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Almost Sharp Conditions for the Existence of Smooth Inertial Manifolds

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Abstract. We consider the nonlinear evolution equation $\dot{u} + Au = f(u)$ in a separable, real Hilbert space $\mathbb{H}$ assuming that A is a linear, self-adjoint, positive operator on $\mathbb{H}$ with compact resolvent. The nonlinearity f is assumed to belong to $C^k_{\mathrm{b}}(\mathcal{D}(A^\alpha), \mathcal{D}(A^\beta))$ with $k \in \mathbb{N}_{>0} \cup \{1-\}$ and nonnegative α, β satisfying $0 \le \alpha - \beta \le \frac{1}{2}$. Let P_N be the orthogonal projection of $\mathbb{H}$ onto the subspace generated by the eigenvectors corresponding to the first N eigenvalues λ_i of A. We state an existence theorem for an inertial C^k manifold graph(φ) with $\varphi \in C^k_{\mathrm{b}}(P_N\mathcal{D}(A^\alpha), (I - P_N)\mathcal{D}(A^\alpha))$ using an almost sharp spectral gap condition

$$\lambda_{N+1} - k\lambda_N > \sqrt{2}\,\mathrm{Lip}\,(f)\left(\lambda^{\alpha-\beta}_{N+1} + k\lambda^{\alpha-\beta}_N\right).$$

Assuming the existence of an absorbing ball $\mathcal{B}_{\mathcal{D}(A^\alpha)}(\underline{r})$ in $\mathrm{dom}(A^\alpha)$, and assuming only $f|\mathcal{B}_{\mathcal{D}(A^\alpha)}(\sqrt{2}\overline{r}) \in C^k_{\mathrm{b}}(\mathcal{B}_{\mathcal{D}(A^\alpha)}(\sqrt{2}\overline{r}), \mathcal{D}(A^\beta))$, we state the existence of a globally attracting, locally positively invariant C^k manifold graph$(\varphi) \cap \mathcal{B}_{\mathcal{D}(A^\alpha)}(\underline{r})$ using the spectral gap condition

$$\lambda_{N+1} - k\lambda_N > \sqrt{2}\,\mathrm{Lip}\left(f|\mathcal{B}_{\mathcal{D}(A^\alpha)}(\sqrt{2}\overline{r})\right)\left(\lambda^{\alpha-\beta}_{N+1} + k\lambda^{\alpha-\beta}_N\right)$$

where $\overline{r} > \underline{r}$. For it a special preparation of f is used.
The proofs of the theorems base on comparison theorems for special two-point boundary value problems and for inequalities in ordered Banach spaces.

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1 Introduction

Let $\mathbb{H}$ be a separable, real Hilbert space with inner product $\langle\cdot|\cdot\rangle$ and norm $|\cdot|$. We consider the nonlinear evolution equation

$$\dot{u} + Au = f(u) \tag{1}$$

for $u \in \mathbb{H}$ where A satisfies

This is the final form of the paper.

Assumption 1. A is a linear, self-adjoint, positive operator on $\mathbb{H}$ with compact resolvent.

Thus, $-A$ is the infinitesimal generator of an analytic semigroup on $\mathbb{H}$.

Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ denote the eigenvalues of A repeated with their multiplicities, and let e_i denote corresponding orthonormal eigenvectors of A. By the properties of A the eigenvectors e_i form an orthonormal basis in $\mathbb{H}$.

We can define the fractional powers A^α for $\alpha \in \mathbb{R}$, see [Hen81]. The domains $\mathcal{U}^\alpha := \mathcal{D}(A^\alpha)$ of A^α are Hilbert spaces with respect to the scalar product $\langle u|v\rangle_\alpha := \langle A^\alpha u|A^\alpha v\rangle$, and the corresponding norm $|\cdot|_\alpha$ is equivalent to the graph norm. With P_N we denotes the orthogonal projection of $\mathbb{H}$ onto $\operatorname{span}\{e_1, \dots, e_N\}$. Since $\mathcal{U}^\alpha = \{u \in \mathbb{H} : \sum_{j=1}^{\infty} \langle u, e_j\rangle^2 \lambda_j^{2\alpha} < \infty\}$, we have $P_N\mathbb{H} \cap \mathcal{U}^\alpha = P_N\mathcal{U}^\alpha$ and $(I - P_N)\mathbb{H} \cap \mathcal{U}^\alpha = (I - P_N)\mathcal{U}^\alpha$. Further P_N commutes with A^γ for $\gamma \geq 0$.

The nonlinear term f is assumed to satisfy at least

Assumption 2. There are $k \in \mathbb{N}_{>0} \cup \{1-\}$, $\kappa \in [0, 1[$ with $\kappa = 0$ iff $k = 1-$ and nonnegative constants α, β satisfying $0 \leq \alpha - \beta \leq \frac{1}{2}$ such that $f|\Omega$ belongs to $C^{k+\kappa}_{\mathrm{bu}}(\Omega, \mathcal{U}^\beta)$ for any bounded set $\Omega \subset \mathcal{U}^\alpha$.

Here $f|\Omega$ denotes the restriction of f onto Ω. $C^{1-}_{\mathrm{bu}}(\mathcal{E}, \mathcal{F})$ denotes the Banach space of the bounded continuous functions from $\mathcal{E}$ into $\mathcal{F}$ being uniformly Lipschitz. For $k \geq 1$, $C^{k+\kappa}_{\mathrm{bu}}(\mathcal{E}, \mathcal{F})$ denotes the Banach space of the k-times κ-Hölder continuously differentiable functions from $\mathcal{E}$ into $\mathcal{F}$ with bounded derivatives up to the order k. In the following, we calculate with $1-$ as with 1. We denote by $\operatorname{Lip}(g)$ the smallest Lipschitz constant of g on its domain $\operatorname{dom}(g)$.

For a subspace $\mathcal{U}$ of $\mathcal{U}^\alpha$ endowed with the induced topology let

$$\mathcal{B}_{\mathcal{U}}(r) := \{u \in \mathcal{U} : |u|_\alpha < r\}$$

be the open ball in $\mathcal{U}$ centered at 0 with radius $r \leq \infty$.

Applying the results of [Hen81], equation (1) generates a (local) semigroup S in $\mathcal{U}^\alpha$, such that the (classical) solution at time t in the existence interval through an initial point $u_0 \in \mathcal{U}^\alpha$ is given by $u(t) = S(t)u_0$. For $t > 0$, $u(t)$ is more regular than the initial point, with $u(t) \in \mathcal{U}^{1+\beta} \subseteq \mathcal{D}(A)$ and $\dot{u}(t) \in \mathcal{U}^\beta$. These regularity results make it possible to work with the equation itself and take inner products rather than have to use the variation of constant formula. In particular expressions such as

$$\frac{1}{2}\frac{d}{dt}|u(t)|_\alpha^2 = \left\langle A^\beta \dot{u}(t)|A^{2\alpha-\beta}u(t)\right\rangle$$

make sense for $t > 0$ since $\mathcal{U}^{1+\beta} \subseteq \mathcal{U}^{2\alpha-\beta}$ because of $\alpha - \beta \leq \frac{1}{2}$ and since $\dot{u}(t) \in \mathcal{U}^\beta$ and $u(t) \in \mathcal{U}^{1+\beta}$ for $t > 0$.

Recall that an *inertial C^k manifold* $\mathcal{M}$ is a subset of $\mathbb{H}$ with the following properties (see [MPS88,FST88,Tem88] for $k = 1-$):

1. $\mathcal{M}$ is a finite dimensional C^k manifold in $\mathcal{U}^\alpha \subseteq \mathbb{H}$.
2. $\mathcal{M}$ is positively invariant; i.e., if $u_0 \in \mathcal{M}$ then $S(t)u_0 \in \mathcal{M}$ for all $t \in [0, \infty[$.
3. $\mathcal{M}$ is exponentially attracting; i.e., there is a $\gamma > 0$ such that for every $\eta \in \mathcal{U}^\alpha$ there is a C such that

$$\operatorname{dist}(S(t)\eta, \mathcal{M}) \leq C\mathrm{e}^{-\gamma t} \qquad (t \geq 0).$$

In some papers the exponential attracting property is supplemented by the *exponential tracking property* ([FST89]) or *asymptotical completeness property* ([CFNT89,Rob96,Tem97]):

> There is $\gamma > 0$ such that for every $\eta \in \mathcal{U}^\alpha$ there are $\hat{\eta} \in \mathcal{M}$ and $C \geq 0$ with $|S(t)\eta - S(t)\hat{\eta}|_\alpha \leq C\mathrm{e}^{-\gamma t} \operatorname{dist}(\eta, \mathcal{M})$ for all $t > 0$.

Usually we are looking for an inertial C^k manifold $\mathcal{M}$ which is constructed as the graph $\operatorname{graph}(\varphi) := \{\xi + \varphi(\xi) : \xi \in P_N\mathcal{U}^\alpha\}$ of a C^k function $\varphi : P_N\mathcal{U}^\alpha \to (I - P_N)\mathcal{U}^\alpha$.

Because of the attraction property of $\mathcal{M}$, the asymptotical behavior of the solutions of (1) is governed by the asymptotical behavior of the solutions on the finite-dimensional manifold $\mathcal{M}$. The dynamic on $\mathcal{M}$ is determined by the ordinary differential equation (*inertial form*)

$$\dot{x} + Ax = P_N f(x + \varphi(x))$$

in the N-dimensional Banach space $P_N\mathcal{U}^\alpha$.

Instead of Assumptions 2 usually one assumes

$$f \in C^k_{\mathrm{b}}(\mathcal{U}^\alpha, \mathcal{U}^\beta) \tag{2}$$

with suitable $\alpha \geq \beta$: For $k = 1-$ we have for example $\alpha = 1$, $\beta = \frac{1}{2}$ in [FST88], $\beta = \alpha - \frac{1}{2}$ in [Tem88], $\alpha = \beta = 0$ in [MPS88], $0 = \beta \leq \alpha < 1$ in [Rom94], $0 \leq \alpha - \beta \leq \frac{1}{2}$ in [Rob93], $0 \leq \alpha - \beta < 1$ in [CLS92]. Thus our assumption $0 \leq \alpha - \beta \leq \frac{1}{2}$ assumed for technical reason is not the weakest possible one.

A spectral gap condition mostly of the form

$$\lambda_{N+1} - \lambda_N > C_1 \operatorname{Lip}(f)\,(\lambda_{N+1}^{\alpha-\beta} + \lambda_N^{\alpha-\beta}) \tag{3}$$

plays an important role where C_1 is a number depending on α, β, and $\operatorname{Lip}(f)$.

Romanov [Rom94] found (3) with $C_1 = 1$ ensuring the existence of a Lipschitz inertial manifold for (1) with $0 = \beta \leq \alpha < 1$. He gave counter-examples satisfying a spectral gap condition (3) with $C_1 < 1$ but not having an inertial manifold. That means, the spectral gap condition (3) with $C_1 = 1$ is a sharp condition for Lipschitz inertial manifolds. As corollary of our Theorem 8 we have a spectral gap condition (3) with $C_1 = \sqrt{2}$, i.e. our spectral gap condition is a little stronger than Romanov's one.

The weakest known spectral gap condition in the form (3) for inertial C^1 manifolds was found by Ninomiya [Nin92] with $C_1 = 2$ for $0 \leq \alpha - \beta < 1/2$. Our

Theorem 8 will allow $C_1 = \sqrt{2}$, i.e. our spectral gap condition is a little weaker than Ninomiya's one.

