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ON THE TRANSFORMATION OF LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE n^{th} ORDER

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We call the equation of the following form

$$(0.1)\quad \sum_{i=0}^{n} \binom{n}{i} a_i(x)\, y^{(n-i)}(x) = 0, \quad a_i \in C_0(I_1), \quad i = 0, 1, \ldots, n, \quad a_0 \not\equiv 0 \text{ in } I_1,$$

a *general* homogeneous linear differential equation of the n^{th} order. Instead of "homogeneous linear differential equation" we shall call it simply "equation".

The equation (0.1) is *normal (semi-canonical)* [*canonical*] if $a_0 \equiv 1$ $(a_1 \equiv 0)$ $[a_1 \equiv a_2 \equiv 0]$. If $a_i/a_0 \in C_0(I_1)$, $i = 1, 2, \ldots, n$, then we call the equation

$$(0.2)\quad y^{(n)} + \sum_{i=1}^{n} \binom{n}{i} (a_i/a_0)\, y^{(n-i)} = 0$$

the normal form of the equation (0.1).

We call two equations *quasi-identical* if they have the identical range of definition and the same fundamental system of solution. We denote the quasi-identical equations by the sign $\doteq$. F.i. (0.1) $\doteq$ (0.2).

1. Perturbated equations.

Let us have the equation

$$(\mathrm{a})\quad y^{(n)}(x) + \sum_{i=1}^{n} \binom{n}{i} a_i(x)\, y^{(n-i)}(x) = 0, \quad a_i \in C_{n-i}(I_1), \quad i = 1, 2, \ldots, n$$

Let $u(x)$ be an arbitrary solution of the equation

$$(\mathrm{u})\qquad u'' + \frac{3}{n+1}\,(a_2 - a_1' - a_1^2)\, u = 0.$$

We call the equation (u) the accompanying equation to the equation (a).

By $(n-1)$ fold iteration of the equation of the first order

$$(1.1) \qquad P_1(y) = u^2 y' + [a_1 u^2 - (n-1)\, uu']\, y = 0$$

we obtain an equation of the n^{th} order

$$(1.2) \qquad P_n(y) = P_1[P_{n-1}(y)] = u^{2n} \sum_{i=0}^{n} \binom{n}{i} f_i^n(a_1, a_2)\, y^{(n-i)} = $$
$$= u^{2n} I_n(y; a_1, a_2) = 0,$$

where the function

$$(1.3) \qquad f_i^n(a_1, a_2), \qquad i = 0, 1, \ldots, n$$

is for the given n a polynomial of the elements a_1, a_2 of the dimension i, which we obtain as a solution of a certain difference equation of the first order — see [1; pp. 39—48]. For instance there is

$$f_0^n(a_1, a_2) = 1, \qquad f_1^n(a_1, a_2) = a_1, \qquad f_2^n(a_1, a_2) = a_2,$$
$$f_3^n(a_1, a_2) = \frac{3}{2} a_2' - \frac{1}{2} a_1'' + 3a_1 a_2 - 3a_1' a_1 - 2a_1^3.$$

We call the polynomial $f_i^n(a_1, a_2)$ the *iterated polynomial* of the dimension i, the equation (1.2) we call an *iterated equation.* Let us note yet, that we take for an iterated equation every equation, which is quasi-identical with the equation (1.2).

Put

$$(1.4) \qquad \omega_i^n = a_i - f_i^n(a_1, a_2), \qquad i = 3, 4, \ldots, n.$$

With the aid of (1.4) we can write this in the form

$$(\omega) \quad I_n(y; a_1, a_2) + \sum_{i=3}^{n} \binom{n}{i} \omega_i^n\, y^{(n-i)} = 0,$$

where $I_n(y; a_1, a_2) = 0$ is the normal form of the equation (1.2). We call the function ω_i^n the coefficient of perturbation of the dimension i of the equation (a), the equation (ω) we call the perturbated form of the equation (a) or the perturbated equation of the equation (a), briefly *the perturbated equation.*

The following can be proved — see [1; pp.50]

Theorem 1. *The equation* (a) *is iterated just then when its fundamental system is the function*

$$(1.5) \quad u^{n-k} v^{k-1} \exp\left\{-\int_{x_0}^{x} a_1\, \mathrm{d}s\right\}, \qquad x_0 \in I_1, \qquad k = 1, 2, \ldots, n,$$

where u and v are linearly independent solutions of the equation (u).

