1, \ldots, p_s)$. We shall denote by $\{\delta_h\}$, $\{\varepsilon_h\}$ $(h = 1, \ldots, m,\ m = p_1 + \ldots + p_q)$ the sequences obtained from $\{\delta_k^{(s)}\}$ and $\{\varepsilon_k^{(s)}\}$, respectively, by disposing all their elements in non-decreasing order.

It will be useful to introduce the function $l = l(s, k)$ $(s = 1, \ldots, q;\ k = 1, \ldots, p_s)$, whose range is the set $1, \ldots, m$ such that

$$\delta_{l(s,k)} = \delta_k^{(s)}.$$

This function is not unique if some of the numbers $\delta_k^{(s)}$ coincide. However we suppose to have chosen, amongst the possible ones, a well determined function $l = l(s, k)$.

Let us first consider the following lemma.

Lemma XIII. *Let $h \to \delta_h$ and $k \to \lambda_k$ be two real valued functions, the first defined for $h = 1, \ldots, m$ and the second for $k = 1, \ldots, n$. Assume that $m \geq n$ and*

$$\delta_1 \leq \delta_2 \leq \ldots \leq \delta_m$$
$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n.$$

Let us suppose that there exists a function $k \to q_k$ defined for $k = 1, \ldots, n$ such that

I) *q_k is a positive integer and $1 \leq q_k \leq m$,*

II) *$q_i = q_j$ for $i \neq j$ implies $q_i \geq n$,*

III) *$\lambda_k \geq \delta_{q_k}$ $(k = 1, \ldots, n)$.*

Under the above hypotheses we have

$$\lambda_k \geq \delta_k. \tag{21}$$

Inequality (21) is obvious if $q_k \geq k$. Let us suppose $q_k < k$. It must exist an index s such that $1 \leq s \leq k - 1$, $q_s \geq k$. In fact $q_s < k$ for any $s \leq \leq k - 1$ implies that there exist two indeces i, j such that $i \leq k$, $j \leq k$, $i \neq j$, $q_i = q_j < k \leq n$. That contradicts hypothesis II). Existence of $q_s \geq k$ with $s \leq k - 1$ implies $\lambda_k \geq \lambda_s \geq \delta_{q_s} \geq \delta_k$.

Theorem XIV. *Let $\delta_{p_r}^{(r)}$ be such that $\delta_{p_s}^{(s)} \geq \delta_{p_r}^{(r)}$ for $s = 1, \ldots, q$. We suppose that, if the spaces H_s decomposing S are more than q, then $c_s \geq \delta_{p_r}^{(r)}$ for every $s > q$. Let n be the smallest integer such that $\delta_n = \delta_{p_r}^{(r)}$. We have the following estimates for the first n eigenvectors of problem* (19).

$$\delta_k \leq \lambda_k \leq \varepsilon_k \qquad (k = 1, \ldots, n).$$

Let us associate to every eigenvalue λ_k of problem (19) a unit vector v_k such that

$$Lv_k - \lambda_k v_k = 0, \qquad v_k \in V, \qquad (v_h, v_k) = \delta_{hk}.$$

The sequence $\{v_k\}$ may be considered as the union of the subsequences $\{v_i^{(s)}\}$ such that

$$Lv_i^{(s)} - \lambda_i^{(s)} v_i^{(s)} = 0, \qquad v_i^{(s)} \in V_s;$$

$\{\lambda_i^{(s)}\}$ is the sequence of the eigenvalues of problem (19_s).

Let us consider for $1 \leq k \leq n$ the eigenvalue λ_k. Suppose that $v_k = v_i^{(s)}$. We have $\lambda_k = \lambda_i^{(s)} \geq \delta_i^{(s)}$ if $1 \leq s \leq q$ and $i \leq p_s$. We have $\lambda_k = \lambda_i^{(s)} \geq \delta_n$ either if $1 \leq s \leq q$, $i > p_s$ or if $s > q$. Set

$$q_k \begin{cases} = l(s, i) & \text{if } 1 \leq s \leq q,\ i \leq p_s \\ = n & \text{if } 1 \leq s \leq q,\ i > p_s \text{ or } s > q. \end{cases}$$

The functions $h \to \delta_h$, $k \to \lambda_k$, $k \to q_k$ satisfy hypotheses of lemma XIII. It follows that inequality (21) holds.