For $k > 1$ and (2), Chow et al. [CLS92] have a spectral gap condition of the form

$$\lambda_{N+1} - k\lambda_N > C_1 \left(\lambda_{N+1}^{\alpha-\beta} + \lambda_N^{\alpha-\beta} \right), \quad \lambda_N > C_0$$

but with unknown C_0, C_1 depending on α, β, k and $\mathrm{Lip}\,(f)$. Additionally they get the a priori estimate $\mathrm{Lip}\,(\varphi) \leq 1$. Inserting (10) with $\overline{\mathcal{Q}} = \mathcal{U}^\alpha$ in (14) we obtain the spectral gap condition

$$\lambda_{N+1} - k\lambda_N > \sqrt{2}\,\mathrm{Lip}\,(f) \left(\lambda_{N+1}^{\alpha-\beta} + k\lambda_N^{\alpha-\beta} \right)$$

for the existence of an inertial C^k manifold of (1) even for $k \geq 1$. Moreover, we get the better a priori estimate $\mathrm{Lip}\,(\varphi) \leq \chi_1$ where the number $\chi_1 < 1$ is defined in Lemma 4.

Let $\mathcal{Q}$ be an open set in $\mathcal{U}^\alpha$. In order to include also manifolds which are subsets of $\mathcal{Q}$, we introduce the following notion: A set $\mathcal{M}$ is called *inertial C^k manifold in $\mathcal{Q}$* if

1. $\mathcal{M}$ is a finite dimensional C^k manifold in $\mathcal{U}^\alpha$.
2. $\mathcal{M} \cap \mathcal{Q}$ is locally positively invariant; i.e., if $u_0 \in \mathcal{M} \cap \mathcal{Q}$ then there is $\varepsilon > 0$ such that $S(t)u_0 \in \mathcal{M} \cap \mathcal{Q}$ for all $t \in [0, \varepsilon[$.
3. $\mathcal{M}$ is exponentially attracting for all orbits in $\mathcal{Q}$; i.e., there is a $\gamma > 0$ such that for any u_0 with $S(t)u_0 \in \mathcal{Q}$ for $t > 0$ there is a constant C such that

$$\mathrm{dist}(S(t)\eta, \mathcal{M}) \leq C\mathrm{e}^{-\gamma t} \qquad (t \geq 0).$$

If $\mathcal{M}$ is an inertial manifold in $\mathcal{Q}$ then the asymptotical behavior of the orbits of (1) in $\mathcal{Q}$ is determined by the orbits of (1) in $\mathcal{M} \cap \mathcal{Q}$.

If $Q = \mathcal{U}^\alpha$ and $\mathrm{dom}(\varphi) = P_N\mathcal{U}^\alpha$ then an inertial manifold $\mathcal{M} = \mathrm{graph}(\varphi)$ in $\mathcal{Q}$ is an inertial manifold in the usual sense.

If f does not satisfy (2), it is usually modified by a trunctation method to a new function f so that the asymptotic behavior of the solutions of (1) is not changed but f satisfies (2): If $\mathcal{B}(\underline{r})$ is an absorbing set of (1) then f is modified outside of $\mathcal{B}(\underline{r})$ in such a way that $f(u) = 0$ outside of $\mathcal{B}(2\underline{r})$, and such that $\mathrm{Lip}\,(f)$ of the new function is not greater than $\mathrm{Lip}\,(f|\mathcal{B}(2\underline{r}))$ of the old function. Then an inertial manifold $\mathcal{M}$ of the prepared equation is an inertial manifold in $\mathcal{B}(\underline{r})$ of the original equation (1).

Let $\mathcal{Q} = \mathcal{B}_{P_N\mathcal{U}^\alpha}(r) + \mathcal{B}_{(I-P_N)\mathcal{U}^\alpha}(r)$ and $\overline{\mathcal{Q}} = \mathcal{B}_{P_N\mathcal{U}^\alpha}(\overline{r}) + \mathcal{B}_{(I-P_N)\mathcal{U}^\alpha}(\overline{r})$ with $\overline{r} > r > \underline{r}$ arbitrary close to $\underline{r}$. Theorem 8 allows to use $\mathrm{Lip}\,\left(f|\overline{\mathcal{Q}}\right)$ instead of $\mathrm{Lip}\,(f|\mathcal{B}(2\underline{r}))$ even for $k \geq 1$ such that we get an additional weakening in the spectral gap condition since $\mathrm{Lip}\,\left(f|\overline{\mathcal{Q}}\right) \leq \mathrm{Lip}\,\left(f|\mathcal{B}(\sqrt{2}\underline{r})\right) \leq \mathrm{Lip}\,(f|\mathcal{B}(2\underline{r}))$ for $\overline{r} < \sqrt{2}\underline{r}$.

A crucial role in the proof of Theorem 8 plays comparison system (8). System (8) has a linear inertial manifold in $\mathbb{R}^2_{\geq 0}$ if and only if the spectral gap condition (5) is satisfied.

2 Main Results

2.1 Existence of Smooth Inertial Manifolds in a Set $\mathcal{Q}$

Let $N \in \mathbb{N}$ be suitable chosen. In order to simplify notation we shall use

$$\begin{aligned} \Lambda_1 &:= \lambda_N, & \Lambda_2 &:= \lambda_{N+1}, \\ \pi_1 &:= P_N, & \pi_2 &:= I - P_N, \\ \mathcal{U}_1^\alpha &:= \pi_1 \mathcal{U}^\alpha, & \mathcal{U}_2^\alpha &:= \pi_2 \mathcal{U}^\alpha \end{aligned}$$

such that $\mathcal{U}^\alpha = \mathcal{U}_1^\alpha \oplus \mathcal{U}_2^\alpha$. To avoid repetition, we agree that i always ranges over the integers 1 and 2.

We assume that the set $\mathcal{Q}$ has the special form

$$\mathcal{Q} := \mathcal{B}_{\mathcal{U}_1^\alpha}(r_1) + \mathcal{B}_{\mathcal{U}_2^\alpha}(r_2),$$

where $r_i \in \mathbb{R}_{\geq 0}$ or $r_1 = r_2 = \infty$.

In order to ensure the existence of an inertial manifold in $\mathcal{Q}$, we introduce

Assumption 3. There are numbers $\gamma_i > 0$ and $\overline{r}_i$ with $r_i < \overline{r}_i < \infty$ or $\overline{r}_i = r_i = \infty$ such that the one-sided Lipschitz inequalities

$$\begin{aligned} \left\langle A^{2\alpha-\beta}\pi_1 u_\Delta \middle| A^\beta \pi_1 [f(u_1) - f(u_2)] \right\rangle &\geq -\gamma_1 \Lambda_1^{\beta-\alpha} |\pi_1 u_\Delta|_{2\alpha-\beta} |u_\Delta|_\alpha, \\ \left\langle A^{2\alpha-\beta}\pi_2 u_\Delta \middle| A^\beta \pi_2 [f(u_1) - f(u_2)] \right\rangle &\leq \gamma_2 \Lambda_2^{\beta-\alpha} |\pi_2 u_\Delta|_{2\alpha-\beta} |u_\Delta|_\alpha \end{aligned} \tag{4}$$

hold for any $u_i \in \overline{\mathcal{Q}} \cap \mathcal{U}^{1+\beta}$ where $u_\Delta = u_1 - u_2$ and

$$\overline{\mathcal{Q}} := \mathcal{B}_{\mathcal{U}_1^\alpha}(\overline{r}_1) + \mathcal{B}_{\mathcal{U}_2^\alpha}(\overline{r}_2).$$

The following technical lemma gives a connection between the spectral gap condition (5) and a comparison problem (8) in the plane:

Lemma 4. *Let the spectral gap condition*

$$\Lambda_2 - \Lambda_1 > \left(\gamma_1^{2/3} + \gamma_2^{2/3}\right)^{3/2} \tag{5}$$

be satisfied. Then we have:

1. *There are* $\chi_2 > \sqrt[3]{\gamma_2/\gamma_1} > \chi_1 > 0$ *and* $\varrho_2 < \varrho_1$ *which are uniquely determined by*

$$\varrho_i = -\Lambda_1 - \gamma_1 \sqrt{1 + \chi_i^2} = -\Lambda_2 + \gamma_2 \sqrt{1 + \chi_i^{-2}}. \tag{6}$$

Moreover,

$$\varrho_2 < 0. \tag{7}$$

2. *The sets* $\Psi_i := \{w \in \mathbb{R}^2_{\geq 0} : w^2 = \chi_i w^1\}$ *are integral manifolds of the comparison system*

$$\dot{w}^1 = -\Lambda_1 w^1 - \gamma_1 |w|, \quad \dot{w}^2 = -\Lambda_2 w^2 + \gamma_2 |w|, \tag{8}$$

where $|w| = \sqrt{(w^1)^2 + (w^2)^2}$. *The function* $\psi_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^2_{\geq 0}$ *defined by*

$$\psi_i(t) := \mathrm{e}^{\varrho_i t}(1, \chi_i) \qquad (t \geq 0)$$

is the solution of (8) *through* $(1, \chi_i) \in \Psi_i$ *at* $t = 0$.

Proof. First we note that $\Psi = \{w \in \mathbb{R}^2_{\geq 0} : w^2 = \chi w^1\}$ is an integral manifold of (8) if $\chi \geq 0$ is a zero of the function $p : \mathbb{R}_{>0} \to \mathbb{R}$ defined by

$$p(\chi) = \Lambda_1 - \Lambda_2 + \gamma_1 \sqrt{1 + \chi^2} + \gamma_2 \sqrt{1 + \chi^{-2}}.$$

The function p is strongly convex with $\lim_{\chi \to 0} p(\chi) = \lim_{\chi \to \infty} p(\chi) = +\infty$. Hence p has at most two positive zeroes. p is minimized at $\chi_0 := \sqrt[3]{\gamma_2/\gamma_1}$ and $p(\chi_0) < 0$ because of (5). Therefore, the existence of positive zeroes χ_1, χ_2 of p with $\chi_1 < \chi_0 < \chi_2$ follow. By definition of p, these numbers χ_1, χ_2 satisfy (6). Thus Ψ_i are integral manifolds of the comparison system (8) and the functions ψ_i are solutions on Ψ_i with the stated properties.

Since $\Lambda_1 > 0$ we have $\varrho_2 < \varrho_1 < 0$ and hence (7). □

Remark 5. Ψ_1 is an inertial manifold of (8) in $\mathbb{R}^2_{\geq 0}$.

Remark 6. Requiring $p(1) < 0$ one gets the little stronger gap condition

$$\Lambda_2 - \Lambda_1 > \sqrt{2}(\gamma_2 + \gamma_1). \tag{9}$$

Assuming (9) we have $\chi_1 < 1 < \chi_2$ and

$$\varrho_1 > -\Lambda_1 - \sqrt{2}\gamma_1, \quad \varrho_2 < -\Lambda_2 + \sqrt{2}\gamma_2.$$

Remark 7. If $r_i < \infty$ and $\mathrm{Lip}\left(f|\overline{\mathcal{Q}}\right) > 0$ the existence of numbers γ_i satisfying Assumption 3 follows from Assumption 2: We can choose

$$\gamma_i = \mathrm{Lip}\left(f|\overline{\mathcal{Q}}\right) \Lambda_i^{\alpha - \beta}. \tag{10}$$

The spectral gap conditions (5), (9) read now

$$\Lambda_2 - \Lambda_1 > \mathrm{Lip}\left(f|\overline{\mathcal{Q}}\right) \left(\Lambda_2^{2(\alpha-\beta)/3} + \Lambda_1^{2(\alpha-\beta)/3}\right)^{3/2}$$

and

$$\Lambda_2 - \Lambda_1 > \sqrt{2}\,\mathrm{Lip}\left(f|\overline{\mathcal{Q}}\right) \left(\Lambda_2^{\alpha-\beta} + \Lambda_1^{\alpha-\beta}\right)$$

in the well-known form.

Theorem 8 (Inertial manifold in $\mathcal{Q}$). *Let the Assumption 1, 2, 3 be satisfied. If (5) then there is a $\varphi \in C_{\mathrm{b}}^{1-}(\mathcal{B}_{\mathcal{U}_1^\alpha}(r_1), \mathcal{U}_2^\alpha)$ with $\mathrm{Lip}\,(\varphi) \le \chi_1$ and being uniquely defined if $r_i = \infty$ such that $\mathcal{M} := \mathrm{graph}(\varphi)$ is an inertial C^{1-} manifold in $\mathcal{Q}$ with*

$$\mathrm{dist}(S(t)u_0, \mathcal{M}) \le \frac{\chi_2 + \chi_1}{\chi_2 - \chi_1} |\pi_2 u_0 - \varphi(\pi_1 u_0)|_\alpha \mathrm{e}^{\varrho_2 t} \qquad (t \ge 0) \tag{11}$$

for any u_0 with $S(t)u_0 \in \mathcal{Q}$ for $t \ge 0$.