The perturbated equation (ω) comes in handy for the study of the asymptotic and oscillatory properties of the equation (a). We give at least two examples on the understanding that the equation (a) is semi-canonical in the interval $I_1 \equiv \equiv \langle x_0, \infty)$, i.e. $a_1 \equiv 0$ in I_1. Let us put for the sake of simplicity $A = \frac{3}{n+1}\, a_2$.

Example 1. Let the following assumptions hold:

$$A^{(r)} = 0(1), \qquad r = 0, 1, \ldots, n-5, \tag{1.6}$$

$$\int_{x_0}^{\infty} x^{-2s}|Ax^{4s} + \varepsilon c^2|\, \mathrm{d}x < \infty, \qquad c > 0, \qquad s < \frac{1}{2}, \qquad (c, s \in E_1), \qquad \varepsilon = \pm 1,$$

$$\int_{x_0}^{\infty} x^{2s(k+j-1)}|\omega_k^n|\, \mathrm{d}x < \infty, \qquad k = 3, 4, \ldots, n; \qquad j = 0, 2, 3, \ldots, n-k.$$

Hence the equation (a) has in the case of $\varepsilon = 1$ the fundamental system

$$\exp\{\beta(n - 2\nu + 1)\, x^{1-2s}\}\, [1 + o(1)], \qquad \nu = 1, 2, \ldots, n,$$

in the case $\varepsilon = -1$

$$[\sin(\beta x^{1-2s})]^{n-\nu}[\cos(\beta x^{1-2s})]^{\nu-1} + o(1), \qquad \nu = 1, 2, \ldots, n,$$

where $\beta = \frac{c}{1-2s}$ — see [2; pp. 184].

For $s = 0$ we obtain the following statement:
Let the following hold: formula (1.6),

$$\int_{x_0}^{\infty} |A + \varepsilon c^2|\, \mathrm{d}x < \infty, \qquad \int_{x_0}^{\infty} |\omega_k^n|\, \mathrm{d}x < \infty, \qquad k = 3, 4, \ldots, n.$$

Then the equation (a) has in the case $\varepsilon = 1$ the fundamental system

$$e^{c(n-2\nu+1)x}[1 + o(1)], \qquad \nu = 1, 2, \ldots, n,$$

and in the case of $\varepsilon = -1$

$$[\sin cx]^{n-\nu}[\cos cx]^{\nu-1} + o(1), \qquad \nu = 1, 2, \ldots, n.$$

Example 2. Let $\omega_n^n \geqq 0$. If the equation (u) is oscillatory, then every solution of the equation

$$I_n(y; 0, a_2) + \omega_n^n y = 0, \tag{1.7}$$

which has at least one zero point oscillates, too. If n is even, then the equation (1.7) is strictly oscillatory.

We note yet that M. Greguš dealt in his paper with the properties of the integrals of the equation

$$I_n(y; 0, 0) + n\omega_{n-1}^n y' + \omega_n^n y = 0 \tag{1.8}$$

— see [21].

2. Transformation

We denote by the symbol $m(I_{1x})$ where $\varnothing \neq I_{1x} \subset I_1$ the set of the elements which are defined as follows: The ordered pair of functions $\{T(x), u(x)\}$ is an element of the set $m(I_{1x})$ if

$$T(x) \in C_{n+1}(I_{1x}), \qquad u(x) \in C_n(I_{1x}), \qquad T'(x) \,.\, u(x) \neq 0 \text{ in } I_{1x}.$$