Let $w_1, \ldots, w_m$ be the m vectors of V obtained by disposing in a unique sequence the vectors of the q sequences $\{w_i^{(s)}\}$ $(s = 1, \ldots, q;\ i = 1, \ldots, p_i)$. The m roots of the determinantal equation

$$\det \{(Lw_i, w_j) - \lambda(w_i, w_j)\} = 0$$
$$(i, j = 1, \ldots, m)$$

are $\varepsilon_1 \leq \varepsilon_2 \leq \ldots \leq \varepsilon_m$. From the theory of the Rayleigh-Ritz method it follows $\lambda_k \leq \varepsilon_k$ $(k = 1, \ldots, m)$.

The following tables show the estimates, which is possible to deduce (for the eigenvalues of a square plate and of a circular plate) from the estimates already known for invariant subspaces. In both cases the lower bounds and the upper bounds have been compared with the asymptotic values given by a formula due to R. Courant and A. Pleijel (Comm. on Pure and Applied Math. III, 1, 1950, p. 1—10). These numerical results suggest that the use of asymptotic formulas for the numerical evaluation of eigenvalues, even of rather high index, could be misleading.

Square plate

	lower bound	upper bound	asymptotic value		lower bound	upper bound	asymptotic value
λ_1	13.29376	13.29378	1.6211	λ_{24}	1833	1875	933.77
λ_2	55.2982	55.29934	6.4845	λ_{25}	2111.8	2155.6	1013.2
λ_3	55.2982	55.29934	14.590	λ_{26}	2133.1	2161.2	1095.8
λ_4	120.2143	120.2232	25.938	λ_{27}	2171	2242	1181.8
λ_5	177.7113	177.7401	40.528	λ_{28}	2171	2242	1270.9
λ_6	179.408	179.431	58.361	λ_{29}	2560	2677	1363.3
λ_7	279.35	279.50	79.435	λ_{30}	2560	2677	1459.0
λ_8	279.35	279.50	103.75	λ_{31}	3120	3244	1557.9
λ_9	454.37	454.99	131.31	λ_{32}	3155	3282	1660.0
λ_{10}	454.37	454.99	162.11	λ_{33}	3158	3284	1765.4
λ_{11}	496.55	497.03	196.15	λ_{34}	3306	3491	1874.0
λ_{12}	601.488	601.983	233.44	λ_{35}	3371	3506	1985.8
λ_{13}	605.792	606.920	273.97	λ_{36}	3371	3652	2100.9
λ_{14}	896.8	901.6	317.74	λ_{37}	3398	3652	2219.3
λ_{15}	896.8	901.6	364.75	λ_{38}	4037	4306	2340.9
λ_{16}	976.13	979.59	415.01	λ_{39}	4038	4317	2465.7
λ_{17}	977.64	981.25	468.50	λ_{40}	4154	4716	2593.8
λ_{18}	1180	1191	525.24	λ_{41}	4154	4716	2725.1
λ_{19}	1180	1191	585.23	λ_{42}	4556	5329	2859.6
λ_{20}	1401	1415	648.45	λ_{43}	4556	5329	2997.4
λ_{21}	1569	1584	714.92	λ_{44}	4582	5372	3138.5
λ_{22}	1577	1593	784.63	λ_{45}	4582	5372	3282.8
λ_{23}	1833	1875	857.58				