Moreover, for any $\underline{\mathcal{Q}} \subseteq \mathcal{Q}$ with positive distance to $\partial\mathcal{Q}$ if $r_i < \infty$ and any u_0 with $S(t)u_0 \in \underline{\mathcal{Q}}$ for $t \ge 0$ there are $\hat{u}_0 \in \mathcal{M} \cap \mathcal{Q}$ and $T \ge 0$ with $S(t)\hat{u}_0 \in \mathcal{M} \cap \mathcal{Q}$ for $t \ge 0$ and

$$|\pi_i[S(t+T)u_0 - S(t)\hat{u}_0]|_\alpha \le \frac{|\pi_2 u_0 - \varphi(\pi_1 u_0)|_\alpha}{\chi_2 - \chi_1} \psi_2^i(t+T) \qquad (t \ge 0) \tag{12}$$

where $T = 0$ if $r_2 = \infty$. If in addition $k \ge 1$ and

$$\varrho_2 > k\varrho_1 \tag{13}$$

then $\varphi \in C_{\mathrm{b}}^k(\mathcal{B}_{\mathcal{U}_1^\alpha}(r_1), \mathcal{B}_{\mathcal{U}_2^\alpha}(r_2))$.

Theorem 8 will be proved by means of Theorem 11 concerning the existence of special overflowing invariant manifolds.

Remark 9. Since $\chi_1 < \sqrt[3]{\gamma_2/\gamma_1} < \chi_2$ we have

$$\varrho_1 > -\Lambda_1 - \gamma_1^{2/3}\sqrt{\gamma_1^{2/3} + \gamma_2^{2/3}}, \quad \varrho_2 < -\Lambda_2 + \gamma_2^{2/3}\sqrt{\gamma_1^{2/3} + \gamma_2^{2/3}}$$

such that (13) can be replaced by

$$\Lambda_2 - k\Lambda_1 > (\gamma_2^{2/3} + k\gamma_1^{2/3})\sqrt{\gamma_1^{2/3} + \gamma_2^{2/3}}.$$

Assuming (9), this inequality can be replaced by the stronger condition

$$\Lambda_2 - k\Lambda_1 > \sqrt{2}(\gamma_2 + k\gamma_1). \tag{14}$$

2.2 Overflowing Invariant Manifolds

Theorem 8 will be reduced to the following Theorem 11 concerning the existence of an overflowing invariant manifold for the prepared evolution equation

$$\dot{u} + Au = \tilde{f}(u). \tag{15}$$

A set $\overline{\mathcal{M}}^* = \mathrm{graph}(\varphi^*)$ with $\varphi^* : \mathrm{cl}\,\mathcal{W}_0 \to \mathcal{U}_2^\alpha$ and $\mathcal{W}_0 \subseteq \mathcal{U}_1^\alpha$ is called *overflowing invariant* with respect to (15) (compare [Wig94]) if:

– $\mathcal{M}^* := \mathrm{graph}(\varphi^*|\mathcal{W}_0)$ is locally positively invariant with respect to (15).

- The vector field of (15) is pointing strictly outward on the boundary $\partial\mathcal{M}^* = \overline{\mathcal{M}}^* \setminus \mathcal{M}^*$.
- The vector field of (15) is nonzero on $\partial\mathcal{M}^*$.

Besides Assumption 1 we need

Assumption 10. There are $\hat{r}_i$ with $0 < \hat{r}_1 < \overline{r}_1 < \infty$ or $0 < \hat{r}_1 \le \overline{r}_1 \le \infty$ and $0 < \hat{r}_2 < \overline{r}_2 \le \infty$ such that $\tilde{f}|\overline{\mathcal{Q}}$ belongs to $C_{\mathrm{b}}^k(\overline{\mathcal{Q}}, \mathcal{U}^\beta)$, and such that $\tilde{f}$ satisfies

$$\begin{aligned} \left\langle A^{2\alpha-\beta}\pi_1 u_\Delta | A^\beta \pi_1[\tilde{f}(u_1) - \tilde{f}(u_2)]\right\rangle &\ge -\gamma_1 \Lambda_1^{\beta-\alpha}|\pi_1 u_\Delta|_{2\alpha-\beta}|u_\Delta|_\alpha, \\ \left\langle A^{2\alpha-\beta}\pi_2 u_\Delta | A^\beta \pi_2[\tilde{f}(u_1) - \tilde{f}(u_2)]\right\rangle &\le \gamma_2 \Lambda_2^{\beta-\alpha}|\pi_2 u_\Delta|_{2\alpha-\beta}|u_\Delta|_\alpha \end{aligned} \tag{16}$$

for $u_i \in \overline{\mathcal{Q}} \cap \mathcal{U}^{1+\beta}$ where $u_\Delta = u_1 - u_2$, and

$$\begin{aligned} \left\langle A^{2\alpha-\beta}\pi_1 u| - A^{1+\beta}\pi_1 u + A^\beta \pi_1 \tilde{f}(u)\right\rangle &> 0 \quad \text{if } |\pi_1 u|_\alpha = \hat{r}_1, \\ \left\langle A^{2\alpha-\beta}\pi_2 u| - A^{1+\beta}\pi_2 u + A^\beta \pi_2 \tilde{f}(u)\right\rangle &< 0 \quad \text{if } |\pi_2 u|_\alpha = \hat{r}_2 \end{aligned} \tag{17}$$

for $u \in \overline{\mathcal{Q}} \cap \mathcal{U}^{1+\beta}$.

The inequalities (17) ensure some inflowing and outflowing properties of the vector field on the boundary of

$$\hat{\mathcal{Q}} := \mathcal{B}_{\mathcal{U}_1^\alpha}(\hat{r}_1) + \mathcal{B}_{\mathcal{U}_2^\alpha}(\hat{r}_2).$$

Let $\tilde{S}$ denote the local semiflow of (15) in $\overline{\mathcal{Q}}$.

Theorem 11 (Overflowing invariant manifold). *Let Assumption 1 and 10 as well as (5) be satisfied and let $\mathcal{W}_0 := \mathcal{B}_{\mathcal{U}_1^\alpha}(\hat{r}_1)$. Then there is a unique $\varphi^* \in C_{\mathrm{b}}^{1-}(\mathrm{cl}\,\mathcal{W}_0, \mathcal{U}_2^\alpha)$ with $\mathrm{Lip}\,(\varphi) \le \chi_1$ and $|\varphi(\xi)|_\alpha \le \hat{r}_2$ for $\xi \in \mathrm{cl}\,\mathcal{W}_0$ such that $\overline{\mathcal{M}}^* := \mathrm{graph}(\varphi^*)$ is overflowing invariant with respect to the prepared evolution equation (15). Moreover, for any $u_0 \in \hat{\mathcal{Q}}$ with $\tilde{S}(t)u_0 \in \hat{\mathcal{Q}}$ for $t \ge 0$ there is $\hat{u}_0 \in \mathcal{M}^*$ with $\tilde{S}(t)\hat{u}_0 \in \mathcal{M}^*$ for $t \ge 0$ and*

$$|\pi_i[\tilde{S}(t)u_0 - \tilde{S}(t)\hat{u}_0]|_\alpha \le \frac{|\pi_2 u_0 - \varphi^*(\pi_1 u_0)|_\alpha}{\chi_2 - \chi_1}\psi_2^i(t) \qquad (t \ge 0).$$

If $k \ge 1$ and (13) then $\varphi^* \in C_{\mathrm{b}}^k(\mathrm{cl}\,\mathcal{W}_0, \mathcal{U}_2^\alpha)$.

In order to show the existence of a C^{1-} manifold with the properties stated in Theorem 11 we proceed as follows. For fixed $\gamma \in [\alpha, \beta+1[$ we introduce the Banach space $\mathbb{G}_0 := C_{\mathrm{b}}^0(\mathrm{cl}\,\mathcal{W}_0, \mathcal{U}_2^\gamma)$ equipped with the supremum norm $||\varphi_0||_0 := \sup_{\xi \in \mathrm{cl}\,\mathcal{W}_0} |\varphi_0(\xi)|_\gamma$. Let

$$\Phi_0 := \{\varphi_0 \in \mathbb{G}_0 \,:\, ||\varphi_0||_0 \le \hat{r}_2,\ \mathrm{Lip}\,(\varphi_0) \le \chi_1\}.$$

Note that Φ_0 is a closed subset of $\mathbb{G}_0$.

We introduce the two-point boundary value problems

$$\dot{u} + Au = \tilde{f}(u) \tag{18a}$$

$$\pi_1 u(\vartheta) = \xi,\ \pi_2 u(0) = \varphi_0(\pi_1 u(0)) \tag{18b}$$

on $[0, \vartheta]$ with $\xi \in \mathrm{cl}\, \mathcal{W}_0$, $\vartheta > 0$, and $\varphi_0 \in \Phi_0$. Showing that (18) has a unique solution $U_0(\cdot, \vartheta, \xi, \varphi_0)$ satisfying

$$U_0(t, \vartheta, \xi, \varphi_0) \in \overline{\mathbb{Q}} \qquad (t \in [0, \vartheta])$$

for any $\vartheta > 0$, $\xi \in \mathrm{cl}\, \mathcal{W}_0$, $\varphi_0 \in \Phi_0$, we can define the $G_0(\vartheta) : \Phi_0 \to \mathbb{G}_0$ by

$$(G_0(\vartheta)\varphi_0)(\xi) = \pi_2 U_0(\vartheta, \vartheta, \xi, \varphi_0) \qquad (\vartheta \geq 0,\ \xi \in \mathrm{cl}\, \mathcal{W}_0,\ \varphi_0 \in \Phi_0).$$

Using some properties of $U_0(t, \vartheta, \xi, \varphi_0)$ we can show that $G_0(\vartheta)$ maps Φ_0 into itself and that $G_0(\vartheta)$ is uniformly contractive for $\vartheta \geq T_0$ and sufficiently large T_0. Hence there is a unique fixed-point $\varphi_0^*(\vartheta)$ in Φ_0 for $\vartheta \geq T_0$. Showing the existence of these fixed-points for all $\vartheta > 0$ and showing their independence of ϑ we get the locally positive invariance of graph$(\varphi_0^* | \mathcal{W}_0)$.

The exponential tracking property can also be proved reducing it to the estimation of solutions of boundary value problems.

In order to show higher smoothness of φ_0 assuming the spectral gap condition (13), we shall use the fiber contraction principle [Van89,CLS92,Tem97]. Since C^k-smoothness for $k \geq 3$ can be proved similarly to the C^2-smoothness, we restrict us to $k \leq 2$.

First let $k = 2$. Let the spectral gap condition (13) be satisfied and let $\gamma \in]\alpha, \beta + 1[$ be fixed. Applying the implicit function theorem one can show that $U_0(t, \vartheta, \cdot, \varphi_0)$ is twice continuously differentiable for $t \in [0, \vartheta]$, $\vartheta > 0$, $\varphi_0 \in \Phi_0$.