Let us choose an arbitrary element $\{T(x), u(x)\} \in m(I_{1x})$. If we put into the equation (a)

$$y(x) = u(x)\, Z(x), \qquad t = T(x),$$

we have the equation

$$(\bar{\mathrm{a}}) \quad u(x)\,[T'(x)]^n [z^{(n)}(t) + \sum_{i=1}^{n} \binom{n}{i} \bar{a}_i(t)\, z^{(n-i)}(t)] = 0, \qquad t \in I_{2t} = T(I_{1x}),$$

where we put $x = T_{-1}(t)$ [$T_{-1}(t)$ is the inverse function to the function $T(x)$], $z(t) = Z[T_{-1}(t)]$. We call the equation ($\bar{\mathrm{a}}$) *the image of the equation* (a) *in the interval I_{1x} with the coordinates* $T(x)$, $u(x)$ and we denote it by the sign ($\bar{\mathrm{a}}$) $\{T(x), u(x)\}$. It can be proved that in the interval I_{2t} the following relations hold

$$\bar{a}_i(t) = [T'(x)]^{-i} \sum_{k=0}^{i} \binom{i}{k} a_k(x)\, \Phi^{n,i}_{i-k}[\eta(x), \zeta(x)], \qquad x = T_{-1}(t), \qquad i = 0, 1, \ldots, n,$$

where $\eta = T''/T'$, $\zeta = u'/u$ are the transformed coordinates of the image ($\bar{\mathrm{a}}$) $\{T(x), u(x)\}$ and

$$\Phi^{n,i}_{i-k}(\eta, \zeta) = \sum_{j=k}^{i} \binom{i-k}{j-k} \varphi^{n-j}_{i-j}(\eta)\, \chi_{j-k}(\zeta),$$

see [3; 3,1.10]. The function φ^{n-j}_{i-j} resp. χ_{j-k} is the polynomial of the element η resp. ζ of the dimension $i - j$ resp. $j - k$. We obtain both functions as a solution of certain linear difference equations of the first order — see [3; (2,1.6), (2,2.3)]. The difference equation which satisfies the polynomial χ is specially simple and therefore we write it here:

$$\chi_k(\zeta) = \zeta\chi_{k-1}(\zeta) + [\chi_{k-1}(\zeta)]', \qquad \chi_0(\zeta) = 1.$$

From this follows f.i. $\chi_1(\zeta) = \zeta$, $\chi_2(\zeta) = \zeta^2 + \zeta'$, and so on.

We introduce yet some explicit polynomials: $\varphi^m_0(\eta) = 1$,

$$\varphi^m_1(\eta) = \frac{m-1}{2}\,\eta, \qquad \varphi^m_2(\eta) = \frac{m-2}{3}\left(\frac{3m-5}{4}\,\eta^2 + \eta'\right),$$

$$\Phi^{n,i}_0(\eta, \zeta) = 1, \qquad \Phi^{n,i}_1(\eta, \zeta) = \frac{n-i}{2}\,\eta + \zeta,$$

$$\Phi^{n,i}_2(\eta, \zeta) = (n - i)\left[\frac{3n - 3i + 1}{12}\,\eta^2 + \frac{1}{3}\,\eta' + \eta\zeta\right] + \zeta^2 + \zeta'.$$

By the symbol $o_a(I_{1x})\,[p_a(I_{1x})]\,\{k_a(I_{1x})\}$ we denote the set of all images [semi-canonical images] {canonical images} of the equation (a) in the interval I_{1x}.

If we choose

$$U(x) = c|T'(x)|^{\frac{1-n}{2}} \exp\left\{-\int_{x_0}^{x} a_1\, ds\right\}, \qquad 0 \neq c \in E_1, \tag{2.1}$$

then the image $(\bar{a})\,\{T(x),\ U(x)\} \in o_a(I_{1x})$ is semicanonic. As the semicanonical image is following (2.1) determined by the coordinate $T(x)$ we write instead of $(\bar{a})\,\{T(x),\ U(x)\} \in p_a(I_{1x})$ in a shorter way $(\bar{a})\,\{T(x)\} \in p_a(I_{1x})$.

We call the image $(A)\,\{x\} \in p_a(I_1)$ the *fundamental semicanonic image* or also the *semi-canonical fundamental form* of the equation (a).