Circular plate

	lower bound	upper bound	asymptotic value		lower bound	upper bound	asymptotic value
λ_{1}	104.363	104.364	16	λ_{47}	58870.7	58913.3	35344
λ_{2}	452.004	452.005	64	λ_{48}	58870.7	58913.3	36864
λ_{3}	452.004	452.005	144	λ_{49}	58973.7	58989.9	38416
λ_{4}	1216.40	1216.41	256	λ_{50}	58973.7	58989.9	40000
λ_{5}	1216.40	1216.41	400	λ_{51}	60939.5	61012.2	41616
λ_{6}	1581.74	1581.75	576	λ_{52}	71103.5	71117.8	43264
λ_{7}	2604.06	2604.07	784	λ_{53}	71103.5	71117.8	44944
λ_{8}	2604.06	2604.07	1024	λ_{54}	73627.7	73673.3	46656
λ_{9}	3700.11	3700.13	1296	λ_{55}	73627.7	73673.3	48400
λ_{10}	3700.11	3700.13	1600	λ_{56}	79602.4	79635.6	50176
λ_{11}	4853.31	4853.33	1936	λ_{57}	79602.4	79635.6	51984
λ_{12}	4853.31	4953.33	2304	λ_{58}	83526.1	83625.1	53824
λ_{13}	7154.14	7154.23	2704	λ_{59}	83526.1	83625.1	55696
λ_{14}	7154.14	7154.23	3136	λ_{60}	88482.2	88661.1	57600
λ_{15}	7939.38	7939.55	3600	λ_{61}	88482.2	88661.1	59536
λ_{16}	8233.49	8233.57	4096	λ_{62}	92426.2	92452.6	61504
λ_{17}	8233.49	8233.57	4624	λ_{63}	92426.2	92452.6	63504
λ_{18}	12325.4	12325.76	5184	λ_{64}	99763.2	99857.0	65536
λ_{19}	12325.4	12325.76	5776	λ_{65}	99763.2	99857.0	67600
λ_{20}	13044.2	13044.5	6400	λ_{66}	104914	104979	69696
λ_{21}	13044.2	13044.5	7056	λ_{67}	104914	104979	71824
λ_{22}	14418.2	14420.0	7744	λ_{68}	114314	114523	73984
λ_{23}	14418.2	14420.0	8464	λ_{69}	114314	114523	76176
λ_{24}	19615.1	19615.8	9216	λ_{70}	118085	118113	78400
λ_{25}	19615.1	19615.8	10000	λ_{71}	118085	118133	80656
λ_{26}	19629.1	19630.3	10816	λ_{72}	123047	123437	82944
λ_{27}	19629.1	19630.3	11664	λ_{73}	123047	123437	85264
λ_{28}	23656.3	23659.1	12544	λ_{74}	125786	126430	87616
λ_{29}	23656.3	23659.1	13456	λ_{75}	131741	131921	90000
λ_{30}	25017.2	25022.3	14400	λ_{76}	131741	131921	92416
λ_{31}	28304.7	28306.3	15376	λ_{77}	135509	135625	94864
λ_{32}	28304.7	28306.3	16384	λ_{78}	135509	135625	97344
λ_{33}	29513.3	29516.3	17424	λ_{79}	148607	148686	99856
λ_{34}	29513.3	29516.3	18496	λ_{80}	148607	148686	102400
λ_{35}	36207.4	36215.6	19600	λ_{81}	152001	152403	104976
λ_{36}	36207.4	36215.6	20736	λ_{82}	152001	152403	107584
λ_{37}	39500.6	39504.1	21904	λ_{83}	165470	166243	110224
λ_{38}	39500.6	39504.1	23104	λ_{84}	165470	166243	112896
λ_{39}	39606.2	39622.3	24336	λ_{85}	170265	170589	115600
λ_{40}	39606.2	39622.3	25600	λ_{86}	170265	170589	118336
λ_{41}	42457.8	42465.1	26896	λ_{87}	171901	172214	121104
λ_{42}	42457.8	42465.1	28224	λ_{88}	171901	172214	123904
λ_{43}	52658.5	52678.8	29584	λ_{89}	172014	173224	126736
λ_{44}	52658.5	52678.8	30976	λ_{90}	172014	173224	129600
λ_{45}	53618.9	53626.1	32400	λ_{91}	184542	184673	132496
λ_{46}	53618.9	53626.1	33856	λ_{92}	184542	184673	135424

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