We introduce

$$\mathbb{G}_1 := C_{\mathrm{b}}^0(\mathrm{cl}\, \mathcal{W}_0, \mathcal{L}(\mathcal{U}_1^\gamma, \mathcal{U}_2^\gamma)),$$

$$\mathbb{G}_2 := C_{\mathrm{b}}^0(\mathrm{cl}\, \mathcal{W}_0, \mathcal{L}(\mathcal{U}_1^\gamma \times \mathcal{U}_1^\gamma, \mathcal{U}_2^\gamma)).$$

$\mathbb{G}_1$, $\mathbb{G}_2$ are complete with respect to the norms $||\cdot||_1$ and $||\cdot||_2$ defined by

$$||\varphi_1||_1 := \sup_{\xi \in \mathrm{cl}\, \mathcal{W}_0} \max_{h \in \mathrm{cl}\, \mathcal{B}_{\mathcal{U}_1^\gamma}(1)} |\varphi_1(\xi) h|_\gamma,$$

$$||\varphi_2||_2 := \sup_{\xi \in \mathrm{cl}\, \mathcal{W}_0} \max_{h_i \in \mathrm{cl}\, \mathcal{B}_{\mathcal{U}_1^\gamma}(1)} |\varphi_2(\xi)(h_1, h_2)|_\gamma$$

for $\varphi_1 \in \mathbb{G}_1$, $\varphi_2 \in \mathbb{G}_2$. Further we introduce the closed sets $\Phi_1 := \{\varphi_1 \in \mathbb{G}_1 : ||\varphi_1||_1 \leq \chi_1\}$, $\Phi_2 := \mathbb{G}_2$.

One can show that for any $\vartheta > 0$, $\xi \in \mathrm{cl}\, \mathcal{W}_0$, $(\varphi_0, \varphi_1, \varphi_2) \in \Phi_0 \times \Phi_1 \times \Phi_2$, $h_1, h_2 \in \mathcal{U}_1^\gamma$ there are a unique classical solution $U_1(\cdot, \vartheta, \xi, \varphi_0, \varphi_1, h_1)$ of

$$\begin{aligned} &\dot{u} + Au = D\tilde{f}(U(t))u, \\ &\pi_1 u(\vartheta) = h_1, \quad \pi_2 u(0) = \varphi_1(\pi_1 U(0)) \pi_1 u(0) \end{aligned} \tag{19}$$

on $[0,\vartheta]$ and a unique classical solution $U_2(\cdot,\vartheta,\xi,\varphi_0,\varphi_1,\varphi_2,h_1,h_2)$ of

$$\begin{aligned} \dot u + Au &= D\tilde f(U(t))u + R_1(t), \\ \pi_1 u(\vartheta) = 0, \quad \pi_2 u(0) &= \varphi_1(\pi_1 U(0))\pi_1 u(0) + R_2 \end{aligned} \tag{20}$$

on $[0,\vartheta]$ where $U(t) = U_0(t,\vartheta,\xi,\varphi_0)$,

$$R_1(t) = D^2\tilde f(U(t))(U_1(t,\vartheta,\xi,\varphi_0,\varphi_1,h_1), U_1(t,\vartheta,\xi,\varphi_0,\varphi_1,h_2)),$$

$$R_2 = \varphi_2(\pi_1 U(0))(U_1(0,\vartheta,\xi,\varphi_0,\varphi_1,h_1), U_1(0,\vartheta,\xi,\varphi_0,\varphi_1,h_2)).$$

We define $G_1(\vartheta) : \Phi_0 \times \Phi_1 \to \mathbb{G}_1$, $G_2(\vartheta) : \Phi_0 \times \Phi_1 \times \Phi_2 \to \mathbb{G}_2$ by

$$\begin{aligned} &(G_1(\vartheta)(\varphi_0,\varphi_1))(\xi,h_1) = \pi_2 U_1(\vartheta,\vartheta,\xi,\varphi_0,\varphi_1,h_1), \\ &(G_2(\vartheta)(\varphi_0,\varphi_1,\varphi_2))(\xi,h_1,h_2) = \pi_2 U_2(\vartheta,\vartheta,\xi,\varphi_0,\varphi_1,\varphi_2,h_1,h_2) \end{aligned} \tag{21}$$

for $\vartheta > 0$, $\xi \in \operatorname{cl}\mathcal{W}_0$, $(\varphi_0,\varphi_1,\varphi_2) \in \Phi_0 \times \Phi_1 \times \Phi_2$, $h_i \in \mathfrak{U}_1^\gamma$. There are $T_2 \geq 0$ and closed $\tilde\Phi_j \subset \Phi_j$ such that $G_0(T_2)$, $G_1(T_2)(\varphi_0,\cdot)$, $G_2(T_2)(\varphi_0,\varphi_1,\cdot)$ are uniformly contractive selfmappings on $\tilde\Phi_0$, $\tilde\Phi_1$, $\tilde\Phi_2$ respectively, for $(\varphi_0,\varphi_1) \in \tilde\Phi_0 \times \tilde\Phi_1$. Because of these contraction properties, the mapping $G : \tilde\Phi_0 \times \tilde\Phi_1 \times \tilde\Phi_2 \to \tilde\Phi_0 \times \tilde\Phi_1 \times \tilde\Phi_2$ defined by

$$G(\varphi_0,\varphi_1,\varphi_2) := (G_0(T_2)(\varphi_0), G_1(T_2)(\varphi_0,\varphi_1), G_2(T_2)(\varphi_0,\varphi_1,\varphi_2))$$

for $(\varphi_0,\varphi_1,\varphi_2) \in \tilde\Phi_0 \times \tilde\Phi_1 \times \tilde\Phi_2$ has a unique fixed-point $(\varphi_0^*,\varphi_1^*,\varphi_2^*) \in \tilde\Phi_0 \times \tilde\Phi_1 \times \tilde\Phi_2$. Showing the continuity of $G_1(\cdot,\varphi_1)$, $G_2(\cdot,\cdot,\varphi_2)$ for $(\varphi_1,\varphi_2) \in \tilde\Phi_1 \times \tilde\Phi_2$, the fiber contraction principle implies the attractivity of $(\varphi_0^*,\varphi_1^*,\varphi_2^*)$, i.e., the convergence of the iterates

$$(\varphi_0^{(n)},\varphi_1^{(n)},\varphi_2^{(n)}) := G^n(\varphi_0,\varphi_1,\varphi_2)$$

to $(\varphi_0^*,\varphi_1^*,\varphi_2^*) \in \tilde\Phi_0 \times \tilde\Phi_1 \times \tilde\Phi_2$ for any $(\varphi_0,\varphi_1,\varphi_2) \in \tilde\Phi_0 \times \tilde\Phi_1 \times \tilde\Phi_2$.

Choosing $(\varphi_0,\varphi_1,\varphi_2) = (0,0,0)$ we have

$$D\varphi_0^{(n)} = \varphi_1^{(n)}, \quad D^2\varphi_0^{(n)} = \varphi_2^{(n)} \qquad (n \in \mathbb{N}).$$

This and $\varphi_0^{(n)} \to \varphi_0^*$, $\varphi_1^{(n)} \to \varphi_1^*$, $\varphi_1^{(n)} \to \varphi_1^*$ imply

$$D\varphi_0^* = \varphi_1^*, \quad D^2\varphi_0^* = \varphi_2^*,$$

i.e., the C^2-smoothness of $\operatorname{graph}(\varphi_0^*|\mathcal{W}_0)$.

For $k = 1$ the proof proceeds similar to the case $k = 2$ where we use $G : \tilde\Phi_0 \times \tilde\Phi_1 \to \tilde\Phi_0 \times \tilde\Phi_1$ defined by $G(\varphi_0,\varphi_1) := (G_0(T_2)(\varphi_0), G_1(T_2)(\varphi_0,\varphi_1))$.

In order to study (18), (19), (20) we shall develop and use comparison theorems for such boundary value problems. The main difficulties are here that the comparison problem in $\mathbb{R}^2_{\geq 0}$ will be a nonlinear one (in order to get an almost sharp gap condition) and that the differential inequality in general holds only in a part of $\mathbb{R}^2_{\geq 0}$ (because of the nonequivalence of $|\cdot|_\alpha$ and $|\cdot|_{2\alpha-\beta}$ for $\alpha > \beta$.)

2.3 Proof of Theorem 8

In order to apply Theorem 11, we have to determine numbers $\hat{r}_i$ and a suitable modification $\tilde{f}$ of f satisfying Assumption 10.

Let the assumptions of Theorem 8 be satisfied.

First let $r_i = \overline{r}_i = \infty$. In this case we can choose $\tilde{f} = f$ and $\hat{r}_1 = \infty$. Remains the choice of $\hat{r}_2 < \infty$ satisfying (17).

Because of Assumption 2, there is a constant K_0 with $|f(u)|_\beta \leq K_0$ for $u \in \mathcal{U}^\alpha$. One can show that a any $\hat{r}_2 > \Lambda_2^{-1+\alpha-\beta} K_0$ satisfies (17) since

$$\left\langle -A^{1+\beta}\pi_2 u + A^\beta \pi_2 \tilde{f}(u) | A^{2\alpha-\beta}\pi_2 u \right\rangle \leq (-\Lambda_2 \hat{r}_2 + K_0 \Lambda_2^{\alpha-\beta})\hat{r}_2 < 0$$

for any $u \in \overline{\mathcal{Q}} \cap \mathcal{U}^{1+\beta}$ with $|\pi_2 u|_\alpha \geq \hat{r}_2$. Thus Assumption 10 is satisfied. Theorem 11 implies the existence of an inertial manifold $\mathcal{M} = \text{graph}(\varphi)$ with $\varphi \in C_{\text{b}}^{1-}(\mathcal{U}_1^\alpha, \pi_2 \mathcal{U}^\alpha)$ and $\text{Lip}\,(\varphi) \leq \chi_1$.

Let $\text{graph}(\varphi')$ with $\varphi' \in C_{\text{b}}^{1-}(\mathcal{U}_1^\alpha, \mathcal{U}_2^\alpha)$ be another inertial manifold with $\text{Lip}\,(\varphi') \leq \chi_1$. Choosing $\hat{r}_2 > \max\{||\varphi||, ||\varphi'||, \Lambda_2^{-1+\alpha-\beta} K_0\}$, Theorem 11 implies $\varphi = \varphi'$. Thus Theorem 8 is proved in the case $\overline{r}_i = \infty$.

Let now $r_i < \overline{r}_i < \infty$. Let $\hat{r}_i$ with $r_i < \hat{r}_i < \overline{r}_i$ be arbitrary.

In order to construct the function $\tilde{f}$ let $b \in C^\infty(-\infty, \infty)$ be a bump function with the following properties: $b(w) = 0$ for $w \leq 0$, $b(w) = 1$ for $w \geq 1$, $Db(w) \geq 0$. We introduce $\hat{f}_i \in C^\infty(\mathcal{U}^\alpha, \mathcal{U}^\alpha)$ defined by

$$\hat{f}_i(u) := b\left(\frac{|\pi_i u|_\alpha^2 - r_i^2}{\hat{r}_i^2 - r_i^2}\right)\pi_i u \qquad (u \in \mathcal{U}^\alpha).$$

Then for any $u \in \mathcal{U}^\alpha$ we have

$$\pi_i \hat{f}_j(u) = 0 \text{ if } i \neq j,\ \hat{f}_i(u) = 0 \text{ if } |\pi_i u|_\alpha \leq r_i,\ \hat{f}_i(u) = \pi_i u \text{ if } |\pi_i u|_\alpha \geq \hat{r}_i. \quad (22)$$

Further

$$\left\langle A^\alpha \pi_i h | A^\alpha \pi_i D\hat{f}_i(u)h \right\rangle \geq 0 \qquad (u, h \in \mathcal{U}^\alpha).$$

Applying the mean value theorem to the scalar-valued function

$$\tau \mapsto \left\langle A^\alpha \pi_i h | A^\alpha \pi_i \hat{f}_i(u + \tau h) \right\rangle$$

we obtain

$$\left\langle A^\alpha \pi_i h | A^\alpha \pi_i [\hat{f}_i(u+h) - \hat{f}_i(u)] \right\rangle \geq 0 \qquad (u, h \in \mathcal{U}^\alpha)$$

and hence

$$\left\langle A^{2\alpha-\beta} \pi_i h | A^\beta \pi_i [\hat{f}_i(u+h) - \hat{f}_i(u)] \right\rangle \geq 0 \qquad (u \in \mathcal{U}^\alpha,\ h \in \mathcal{U}^{2\alpha-\beta}). \quad (23)$$

There is $\mu_1 > 0$ satisfying $-\Lambda_1 r^2 - \gamma_1 r + \mu_1 r^2 \geq 1$ for $r \in [\hat{r}_1, \overline{r}_1]$. Further let $\mu_2 := (\frac{1}{4} - \Lambda_2^{-1}\gamma_2^2 + 1)/\hat{r}_2$.