If we put

$$A_i = \sum_{k=0} \binom{i}{k} a_k \chi_{i-k}(-a_1), \qquad i = 2, 3, \ldots, n, \qquad x \in I_1, \tag{2.2}$$

then we can write the semicanonical image in the form

$$\text{(A)} \quad U_1(x)\left[Z^{(n)}(x) + \sum_{i=2}^{n} \binom{n}{i} A_i(x)\, Z^{(n-i)}(x)\right] = 0,$$

$$\text{where } U_1(x) = c\,.\exp\left\{-\int_{x_0}^{x} a_1\, ds\right\},\ 0 \neq c \in E_1.$$

We call the function (2.2) the *fundamental coefficients* of the equation (a).

Let us put

$$f_i^n(A_2) = f_i^n(0, A_2), \qquad i = 0, 1, \ldots, n, \tag{2.3}$$

$$I_n(Z; A_2) = \sum_{i=0}^{n} \binom{n}{i} f_i^n(A_2)\, Z^{(n-i)}, \tag{2.4}$$

$$\Omega_i^n = A_i - f_i^n(A_2), \qquad i = 3, 4, \ldots, n.$$

There is for instance

$$f_0^n(A_2) = 1, \qquad f_1^n(A_2) = 0, \qquad f_2^n(A_2) = A_2,$$

$$f_3^n(A_2) = \frac{3}{2} A_2', \qquad f_4^n(A_2) = \frac{9}{5} A_2'' + \frac{3(5n+7)}{5(n+1)} A_2^2. \tag{2.5}$$

Let us introduce yet the formula (2.4) for $n = 3, 4$:

$$I_3(y; A_2) = y''' + 3A_2 y' + \frac{3}{2} A_2' y. \tag{2.6}$$

$$I_4(y; A_2) = y^{(4)} + 6A_2 y'' + 6A_2' y' + \frac{9}{5}\left(A_2'' + \frac{9}{5} A_2^2\right) y. \tag{2.7}$$

Then we can write the normal form of the equation (A) in the perturbated form

$$(\Omega) \quad I_n(Z; A_2) + \sum_{i=3}^{n} \binom{n}{i} \Omega_i^n Z^{(n-i)} = 0.$$

We call the function $f_i^n(A_2)$ resp. Ω_i^n the fundamental iterated polynomial — briefly the *fundamental polynomial* — resp. the *fundamental semiinvariant* of the dimension i of the equation (a). We call the equation (Ω) the perturbated fundamental semicanonical form of the equation (a) or briefly the *perturbated fundamental equation.*

We introduce the perturbated fundamental equations of the order 3 and 4 in their most often occurring arrangements. If we put $A = \frac{3}{2} A_2$, $\omega_3 = \omega_3^3 = A_3 - \frac{3}{2} A_2'$ we obtain with the aid of (2.5), (2.6) the perturbated fundamental equation of the 3^d order in the form

$$y''' + 2Ay' + (A' + \omega_3)\, y = 0.$$

If we put $A = \frac{3}{5} A_2$, $\omega_3 = 4\omega_3^4 = 4\left(A_3 - \frac{3}{2} A_2'\right)$, $\omega_4 = \omega_4^4 = A_4 - \frac{9}{5}\left(A_2'' + \frac{9}{5} A_2^2\right)$, we obtain with the aid of (2.5), (2.7) a perturbated fundamental equation of the 4^{th} order in the form

$$y^{(4)} + 10Ay'' + (10A' + \omega_3)\, y' + [3(A'' + 3A^2) + \omega_4]\, y = 0,$$

see f.i. [17; pp. 511—3·26, pp. 528—4·11], [20], [11], [7].

Between the functions (2.3) and (1.3) resp. (2.5) and (1.4) hold the following relations:

$$f_i^n(A_2) = \sum_{k=0}^{i} \binom{i}{k} f_k^n(a_1, a_2)\, \chi_{i-k}(-a_1), \qquad i = 3, 4, \dots, n,$$

$$\Omega_i^n = \sum_{k=0}^{i} \binom{i}{k} \omega_k^n \chi_{i-k}(-a_1), \qquad i = 3, 4, \dots, n,$$

see [6; (2.5)].

The semicanonic image $(\bar{A})\, \{T(x)\} \in p_a(I_{1x})$ can be written in the form

$$(\bar{A}) \quad U(x)\, [T'(x)]^n [z^{(n)}(t) + \sum_{i=2}^{n} \binom{n}{i} \bar{A}_i(t)\, z^{(n-i)}(t)] = 0, \quad x = T_{-1}(t), \quad \text{where}$$

$$(2.8)\ \bar{A}_i(t) = [T'(x)]^{-i} \sum_{k=0}^{i} \binom{i}{k} A_k(x)\, \Phi_{i-k}^{n,i}(\eta), \ x = T_{-1}(t), \qquad i = 2, 3, \dots, n,$$

where the functions A_k, $k = 2, 3, \dots, n$ are the fundamental coefficients of the equation (a), $A_0 \equiv 1$, $A_1 \equiv 0$, $\Phi_{i-k}^{n,i}(\eta) = \Phi_{i-k}^{n,i}\left(\eta, -\frac{n-1}{2}\eta\right)$ and (2.1) holds, see [3; 3, 2.15].

If the function $T(x)$ is in the interval I_{1x} the solution of the equation $\{T, x\} = (3/n + 1) A_2$ [the symbol $\{T, x\}$ stands for the SCHWARZ derivative of the function $T(x)$], then the image $(\bar{\mathrm{A}}) \{T(x)\} \in p_a(I_{1x})$ is canonical and it can be written in the form $(\bar{\mathrm{A}})$ where (2.1), (2.8), $\bar{A}_2 \equiv 0$ hold and

$$\Phi_{i-k}^{n,i}(\eta) = \frac{(n-i)!}{(n-k)!\,(i-k)!} \sum_{\varrho=0}^{i-k} \eta^{i-k-\varrho} F_{\varrho}^{n,i,k}(A_2) \tag{2.9}$$

— see [3; 3, 3.5]. The function $F_{\varrho}^{n,i,k}$ is the polynomial of the element A_2 of the dimension ϱ, which is defined like the polynomial $\Phi_{i-k}^{n,i}$ — see [3; 2, 3.4].

If the equation (a) is canonical, i.e. if $a_1 \equiv a_2 \equiv 0$, then the canonical image $(\bar{\alpha})$ $\{T(x)\} \in k_a(I_{1x})$ is of the form

$$(\bar{\alpha}) \quad U_2(x)\, [T'(x)]^n [z^{(n)}(t) + \sum_{i=3}^{n} \binom{n}{i} (\bar{\alpha}_i(t)\, z^{(n-i)}(t)] = 0, \qquad x = T_{-1}(t),$$

where

$$U_2(x) = c|T'(x)|^{\frac{1-n}{2}}, \qquad 0 \neq c \in E_1, .$$

$$\bar{\alpha}_i(t) = [T'(x)]^{-i} \sum_{\nu=0}^{i-3} [\eta(x)]^{\nu} \left(-\frac{1}{2}\right)^{\nu} \binom{i}{\nu} \binom{i-1}{\nu} \nu!\, a_{i-\nu}(x), \tag{2.10}$$

$$i = 3, 4, \ldots, n, \qquad x = T_{-1}(t).$$

The function $T(x)$ is in the interval I_{1x} the solution of the equation $\{T, x\} = 0$, see [3; 3, 3.6].

We shall introduce yet the perturbated forms of the images of the equation (a).

The equation

$$(\bar{\omega}) \quad uT'^n [I_n(z; \bar{a}_1, \bar{a}_2) + \sum_{i=3}^{n} \binom{n}{i} \bar{\omega}_i^n(t)\, z^{(n-i)}(t)] = 0,$$

where

$$\bar{a}_1 = (T')^{-1} [\Phi_1^{n,1}(\eta, \zeta) + a_1],$$

$$\bar{a}_2 = (T')^{-2} [\Phi_2^{n,2}(\eta, \zeta) + 2\Phi_1^{n,2}(\eta, \zeta)\, a_1 + a_2],$$

$$I_n(z; \bar{a}_1, \bar{a}_2) = \sum_{i=0}^{n} \binom{n}{i} f_i^n(\bar{a}_1, \bar{a}_2)\, z^{(n-i)},$$

$$f_i^n(\bar{a}_1, \bar{a}_2) = (T')^{-i} \sum_{k=0}^{i} \binom{i}{k} f_k^n(a_1, a_2)\, \Phi_{i-k}^{n,i}(\eta, \zeta), \qquad i = 0, 1, \ldots, n,$$

$$\bar{\omega}_i^n = (T')^{-i} \sum_{k=3}^{i} \binom{i}{k} \omega_k^n \Phi_{i-k}^{n,i}(\eta, \zeta), \qquad i = 3, 4, \ldots, n,$$

is the perturbated form of the image $(\bar{\mathrm{a}}) \{T(x), u(x)\} \in o_a(I_{1x})$. The equation