Now we are in position to introduce $\tilde{f} : \mathcal{U}^\alpha \to \mathcal{U}^\beta$ defined by

$$\tilde{f}(u) := f(u) + \mu_1 \hat{f}_1(u) - \mu_2 \hat{f}_2(u) \qquad (u \in \mathcal{U}^\alpha)$$

satisfying Assumption 10: The inequalities (16) follows from (23) and (4). For any $u \in \overline{\mathcal{Q}} \cap \mathcal{U}^{1+\beta}$ with $|\pi_1 u|_\alpha \geq \hat{r}_1$, we have

$$\left\langle -A^{1+\beta}\pi_1 u + A^\beta \pi_1 \tilde{f}(u) | A^{2\alpha-\beta}\pi_1 u \right\rangle \geq 1$$

by choice of μ_1 and hence the first inequality in (17). Further one can show

$$\left\langle -A^{1+\beta}\pi_2 u + A^\beta \pi_2 \tilde{f}(u) | A^{2\alpha-\beta}\pi_2 u \right\rangle \leq -1$$

for any $u \in \overline{\mathcal{Q}} \cap \mathcal{U}^{1+\beta}$ with $|\pi_2 u|_\alpha \geq \hat{r}_2$. Thus the second inequality in (17) is satisfied, too.

Applying Theorem 11 to the prepared evolution equation (15) we get an overflowing invariant manifold $\overline{\mathcal{M}}^* = \operatorname{graph}(\varphi^*)$ with the properties stated in this theorem. Let $\varphi := \varphi^* | \mathcal{B}_{\mathcal{U}_1^\alpha}(r_1)$ and $\mathcal{M} := \operatorname{graph}(\varphi)$. Then $\operatorname{Lip}(\varphi) \leq \chi_1$. Because of (22), we have

$$\tilde{f}(u) = f(u) \qquad (u \in \mathcal{Q}).$$

Therefore, the manifold $\mathcal{M} \cap \mathcal{Q}$ is locally positively invariant with respect to (1). Let $u_0 \in \mathcal{Q}$ with $\tilde{S}(t)u_0 = S(t)u_0 \in \mathcal{Q}$ for $t \geq 0$. By means of Theorem 11 there is $\tilde{u}_0 \in \mathcal{M}^*$ with $\tilde{S}(t)\tilde{u}_0 \in \hat{\mathcal{Q}}$ and

$$|\pi_i[\tilde{S}(t)u_0 - \tilde{S}(t)\tilde{u}_0]|_\alpha \leq |\pi_2 u_0 - \varphi^*(\pi_1 u_0)|_\alpha \psi_2^i(t)$$

for $t \geq 0$. Thus

$$\begin{aligned}
\operatorname{dist}(S(t)u_0, \mathcal{M}) &\leq |\pi_2 S(t)u_0 - \varphi(\pi_1 S(t)u_0)|_\alpha \\
&\leq |\pi_2 S(t)u_0 - \pi_2 \tilde{S}(t)\tilde{u}_0|_\alpha + |\varphi^*(\pi_1 \tilde{S}(t)\tilde{u}_0) - \varphi^*(\pi_1 S(t)u_0)|_\alpha \\
&\leq |\pi_2 u_0 - \varphi^*(\pi_1 u_0)|_\alpha (\chi_1 \psi_2^1(t) + \psi_2^2(t))
\end{aligned}$$

for $t \geq 0$ such that (11) follows.

If $\underline{\mathcal{Q}} = \mathcal{Q} = \mathcal{U}^\alpha$, the exponential attracting property follows directly from Theorem 11.

Let $\underline{\mathcal{Q}} \subset \mathcal{Q}$ have positive distance to $\partial\mathcal{Q}$ and let u_0 satisfy $\tilde{S}(t)u_0 = S(t)u_0 \in \underline{\mathcal{Q}}$ for $t \geq 0$. By means of Theorem 11 there is $\tilde{u}_0$ with $\tilde{S}(t)\tilde{u}_0 \in \mathcal{M}^*$ for $t \geq 0$ and

$$|\pi_i[S(t)u_0 - \tilde{S}(t)\tilde{u}_0]|_\alpha \leq |\pi_2 u_0 - \varphi^*(\pi_1 u_0)|_\alpha \psi_2^i(t)$$

for $t \geq 0$. Using these inequalities the existence of $T \geq 0$ follows with $\tilde{S}(t)\tilde{u}_0 \in \mathcal{Q}$ for $t \geq T$. Let $\hat{u}_0 := \tilde{S}(T)\tilde{u}_0$. Then $S(t)\hat{u}_0 \in \mathcal{M} \cap \mathcal{Q}$ for $t \geq 0$ and the inequalities (12) follow.

The smoothness properties of φ follow directly from Theorem 11. Thus Theorem 8 is proved. □

3 Some Comparison Theorems for Two-point Boundary Value Differential Inequalities

Let the assumptions of Theorem 11 be satisfied.

The following Lemmas 12, 13, 14 give a connection between solutions or the difference of solutions of the boundary value problems (18), (19), (20) and a solution $v \in C([0,\vartheta], \mathbb{R}^2_{\geq 0}$ of the boundary value differential inequality

$$\begin{aligned} &\dot{v}^1(t) \geq (-\Lambda_1 - \varrho)v^1(t) - \gamma_1|v(t)| - A_1 \text{ for a.e. } t \in [0,\vartheta], \\ &\dot{v}^2(t) \leq (-\Lambda_2 - \varrho)v^2(t) + \gamma_2|v(t)| + A_2 \text{ if } v(t) \in \mathcal{V}_+(\varrho, A_2), \\ &v^1(\vartheta) \leq B_1, \quad v^2(0) \leq \chi_1 v^1(0) + B_2 \end{aligned} \tag{24}$$

where A_1, A_2, B_1, B_2 are nonnegative numbers, $\varrho > -\Lambda_2$, and

$$\mathcal{V}_+(\varrho, A_2) := \{v \in \mathbb{R}^2_{\geq 0} \, : \, -2(-\Lambda_2 - \varrho)v^2 > \gamma_2|v| + A_2\}.$$

For a compact time interval $\mathcal{T}$ let $\min \mathcal{T}$ ($\max \mathcal{T}$) denote the lower (upper) boundary point of $\mathcal{T}$.

The main goal of this section is to develop Theorem 16 and 17 for the comparison of solutions $v \in C([0,\vartheta], \mathbb{R}^2_{\geq 0})$ of (24) with solutions $w \in C([0,\vartheta], \mathbb{R}^2_{\geq 0})$ of the boundary value problem

$$\begin{aligned} &\dot{w}^1(t) = (-\Lambda_1 - \varrho)w^1(t) - \gamma_1|w(t)| - a_1, \\ &\dot{w}^2(t) = (-\Lambda_2 - \varrho)w^2(t) + \gamma_2|w(t)| + a_2, \end{aligned} \qquad (t \in \mathcal{T}), \tag{25a}$$

$$w^1(\max \mathcal{T}) = b_1, \quad w^2(\min \mathcal{T}) = \chi_1 w^1(\min \mathcal{T}) + b_2 \tag{25b}$$

where $a_i = A_i$, $b_i = B_i$, $\mathcal{T} = [0,\vartheta]$, or with solutions $\hat{w} \in C([0,\vartheta], \mathbb{R}^2_{\geq 0})$ of the boundary value differential inequality

$$\begin{aligned} &\dot{w}^1(t) \leq (-\Lambda_1 - \varrho)w^1(t) - \gamma_1|w(t)| - a_1, \\ &\dot{w}^2(t) \geq (-\Lambda_2 - \varrho)w^2(t) + \gamma_2|w(t)| + a_2 \qquad (t \in \mathcal{T}), \\ &w^1(\max \mathcal{T}) \geq b_1, \quad w^2(\min \mathcal{T}) \geq \chi_1 w^1(\min \mathcal{T}) + b_2 \end{aligned} \tag{26}$$

where $a_i = A_i$, $b_i = B_i$, $\varrho \in \mathbb{R}$, $\mathcal{T} = [0,\vartheta]$. In an intermediate step we shall compare solutions $v \in C(\mathcal{T}, \mathbb{R}^2_{\geq 0})$ of

$$\begin{aligned} &\dot{v}^1(t) \geq (-\Lambda_1 - \varrho)v^1(t) - \gamma_1|v(t)| - a_1, \\ &\dot{v}^2(t) \leq (-\Lambda_2 - \varrho)v^2(t) + \gamma_2|v(t)| + a_2 \qquad (t \in \operatorname{int} \mathcal{T}), \\ &v^1(\max \mathcal{T}) \leq b_1, \quad v^2(\min \mathcal{T}) \leq \chi_1 v^1(\min \mathcal{T}) + b_2. \end{aligned} \tag{27}$$

with solutions w and $\hat{w}$ of (25) or (26), respectively.

Lemma 12. *Let $u_1 : [0,\vartheta] \to \overline{\mathfrak{Q}}$ be a solution of the boundary value problem* (18) *and let $u_2 : [0,\vartheta] \to \overline{\mathfrak{Q}}$ be a solution of* (18a)*. Then $v \in C([0,\vartheta],\mathbb{R}^2_{\geq 0})$ defined by $v(t) = (|\pi_1[u_1(t) - u_2(t)]|_\alpha, |\pi_2[u_1(t) - u_2(t)]|_\alpha)$ satisfies the boundary differential inequality* (24) *with $\varrho = 0$, $A_1 = A_2 = 0$, $B_1 \geq |\pi_1 u_2(\vartheta) - \xi|_\alpha$, $B_2 \geq |\pi_2 u_2(0) - \varphi_0(\pi_1 u_2(0))|_\alpha$.*

Proof. 1. First we want to show

$$\begin{aligned} &\left\langle -A^{1+\beta}\pi_1[u_1 - u_2] + A^\beta \pi_1[\tilde{f}(u_1) - \tilde{f}(u_2)] | A^{2\alpha-\beta}\pi_1[u_1 - u_2]\right\rangle \\ &\qquad \geq -\Lambda_1 |\pi_1[u_1 - u_2]|_\alpha^2 - \gamma_1 |u_1 - u_2|_\alpha |\pi_1[u_1 - u_2]|_\alpha \end{aligned} \tag{28}$$

for any u_1, $u_2 \in \overline{\mathfrak{Q}} \cap \mathfrak{U}^{1+\beta}$. Moreover, we will show

$$\begin{aligned} &\left\langle -A^{1+\beta}\pi_2[u_1 - u_2] + A^\beta \pi_2[\tilde{f}(u_1) - \tilde{f}(u_2)] | A^{2\alpha-\beta}\pi_2[u_1 - u_2]\right\rangle \\ &\qquad \leq -\Lambda_2 |\pi_2[u_1 - u_2]|_\alpha^2 + \gamma_2 |u_1 - u_2|_\alpha |\pi_2[u_1 - u_2]|_\alpha \end{aligned} \tag{29}$$

for any u_1, $u_2 \in \overline{\mathfrak{Q}} \cap \mathfrak{U}^{1+\beta}$ with

$$(|\pi_1[u_1 - u_2]|_\alpha, |\pi_2[u_1 - u_2]|_\alpha) \in \mathcal{V}_+(0,0). \tag{30}$$

Let u_1, $u_2 \in \overline{\mathfrak{Q}} \cap \mathfrak{U}^{1+\beta}$ be arbitrary. For shortness let $u_\Delta := u_1 - u_2$. Inequality (28) follows directly from (16) and $|\pi_1 u_\Delta|_{2\alpha-\beta} \leq \Lambda_1^{\alpha-\beta} |\pi_1 u_\Delta|_\alpha$.