$(\bar{\Omega})$ $$UT'^{n}[I_n(z; \bar{A}_2) + \sum_{i=3}^{n} \binom{n}{i} \bar{\Omega}_i^n z^{(n-i)} = 0,$$
where
$$\bar{A}_2 = (T')^{-2}\left[\frac{n+1}{6}\left(\frac{1}{2}\eta^2 - \eta'\right) + A_2\right],$$
$$I_n(z; \bar{A}_2) = z^{(n)} + \sum_{i=2}^{n} \binom{n}{i} f_i^n(\bar{A}_2)\, z^{(n-i)},$$
$$f_i^n(\bar{A}_2) = (T')^{-i} \sum_{k=0}^{i} f_k^n(A_2)\, \Phi_{i-k}^{n,i}(\eta),$$
$$\bar{\Omega}_i^n = (T')^{-i} \sum_{k=3}^{i} \binom{i}{k} \Omega_k^n \Phi_{i-k}^{n,i}(\eta),$$
is the perturbated form of the image $(\bar{\mathrm{A}})\ \{T(x)\} \in p_a(I_{1x})$.

If the function $T(x)$ is the solution of the equation $\{T, x\} = \frac{3}{n+1} A_2$, then the equation $(\bar{\Omega})$ is the perturbated form of the canonical image $(\bar{\alpha})\ \{T(x)\} \in k_a(I_{1x})$, where we put $\bar{A}_2 \equiv 0$, $I_n(z; 0) = z^{(n)}$,
$$\bar{\Omega}_i^n = i!\,(n-i)!\,(T')^{-i} \sum_{k=3}^{i} \frac{1}{k!\,(n-k)!}\, \Omega_k^n \sum_{\varrho=0}^{i-k} \eta^{i-k-\varrho} F_{\varrho}^{n,i,k}(A_2).$$
Between the polynomials (2.3) and $F_{\varrho}^{n,i,k}(A_2)$ hold the relations
$$\sum_{k=0}^{\nu} \frac{1}{k!\,(n-k)!} f_k^n(A_2)\, F_{\nu-k}^{n,i,k}(A_2) = 0, \qquad \nu = 0, 1, \ldots, i, \qquad i = 3, 4, \ldots, n.$$

3. Equivalence

The notion of equivalence is an important notion in the theory of linear differential equations.

Let us have the equation

(b) $$z^{(n)}(t) + \sum_{i=1}^{n} \binom{n}{i} b_i(t)\, z^{(n-i)}(t) = 0, \qquad b_i \in C_{n-i}(I_2), \qquad i = 1, 2, \ldots, n$$

and let $o_b(I_{2t})$ be the set of the images of the equation (b) in the interval $\emptyset \neq I_{2t} \subset I_2$.

We say that the sets $o_a(I_{1x})$, $o_b(I_{2t})$ are *quasi-identical* denoted by the sign

(3.1) $$o_a(I_{1x}) \doteq o_b(I_{2t}),$$

if every element of the set $o_a(I_{1x})$ is quasi-identical with one of the elements of the set $o_b(I_{2t})$. The relation (3.1) is reflexive, symmetrical and transitive and holds just when at least one element of the set $o_a(I_{1x})$ is quasi-identical with some of the elements of the set $o_b(I_{2t})$.

If (3.1) holds, then we say that the equations (a), (b) are in the intervals I_{1x}, I_{2t} equivalent and we denote it by

$$(\mathrm{a})\, I_{1x} \sim (b)\, I_{2t}\{T(x)\},$$

where $T(x)$ is the first coordinate of the image $(\bar{\mathrm{a}}) \in o_a(I_{1x})$, which is in the interval I_{2t} quasi-identical with the equation (b) so that $T(I_{1x}) = I_{2t}$ holds. We call the function $T(x)$ the carrier of the equivalence of the equation (b) to the equation (a).

The second coordinate $u(x)$ of the image $(\bar{\mathrm{a}})$ is given by the formula

$$u(x) = c|T'|^{\frac{1-n}{2}} \exp \{\int_{x_0}^{x} (b_1[T(s)]\, T'(s) - a_1(s)\,)\, ds\}.$$

With the aid of the relations (2.10) the necessary and sufficient conditions for the equivalence of the canonical equations can be proved.

$$(\alpha) \qquad y^{(n)}(x) + \sum_{i=3}^{n} \binom{n}{i} \alpha_i(x)\, y^{(n-i)}(x) = 0, \quad \alpha_i \in C_{n-i}(I_1), \quad i = 3, 4, \ldots, n,$$

$$(\beta) \qquad z^{(n)}(t) + \sum_{i=3}^{n} \binom{n}{i} \beta_i(t)\, z^{(n-i)}(t) = 0, \quad \beta_i \in C_{n-i}(I_2), \quad i = 3, 4, \ldots, n.$$

Let us denote

$$\vartheta_3(\alpha_3) = \alpha_3$$

$$(3.2) \qquad \vartheta_i(\alpha_3, \ldots, \alpha_i) = \sum_{r=3}^{i} (-1)^{i-r} C_r^i \alpha_r{}^{(i-r)}, \qquad i = 4, 5, \ldots, n,$$

where

$$(3.3) \qquad C_r^i = \binom{i+r-2}{i-1} \binom{i}{r} \Big/ \binom{2i-2}{i-1}, \qquad r = 3, 4, \ldots, i.$$

The formula (3.2) is quoted in the literature as the formula of Brioschi, see [16; p. 197], [18; p. 35], [5; 3.7]. Then holds

Theorem 2. $(\alpha)\, I_{1x} \sim (\beta)\, I_{2t}\{T(x)\} \Leftrightarrow \vartheta_i\{\beta_3[T(x)], \ldots, \beta_i[T(x)]\}\, [T'(x)]^i = \vartheta_i(\alpha_3, \ldots, \alpha_i)$, $i = 3, 4, \ldots, n$, $x \in I_{1x}$, *where* $T(x)$ *is the solution of the equation* $\{T, x\} = 0$. See [5; 2.1].

The function $\vartheta_i(\alpha_3, \ldots, \alpha_i)$ *is the canonical invariant of the equation* (α) *of the dimension and weight* i. *As it is a polynomial of the first order it is also called the linear invariant.*

With the aid of the theorem 2 and with the aid of the relations (2.8), (2.9) the necessary and sufficient conditions for the equivalence of the equations (a), (b) can be proved.

Let us denote

$$(3.4) \qquad \Theta_i^n(A_2, \ldots, A_i) = \sum_{r=3}^{i} C_r^i \Psi_i^{n,r,i-r}(A_2, \ldots, A_i), \qquad i = 3, 4, \ldots, i,$$

where the constants C_r^i are determined by the formula (3.3) and $\Psi_i^{n,r,i-r}$ is the polynomial of the elements $A_2, \ldots, A_i$ of the dimension i, which satisfies a certain linear difference equation of the first order — see [5; (1.6)]. Then the theorem 3 holds.

Theorem 3. (a) $I_{1x} \sim$ (b) $I_{2t}\{T(x)\} \Leftrightarrow \Theta_i^n\{B_2[T(x)], \ldots, B_i[T(x)]\} . [T'(x)]^i = = \Theta_i^n(A_2, \ldots, A_i)$, $i = 3, 4, \ldots, n$, $x \in I_{1x}$, *where* A_i *resp.* B_i, $i = 2, 3, \ldots, n$ *are the fundamental coefficients of the equation* (a) *resp.* (b) *and the function* $T(x)$ *is the solution of the equation*

$$\{T, x\} + \frac{3}{n+1} B_2[T(x)] . [T'(x)]^2 = \frac{3}{n+1} A_2.$$

See [5; 3.3].