Further (16) implies

$$\begin{aligned} &(-A^{1+\beta}\pi_2 u_\Delta + A^\beta \pi_2[\tilde{f}(u_1) - \tilde{f}(u_2)] | A^{2\alpha-\beta}\pi_2 u_\Delta) \\ &\qquad \leq -\Lambda_2^{1-2\alpha+2\beta} |\pi_2 u_\Delta|_{2\alpha-\beta}^2 + \gamma_2 \Lambda_2^{\beta-\alpha} |u_\Delta|_\alpha |\pi_2 u_\Delta|_{2\alpha-\beta}. \end{aligned}$$

Thus (29) is shown if $\alpha = \beta$.

Let now $\alpha > \beta$. Since

$$\tau \mapsto -\Lambda_2^{1-2\alpha+2\beta}\tau^2 + \gamma_2 \Lambda_2^{\beta-\alpha} |u_\Delta|_\alpha \tau$$

is monotonously decreasing for $\tau \geq \frac{1}{2}\Lambda_2^{-1+\alpha-\beta}\gamma_2 |u_\Delta|_\alpha$, we can use the estimate

$$|\pi_2 u_\Delta|_{2\alpha-\beta} \geq \Lambda_2^{\alpha-\beta} |\pi_2 u_\Delta|_\alpha$$

in order to get (29) if

$$\Lambda_2^{\alpha-\beta} |\pi_2 u_\Delta|_\alpha \geq \frac{1}{2}\Lambda_2^{-1+\alpha-\beta}\gamma_2 |u_\Delta|_\alpha$$

i.e. if (30) holds.

2. Now let u_1, u_2 be solutions of (18a) with the properties as required in the lemma and let v as defined in the lemma. Futher let $u_\Delta = u_1 - u_2$. Then

$$\begin{aligned} v^i(t)\dot{v}^i(t) &= \tfrac{1}{2}\tfrac{d}{dt}|\pi_i u_\Delta|_\alpha^2 \\ &= \left\langle -A^{1+\beta}\pi_i u_\Delta + A^\beta \pi_i[\tilde{f}(u_1(t)) - \tilde{f}(u_2(t))] | A^{2\alpha-\beta}\pi_i u_\Delta \right\rangle. \end{aligned}$$

Using (28), (29) we find $v^1(t)\dot v^1(t) \ge -\Lambda_1(v^1(t))^2 - \gamma_1|v(t)|v^1(t)$ for a.e. $t > 0$ and $\dot v^2(t) \le -\Lambda_2 v^2(t) + \gamma_2|v(t)|$ for $t > 0$ with $v(t) \in \mathcal{V}_+(0,0)$.

3. We have $v^1(\vartheta) = |\pi_1 u_2(\vartheta) - \xi|_\alpha \le B_1$ and

$$\begin{aligned} v^2(0) &= |[\varphi_0(\pi_1 u_1(0)) - \varphi_0(\pi_1 u_2(0))] + [\varphi_0(\pi_1 u_2(0)) - \pi_2 u_2(0)]|_\alpha \\ &\le \chi_1|\pi_1[u_1(0) - u_2(0)]|_\alpha + |\varphi_0(\pi_1 u_2(0)) - \pi_2 u_2(0)|_\alpha \\ &\le \chi_1 v^1(0) + B_2. \end{aligned}$$

Thus Lemma 12 is proved. □

Similarly to Lemma 12 one can prove the following two lemmas.

Lemma 13. *Let $k \ge 1$ and let $U \in C([0,\vartheta], \overline{\mathcal{Q}})$. Let $u : [0,\vartheta] \to \mathcal{U}^\alpha$ be a solution of (19) on $[0,\vartheta]$ with $\varphi_1 \in \Phi_1$, $h_1 \in \mathcal{U}_1^\alpha$. Then $v \in C([0,\vartheta], \mathbb{R}^2_{\ge 0})$ defined by $v(t) = (|\pi_1 u(t)|_\alpha, |\pi_2 u(t)|_\alpha)$ satisfies (24) with $\varrho = 0$, $\Lambda_1 = \Lambda_2 = 0$, $B_1 \ge |h_1|_\alpha$, $B_2 = 0$.*

Lemma 14. *Let $k \ge 1$ and let $U \in C([0,\vartheta], \overline{\mathcal{Q}})$. Let $u : [0,\vartheta] \to \mathcal{U}^\alpha$ be a solution of (20) on $[0,\vartheta]$ with $\varphi_1 \in \Phi_1$.*

If $R_1 = 0$ then $v(t) = (|\pi_1 u(t)|_\alpha, |\pi_2 u(t)|_\alpha)$ satisfies (24) with $\Lambda_i = \varrho = 0$, $B_1 = 0$, $B_2 \ge |R_2|_\alpha$.

If $|R_1(t)| \le K \mathrm{e}^{\tilde\varrho(t-\vartheta)}$, $|R_2| \le K \mathrm{e}^{-\tilde\varrho\vartheta}$, $K > 0$ then $v \in C([0,\vartheta], \mathbb{R}^2_{\ge 0})$ defined by $v(t) = K^{-1}\mathrm{e}^{-\tilde\varrho(t-\vartheta)}(|\pi_1 u(t)|_\alpha, |\pi_2 u(t)|_\alpha)$ satisfies (24) with $\varrho = \tilde\varrho$, $A_1 = \Lambda_1^{\alpha-\beta}$, $A_2 = \Lambda_2^{\alpha-\beta}$, $B_1 = 0$, $B_2 = 1$.

Now let a_i, b_i nonnegative numbers, $\varrho \in \mathbb{R}$ and let $\mathcal{T}$ be a compact time interval. We introduce the cone $\mathcal{K}_\mathcal{T} := C(\mathcal{T}, \mathbb{R}^2_{\ge 0})$ in the Banach space $C(\mathcal{T}, \mathbb{R}^2)$ equipped with the norm $||\cdot||$ defined by

$$||w|| := \max_{t\in\mathcal{T}} |w(t)|.$$

For v_1 and v_2 belonging to $\mathcal{K}_\mathcal{T}$ we say $v_1 \le v_2$ if and only if $v_2 - v_1 \in \mathcal{K}_\mathcal{T}$. We say $v_1 \ll v_2$ if $v_2 - v_1$ belongs to the interior of $\mathcal{K}_\mathcal{T}$. If $v_1 \le v_2$ and $v_1 \ne v_2$ then we say $v_1 < v_2$. Note that $\mathcal{K}_\mathcal{T}$ is a closed and normal cone. Here the *normality* of the cone means the *semi-monotony* of the norm, i.e. there is a number M such that $||v_1|| \le M||v_2||$ for any $v_1, v_2 \in \mathcal{K}_\mathcal{T}$ with $v_1 \le v_2$.

We introduce the nonlinear but homogene, isotone and completely continuous integral operator $L_{\mathcal{T},\varrho} : \mathcal{K}_\mathcal{T} \to \mathcal{K}_\mathcal{T}$ defined by

$$\begin{aligned} (L^1_{\mathcal{T},\varrho} w)(t) &:= \int_t^{\max\mathcal{T}} \mathrm{e}^{(t-\tau)(-\Lambda_1-\varrho)}\gamma_1|w(\tau)|\,d\tau \\ (L^2_{\mathcal{T},\varrho} w)(t) &:= \int_{\min\mathcal{T}}^{t} \mathrm{e}^{(t-\tau)(-\Lambda_2-\varrho)}\gamma_2|w(\tau)|\,d\tau + \mathrm{e}^{(t-\min\mathcal{T})(-\Lambda_2-\varrho)}\chi_1 w^1(\min\mathcal{T}) \end{aligned}$$

for $w \in \mathcal{K}_{\mathcal{T}}$, and the function $q(\mathcal{T}, \varrho, a_1, a_2, b_1, b_2) \in \mathcal{K}_{\mathcal{T}}$ defined by

$$(q^1(\mathcal{T}, \varrho, a_1, a_2, b_1, b_2))(t) := \int_t^{\max \mathcal{T}} \mathrm{e}^{(t-\tau)(-\Lambda_1-\varrho)} a_1 \, d\tau + \mathrm{e}^{(t-\max \mathcal{T})(-\Lambda_1-\varrho)} b_1,$$
$$(q^2(\mathcal{T}, \varrho, a_1, a_2, b_1, b_2))(t) := \int_{\min \mathcal{T}}^{t} \mathrm{e}^{(t-\tau)(-\Lambda_2-\varrho)} a_2 \, d\tau + \mathrm{e}^{(t-\min \mathcal{T})(-\Lambda_2-\varrho)} b_2$$

for $t \in \mathcal{T}$. Then the fixed-point problem

$$L_{\mathcal{T},\varrho} w + q(\mathcal{T}, \varrho, a_1, a_2, b_1, b_2) = w \qquad (w \in \mathcal{K}_{\mathcal{T}}) \tag{31}$$

is equivalent to the two-point boundary value problem (25) in $\mathcal{K}_{\mathcal{T}}$. A function $v \in \mathcal{K}_{\mathcal{T}}$ is called *lower solution of* (31) if $v \leq L_{\mathcal{T},\varrho} v + q(\mathcal{T}, \varrho, a_1, a_2, b_1, b_2)$. Analogously, a function $v \in \mathcal{K}_{\mathcal{T}}$ is called *upper solution of* (31) if $L_{\mathcal{T},\varrho} v + q(\mathcal{T}, \varrho, a_1, a_2, b_1, b_2) \leq v$. One can show that a solution $\hat{w} \in \mathcal{K}_{\mathcal{T}}$ of (26) is an upper solution of (31) and that a solution $v \in \mathcal{K}_{\mathcal{T}}$ of (27) is a lower solution of (31).

Lemma 15. *Let $a_i \geq 0$, $b_i \geq 0$, $\varrho \in \mathbb{R}$ and let $\mathcal{T}$ be a compact time interval. If $v \in \mathcal{K}_{\mathcal{T}}$ is a solution of (27) then*

$$v \leq w^* \leq \hat{w},$$

where $w^* \in \mathcal{K}_{\mathcal{T}}$ is the unique solution of (25) in $\mathcal{K}_{\mathcal{T}}$ and $\hat{w} \in \mathcal{K}_{\mathcal{T}}$ is a solution of (26).

Proof. Let v be a solution of (27), i.e. a lower solution of (31).

1. We show that there is a solution w^* of (31) with $v \leq w^*$. For it we introduce $w_0 \in \operatorname{int} \mathcal{K}_{\mathcal{T}}$ defined by

$$w_0(t) = \mathrm{e}^{-\varrho(t-\min \mathcal{T})} \psi_2(t - \min \mathcal{T}) \qquad (t \in \mathcal{T}).$$

Note that w_0 is a solution of (25a) with $a_i = b_i = 0$ and $\operatorname{graph}(w_0) \subset \Psi_2$. Since $w_0^1(\max \mathcal{T}) > 0$ and $w_0^2(\min \mathcal{T}) = \chi_2 > \chi_1 = \chi_1 w_0^1(\min \mathcal{T})$ we have

$$q_0 := L_{\mathcal{T},\varrho} w_0 - w_0 \gg 0.$$

There is $\eta > 0$ with $v \leq \eta w_0$ and $q \leq \eta q_0$. Setting $\bar{w} := \eta w_0$ we have $\bar{w} \in \mathcal{K}_{\mathcal{T}}$ and

$$v \leq \bar{w}, \quad L_{\mathcal{T},\varrho} \bar{w} + q \leq \bar{w}.$$

For shortness let $\tilde{L} : \mathcal{K}_{\mathcal{T}} \to \mathcal{K}_{\mathcal{T}}$ be defined by $\tilde{L} w := L_{\mathcal{T},\varrho} w + q$ for $w \in \mathcal{K}_{\mathcal{T}}$. Since v is a lower solution of (31), the isotony of $L_{\mathcal{T},\varrho}$ implies

$$v \leq \tilde{L} v \leq \tilde{L} \bar{w} \leq \bar{w}.$$

The sequence $(\tilde{L}^k \bar{w})_{k \in \mathbb{N}}$ is monotone decreasing in the normal cone $\mathcal{K}_{\mathcal{T}}$. Using [EL75, Theorem 3.1] we get the convergence of $(\tilde{L}^k \bar{w})_{k \in \mathbb{N}}$ to a solution $w^* \in \mathcal{K}_{\mathcal{T}}$ of (31). Since $v \leq \bar{w}$ we have $v \leq w^*$, too.