The function $\Theta_i^n(A_2, \ldots, A_i)$ *is the fundamental invariant of the equation* (a) *of the dimension and weight* i.

The theorem 3 is stated without proof and inexactly in [16; p. 191].

Between the functions ω_i^n and Θ_i^n hold the following relations:

$$(3.5.) \qquad \Theta_j^n \equiv 0, \quad j = 3, 4, \ldots, i \Leftrightarrow \omega_j^n \equiv 0, \quad j = 3, 4, \ldots, i; \quad i = 3, 4, \ldots, n.$$

From these relations follows

Theorem 4. *The equation* (a) *is iterated just then when all its fundamental invariants are identical to zero.*

In [16; p. 204—205) is quoted without proof the theorem of F. Brioschi which is a special case of the theorems 1 and 4: If all the fundamental invariants of the equation (a) are identical to zero, then the equation (a) has a fundamental system of the form (1.5).

The first non-zero coefficient of the perturbation of the equation (a) is a fundamental invariant, which means that if (3.5) holds, then

$$\Theta_{i+1}^n \not\equiv 0 \Leftrightarrow \omega_{i+1}^n \not\equiv 0 \quad \text{and at the same time} \quad \Theta_{i+1}^n \equiv \omega_{i+1}^n.$$

Theorem 5. *Let* $I_1 \equiv I_2$. *The equations* (a), (b) *are mutually adjoint if and only if the relations*

$$\Theta_i^n(B_2, \ldots, B_i) = (-1)^i \Theta_i^n(A_2, \ldots, A_i), \qquad i = 3, 4, \ldots, n.$$

See [10; 1.18].

Corollary: *Let* $(\bar{\mathrm{A}})$ $\{T(x)\} \in p_a(I_{1x})$. *The normal form of the equation* $(\bar{\mathrm{A}})$ *is a self-adjoint equation just when*

$$\Theta^n_{2\nu+1}(A_2, \ldots, A_i) \equiv 0, \qquad \nu = 1, 2, \ldots, \left[\frac{n-1}{2}\right], \qquad x \in I_{1x}.$$

See [10; 2.9].

From the corollary of the theorem 5. follows this statement: If all the fundamental invariants with odd indices of the equation

$$y^{(n)} + \sum_{i=2}^{n} \binom{n}{i} a_i y^{(n-i)} = 0$$

are identical to zero, then this equation is self-adjoint. This theorem is mentioned without proof in [16; p. 224] and [18; p. 235].

It seems that it is convenient to introduce the notion of the genus of homogeneous linear differential equations.

Let $2 \leqq k \leqq n$ be a natural number. If

$$\Theta^n_j(A_2, \ldots, A_i) \equiv 0, \qquad j = 3, 4, \ldots, n + 2 - k, \qquad \Theta^n_{n+3-k} \not\equiv 0$$

(for $k = 2$ we put $\Theta^n_{n+1} \equiv 0$), we say then that the equation (a) is of the genus k.

The theorem 6 holds.

Theorem 6. *The equation* (a) *is of the same genus k if and only if the equation*

$$I_n(y; a_1 a_2) + \sum_{i=n+3-k}^{n} \binom{n}{i} \omega^n_i y^{(n-i)} = 0, \qquad \omega_{n+3-k} \not\equiv 0, \quad \sum_{i=n+1}^{n} \equiv 0$$

is the perturbated form of the equation (a). See [6; (3.1)].

We take note that under the assumption $\omega^n_n \not\equiv 0$ resp. $\omega^n_{n-1} \not\equiv 0$ is the equation (1.6) resp. (1.7) of the genus 3 resp. 4.

From the theorem 5 (corollary) follows that the self-adjoint equation of the n^{th} order can be of the genus not higher than $(n - 1)$. The iterated equations are of the genus 2. The equation (a) is not of a higher genus than 3 if its canonical image is a binomial equation.

Many of the properties which hold for the equations of the second order hold also for the equations of the n^{th} order of the genus 2. It can be expected that some of the properties of the k^{th} order will hold also for the equations of the n^{th} order of the genus k, f.i. see [8].

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