2. Now we show the uniqueness of w^*. Assume there are two different solutions w_1, w_2 of (31). Then w_1 and w_2 are lower solutions of (31). Proceeding as above we get the existence of a third solution w_3 of (31) with $w_1 \leq w_3$, $w_2 \leq w_3$. Therefore, without loss of generality, we can assume $w_1 < w_2$.

Let $w_\Delta = w_2 - w_1$. Then $w_\Delta > 0$. Since $w_\Delta = L_{\mathcal{T},\varrho} w_2 - L_{\mathcal{T},\varrho} w_1 \leq L_{\mathcal{T}m,\varrho} w_\Delta$, w_Δ is a lower solution of

$$w = L_{\mathcal{T},\varrho} w. \tag{32}$$

Thus is a solution $\tilde{w}$ of (32) such that $w_\Delta \leq \tilde{w}$.

Since (32) is equivalent to the boundary value problem (25) with $a_i = b_i = 0$, $\tilde{w}$ is a solution of (25) with $a_i = b_i = 0$. Since $\tilde{w}(\min \mathcal{T})$ belongs to the invariant set Ψ_1, the point $\tilde{w}(\max \mathcal{T})$ belongs to Ψ_1, too. Since $\tilde{w}^1(\max \mathcal{T}) = 0$, this inclusion implies $\tilde{w}(\max \mathcal{T}) = 0$. By uniqueness of the solutions of the corresponding initial value problem, we have $\tilde{w} = 0$ such that the contradiction $w_\Delta = 0$ follows. Thus w^* is the unique solution of (31) and hence of (25) in $\mathcal{K}_\mathcal{T}$.

3. Let $\hat{w} \in \mathcal{K}_\mathcal{T}$ be a solution of (26), i.e. let $\hat{w}$ be an upper solution of (31). Since $(\tilde{L}^k \hat{w})_{k \in \mathbb{N}}$ is monotonously decreasing and converging to a solution of (31), the inequality $w^* \leq \hat{w}$ follows from the uniqueness of w^*. □

Theorem 16. *Let $v \in \mathcal{K}_{[0,\vartheta]}$ satisfy (24) with $A_i = \varrho = 0$. Then*

$$v \leq w \leq \hat{w} \tag{33}$$

where $w \in \mathcal{K}_{[0,\vartheta]}$ is the solution of (25) and $\hat{w} \in \mathcal{K}_{[0,\vartheta]}$ is a solution of (26) with $a_i = A_i$, $b_i = B_i$, $\mathcal{T} = [0, \vartheta]$.

Proof. First we note that the existence and uniqueness of w as well as $w \leq \hat{w}$ follow from Lemma 15 with $\mathcal{T} = [0, \vartheta]$, ϱ_0, $a_i = 0$, $b_i = B_i$ since $\hat{w}$ is an upper solution of (31).

Studying the phase portrait of (8) we find

$$-\Lambda_2 w^2 + \gamma_2 |w| < 0 \text{ if } w^2 > \chi_1 w^1, \; w \in \mathbb{R}^2_{\geq 0}.$$

Hence $\tilde{\mathcal{V}}_+ := \{v \in \mathbb{R}^2_{\geq 0} \,:\, v^2 > \chi_1 v^1\} \subset \mathcal{V}_+(0,0)$. Further we introduce $\tilde{\mathcal{V}}_- := \{v \in \mathbb{R}^2_{\geq 0} \,:\, v^2 < \chi_1 v^1\}$.

1. Assume $v(t) \in \operatorname{cl} \tilde{\mathcal{V}}_+$ for $t \in [0, \vartheta]$. Then v is a solution of (27) with $\mathcal{T} = [0, \vartheta]$, $\varrho = 0$, $a_i = 0$, $b_i = B_i$. The claim of the theorem follows directly from Lemma 15.

2. Assume now there is a $t \in [0, \vartheta]$ with $v(t) \in \tilde{\mathcal{V}}_-$. We want show that there is a $\vartheta_1 \in [0, \vartheta]$ with

$$v(t) \in \tilde{\mathcal{V}}_+ \text{ for } t \in [0, \vartheta_1[, \quad v(t) \in \operatorname{cl} \tilde{\mathcal{V}}_- \text{ for } t \in [\vartheta_1, \vartheta]. \tag{34}$$

Let ϑ_1 be the first time point with $v^2(\vartheta_1) = \chi_1 v^1(\vartheta_1)$. Assume there are $\vartheta_2 \in [\vartheta_1, \vartheta[$, $\vartheta_3 \in]\vartheta_2, \vartheta[$ with $v^2(\vartheta_2) = \chi_1 v^1(\vartheta_2)$ and $v^2(t) > \chi_1 v^1(t)$ for $t \in]\vartheta_2, \vartheta_3]$.

We set $\mathcal{T} = [\vartheta_2, \vartheta_3]$, $b_1 = v^1(\vartheta_3)$, $b_2 = 0$, $a_i = 0$. Then (27) holds with $\varrho = 0$. Applying Lemma 15 we obtain $v|[\vartheta_2, \vartheta_3] \le \bar{w}$ where $\bar{w} \in \mathcal{K}_{[\vartheta_2,\vartheta_3]}$ is the solution of (25) on $[\vartheta_2, \vartheta_3]$. Since $\bar{w}(\vartheta_2) \in \Psi_1$, we have graph$(\bar{w}) \subset \Psi_1$. Hence

$$v^2(\vartheta_3) \le \bar{w}^2(\vartheta_3) = \chi_1 \bar{w}^1(\vartheta_3) = \chi_1 v^1(\vartheta_3)$$

in contrary to the choice of ϑ_2 and ϑ_3. Therefore $v(t) \in \operatorname{cl} \tilde{\mathcal{V}}_-$ for $t \ge \vartheta_1$ and (34) is shown.

Applying Lemma 15 with $\mathcal{T} = [0, \vartheta]$, $\varrho = 0$, $b_1 = B_1 + \varepsilon$, $b_2 = B_2$, $a_i = 0$, $\varepsilon > 0$, the existence of the unique solution $w_\varepsilon \in \mathcal{K}_{[0,\vartheta]}$ of (25) follows. Moreover, Lemma 15 implies $w \le w_\varepsilon$.

Let $\delta = w_\varepsilon^1 - v^1$. Then

$$\begin{aligned} \dot{v}^1(t) &\ge -\Lambda_1 v^1(t) - \gamma_1 |v(t)| &&\ge -\Lambda_1 v^1(t) - \gamma_1 \sqrt{(v^1(t))^2 + (\chi_1 v^1(t))^2}, \\ \dot{w_\varepsilon}^1(t) &= -\Lambda_1 {w_\varepsilon}^1(t) - \gamma_1 |w_\varepsilon(t)| &&\le -\Lambda_1 {w_\varepsilon}^1(t) - \gamma_1 \sqrt{({w_\varepsilon}^1(t))^2 + (\chi_1 {w_\varepsilon}^1(t))^2} \end{aligned}$$

for a.e. $t \in [\vartheta_1, \vartheta]$ and hence

$$\dot{\delta}(t) \le -\Lambda_1 \delta(t) \text{ for a.e. } t \in [\vartheta_1, \vartheta] \text{ with } \delta(t) \ge 0.$$

Since $\varepsilon > 0$, we have $\delta(\vartheta) > 0$. Assume we do not have $\delta(t) > 0$ for any $t \in [\vartheta_1, \vartheta]$. Then there is $\tau \in [\vartheta_1, \vartheta[$ with $\delta(t) > 0$ for $t \in]\tau, \vartheta]$ and $\delta(\tau) = 0$. Thus we have $\dot{\delta}(t) \le -\Lambda_1 \delta(t)$ for a.e. $t \in [\tau, \vartheta]$ and $\delta(\tau) = 0$. This differential inequality implies $\delta(t) \le 0$ for $t \in [\tau, \vartheta]$ in contrary to $\delta(\vartheta) > 0$. Therefore $v^1(t) \le {w_\varepsilon}^1(t)$ for $t \in [\vartheta_1, \vartheta]$. Further $v^2(t) \le \chi_1 v^1(t) \le \chi_1 {w_\varepsilon}^1(t) \le {w_\varepsilon}^2(t)$ for $t \in [\vartheta_1, \vartheta]$. Hence $v|[\vartheta_1, \vartheta] \le w_\varepsilon|[\vartheta_1, \vartheta]$ in $\mathcal{K}_{[\vartheta_1,\vartheta]}$. Since $w_\varepsilon^1(\vartheta_1) \ge v^1(\vartheta_1)$ and $w_\varepsilon^2(0) \ge \chi_1 w_\varepsilon^1(0) + B_2$, we have $v|[0, \vartheta_1] \le w_\varepsilon|[0, \vartheta_1]$ in $\mathcal{K}_{[0,\vartheta]}$ such that $v \le w_\varepsilon$ in $\mathcal{K}_{[0,\vartheta]}$ follows.

Let $\tilde{w} \in \mathcal{K}_{\mathcal{T}}$ be the solution of (25) with $\mathcal{T} = [0, \vartheta]$, $\varrho = 0$, $a_i = 1$, $b_1 = 1$, $b_2 = 0$. Then $w_\varepsilon - w = \varepsilon \tilde{w}$ such that $w_\varepsilon \to w$ as $\varepsilon \to 0$. Since $w \le \hat{w}$, the inequality (33) follows. □

Theorem 17. *Let $A_1 = \lambda_N^{\alpha-\beta}$, $A_2 = \Lambda_2^{\alpha-\beta}$. Then for any $\varrho \in]\varrho_2, \varrho_1[$ we have:*

1. There is a unique nonnegative stationary point $w_0(\varrho)$ of (25a) with $a_i = A_i$.

2. There are $\hat{\chi} \in]\chi_1, \chi_2[$ and $\eta(\varrho)$ such that $\hat{w}_0(\varrho) \in \mathcal{K}_{[0,\vartheta]}$ defined by $\hat{w}_0(\varrho)(t) = (\hat{\eta}, \hat{\eta}\hat{\chi})$ is a constant solution of (26) with $a_i = A_i$, $b_1 = 0$, $b_2 = \max\{1, w_0^2(\varrho)\}$, $\mathcal{T} = [0, \vartheta]$.

3. If $v \in \mathcal{K}_{[0,\vartheta]}$ satisfies (24) with $B_1 = 0$, $B_2 = \max\{1, w_0^2(\varrho)\}$ then $v \le \hat{w}_0(\varrho)$.

Proof. Let $\varrho \in]\varrho_2, \varrho_1[$ be fixed and let $W_1(w) = (-\Lambda_1 - \varrho)w^1 - \gamma_1|w| - A_1$, $W_2(w) = (-\Lambda_2 - \varrho)w^2 + \gamma_2|w| + A_2$.

1. In order to show the existence of $w_0(\varrho)$ we note

$$W_1(0, w^2) < 0, \quad D_1 W_1(w^1, w^2) > -\Lambda_1 - \varrho_1 - \gamma_1 > 0, \quad D_2 W_1(w^1, w^2) < 0,$$

$$W_2(w^1,0) > 0, \quad D_1W_2(w^1,w^2) > 0, \quad D_2W_2(w^1,w^2) < -\Lambda_2 - \varrho_2 + \gamma_2 < 0$$

for $w > 0$. Hence there are strongly increasing functions $\hat{\Psi}_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $W_2(\eta, \hat{\Psi}_1(\eta)) = 0$ and $W_1(\hat{\Psi}_2(\eta), \eta) = 0$ for $\eta \geq 0$ and describing the isoclines $\dot{w}^1 = 0$ and $\dot{w}^2 = 0$ of (25a), respectively. Thus there is exactly one positive stationary point $w_0 = w_0(\varrho)$ of (25a). This stationary point has an unstable manifold graph$(\tilde{\Psi}_1)$ where $\Psi_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is strongly increasing function satisfying $\Psi_1(\tilde{\eta}) > \hat{\Psi}_1(\eta)$ for $\eta < w_0^1$, and $\tilde{\Psi}_1(\eta) < \hat{\Psi}_1(\eta)$ for $\eta > w_0^1$.

2. Let $B_1 = 0$, $B_2 = w_0^2(\varrho)$. We define $p_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by

$$p_1(\chi) := -\Lambda_1 - \varrho - \gamma_1\sqrt{1+\chi^2}, \quad p_2(\chi) := -\Lambda_2 - \varrho + \gamma_2\sqrt{1+\chi^{-2}}.$$

Then p_1 is strongly concave and p_2 is strongly convex with

$$p_1(\chi_1) = p_2(\chi_1) = \varrho_1 - \varrho > 0, \quad p_1(\chi_2) = p_2(\chi_2) = \varrho_2 - \varrho < 0.$$

Thus there are $\hat{\chi}_i \in \,]\chi_1, \chi_2[$ with $\hat{\chi}_2 < \hat{\chi}_1$ and $p_i(\hat{\chi}_i) = 0$. Let $\hat{\chi} \in \,]\hat{\chi}_2, \hat{\chi}_1[$ be arbitrary. Then

$$p_1(\hat{\chi}) > 0, \quad p_2(\hat{\chi}) < 0, \quad \hat{\chi} > \chi_1.$$

Therefore there is $\hat{\eta} > 0$ such that $w = (\eta, \eta\hat{\chi})$ satisfies

$$0 \leq (-\Lambda_1 - \varrho)w^1 - \gamma_1|w| - A_1,$$
$$0 \geq (-\Lambda_2 - \varrho)w^2 + \gamma_2|w| + A_2, \; w^2 \geq \chi_1 w^1 + B_2.$$

Thus $\hat{w}_0(\varrho)$ as defined in the theorem is a solution of (26) with $a_i = A_i$, $b_i = B_i$, $\mathcal{T} = [0, \vartheta]$.

3. Let $B_1 = 0$, $B_2 = w_0^2(\varrho)$. By construction we have

$$\{w \in \mathbb{R}^2_{\geq 0} \,:\, w^2 > \hat{\Psi}_1(w^1)\} \subset \mathcal{V}_+(\varrho, A_2).$$

Let

$$\tilde{\mathcal{V}} := \{w \in \mathbb{R}^2_{\geq 0} \,:\, w^1 \leq \max\{\hat{\Psi}_2(w^2), w_0^1\}\}, \quad \hat{\mathcal{V}}_+ := \{w \in \tilde{\mathcal{V}} \,:\, w^2 \geq \tilde{\Psi}_1(w^1)\}.$$

Then $\dot{v}^1(t) < 0$ if $t > 0$ and $v(t) \neq \tilde{\mathcal{V}}_+$. Further $\hat{\mathcal{V}}_+ \subset \mathcal{V}_+(\varrho, A_2)$.

Assume there is $t_1 \in [0, \vartheta[$ with $v(t) \notin \tilde{\mathcal{V}}$. Because of $v^1(\vartheta) = 0$, there is a $t_2 \in \,]t_1, \vartheta[$ with $v(t) \notin \tilde{\mathcal{V}}$ for $t \in [t_1, t_2[$ and

$$v^1(t_2) = w_0^1, \quad v^2(t_2) \leq w_0^2 \tag{35}$$

or

$$v^1(t_2) = \hat{\Psi}_2(v^2(t_2)), \quad v^2(t_2) > w_0^2. \tag{36}$$

If (35) then there is $\tau \in [t_1, t_2]$ with $v(t_2) - v(t_1) = \dot{v}(\tau)(t_2 - t_1) \leq 0$ in contrary to $v(t_2) > v(t_1)$.

If (36) then $v(t) \neq \hat{\mathcal{V}}_+$ implies the existence of $t_3 \in [t_1, t_2[$ with $v(t_3) \neq \tilde{\mathcal{V}}$ and $v(t) \in \hat{\mathcal{V}}_+$, i.e. with $\dot{v}^2(t) \le 0$ for $t \in [t_3, t_2]$. Thus there are τ_1, τ_2 with $\tau_i \in [t_3, t_2]$ and

$$v^1(t_2) - v^1(t_3) = \dot{v}^1(\tau_1)(t_2 - t_3) \ge 0,$$

$$v^2(t_2) - v^2(t_3) = \dot{v}^2(\tau_2)(t_2 - t_3) \le 0$$

which would imply $v(t_1) \in \tilde{\mathcal{V}}$ in contradiction to the choice of t_3.

Therefore $v(t) \in \tilde{\mathcal{V}}$ for $t \in [0, \vartheta]$.

Let

$$\tilde{\mathcal{V}}_+ := \hat{\mathcal{V}}_+ \cap \tilde{\mathcal{V}}, \quad \tilde{\mathcal{V}}_- := \hat{\mathcal{V}}_- \cap \tilde{\mathcal{V}}.$$

By the choice of B_2, the solution $w \in \mathcal{K}_{[0,\vartheta]}$ of (25) satisfies $w(t) \in \tilde{\mathcal{V}}_+$ for $t \in [0, \vartheta]$. Thus we can proceed as in the proof of Theorem 16 in order to infer $v \le w \le \hat{w}_0(\varrho)$. □

4 Proof of Theorem 11

4.1 Existence and Properties of $G_0(\vartheta)$

4.1.1 Uniqueness and Estimates of $U_0(\cdot, \vartheta, \xi, \varphi_0)$. For $\varphi_0 \in \Phi_0$ let

$$\mathcal{W}_\vartheta(\varphi_0) := \{\pi_1 \tilde{S}(\vartheta)(\zeta + \varphi_0(\zeta)) \,:\, \zeta \in \mathcal{W}_0\} \qquad (\vartheta \ge 0).$$

Then for any $\varphi_0 \in \Phi_0$, $\vartheta \ge 0$, $\xi \in \operatorname{cl} \mathcal{W}_\vartheta(\varphi_0)$ there is at least one solution $U_0(\cdot, \vartheta, \xi, \varphi_0)$ of the boundary value problem (18).

Our goal is to prove that for any $\varphi_0 \in \Phi_0$, $\vartheta > 0$, $\xi \in \operatorname{cl} \mathcal{W}_\vartheta(\varphi_0)$ there is at most one solution $U_0(\cdot, \vartheta, \xi, \varphi_0)$ of the boundary value problem (18) with maximal existence interval satisfying

$$U_0(t, \vartheta, \xi, \varphi) \in \overline{\mathcal{Q}} \qquad (t \in [0, \vartheta]). \tag{37}$$

Further we show some estimates which we need for $G_0(\vartheta)$.

Lemma 18. *There hold:*

1. Let u_i be solutions of (18a) with $u_i(t) \in \overline{\mathcal{Q}}$ for $t \in [0, T]$ and with $\pi_1 u_i(\vartheta_i) = \xi_i$, $\pi_2 u_i(0) = \varphi_0(\pi_1 u_i(0))$ where $\vartheta_i \in [\underline{\vartheta}, \overline{\vartheta}] \subset [0, T]$, $\varphi_0 \in \Phi_0$, $\xi_i \in \operatorname{cl} \mathcal{W}_{\vartheta_i}(\varphi_0)$. Then there is a constant K such that

$$|\pi_i[u_1(t) - u_2(t)]|_\alpha \le (K|\vartheta_1 - \vartheta_2| + |\xi_1 - \xi_2|_\alpha) \max_{\Theta \in [\underline{\vartheta}, \overline{\vartheta}]} \psi_1^i(t - \Theta) \tag{38}$$

for $t \in [0, \max\{\vartheta_1, \vartheta_2\}]$.

2. Let u_i be solutions of (18a) with $u_i(t) \in \overline{\mathcal{Q}}$ for $t \in [0,T]$ and with $\pi_1 u_i(\vartheta) = \xi$, $\pi_2 u_1(0) = \varphi_0(\pi_1 u_1(0))$, $\pi_2 u_2(0) = \varphi_0'(\pi_1 u_2(0))$ where $\vartheta \in [0,T]$, $\varphi_0, \varphi_0' \in \Phi_0$, $\xi_i \in \operatorname{cl}\mathcal{W}_\vartheta(\varphi_0) \cap \operatorname{cl}\mathcal{W}_\vartheta(\varphi_0')$. Then

$$|\pi_i[u_1(t) - u_2(t)]|_\alpha \le \frac{\psi_2^i(t)}{\chi_2 - \chi_1}||\varphi_0 - \varphi_0'|| \qquad (t \in [0,\vartheta]). \tag{39}$$

3. For any $\varphi_0 \in \Phi_0$, $\vartheta > 0$, and $\xi \in \mathcal{W}_\vartheta(\varphi_0)$ there is at most one solution $U_0(\cdot,\vartheta,\xi,\varphi_0)$ of (18) satisfying (37).

Proof. 1. Let u_1, u_2 have the properties as required in the first claim. Without loss of generality we can assume $\vartheta_1 \ge \vartheta_2$. Set $\vartheta := \vartheta_1$, $\xi := \xi_1$.

We have

$$|\pi_1[u_2(\vartheta) - \xi]|_\alpha \le |\pi_1[u_2(\vartheta_1) - u_2(\vartheta_2)]|_\alpha + |\pi_1 u_2(\vartheta_2) - \xi|_\alpha.$$

Since $-A\pi_1$ is a linear, bounded operator and since $\tilde{f}$ maps bounded sets into bounded sets, there is a constant K with

$$K \ge |-A\pi_1 u + \pi_1 f(u)|_\alpha \qquad (u \in \overline{\mathcal{Q}}).$$

Because of $|\pi_1[u_2(\vartheta_1) - u_2(\vartheta_2)]|_\alpha \le K|\vartheta_1 - \vartheta_2|$, $\pi_1 u_2(\vartheta_2) = \xi_2$, the estimate

$$|\pi_1[u_2(\vartheta) - \xi]|_\alpha \le B_1$$

follows where $B_1 := K|\vartheta_1 - \vartheta_2| + |\xi_1 - \xi_2|_\alpha$. Moreover $|\pi_2 u_2(0) - \varphi_0(\pi_1 u_2(0))|_\alpha = B_2 := 0$. Lemma 12 and Theorem 16 imply $|\pi_i[u_1(t) - u_2(t)]| \le w_1^i(t)$ for $t \in [0,\vartheta]$ where $w_1 \in \mathcal{K}_{[0,\vartheta]}$ defined by $w_1(t) = (M|\vartheta_1 - \vartheta_2| + |\xi_1 - \xi_2|_\alpha)\psi_1(t - \vartheta_1)$ is the solution of (25) for these values of B_1 and B_2. Thus (38) follows.

2. Let u_1, u_2 have the properties as required in the second claim. We have

$$|\pi_1[u_2(\vartheta) - \xi]|_\alpha = B_1 := 0.$$

Further $|\pi_2 u_2(0) - \varphi_0(\pi_1 u_2(0))|_\alpha = |\varphi_0'(\pi_1 u_2(0)) - \varphi_0(\pi_1 u_2(0))|_\alpha \le ||\varphi_0 - \varphi_0'||_0 =: B_2$.

The function $w_2 : [0,\vartheta] \to \mathbb{R}^2_{\ge 0}$ defined by

$$w_2 := \psi_2(t)(\chi_2 - \chi_1)^{-1}||\varphi_0 - \varphi_0'||_0 \qquad (t \in [0,\vartheta])$$

is a solution of (26) for $a_i = 0$, $b_i = B_i$, $\mathcal{T} = [0,\vartheta]$. Lemma 12 and Theorem 16 imply $|\pi_i[u_1(t) - u_2(t)]| \le w_2^i(t)$ for $t \in [0,\vartheta]$, i.e. (39).

3. Let $\varphi_0 \in \Phi_0$, $\vartheta \in [0,T]$, $\xi \in \mathcal{W}_\vartheta(\varphi_0)$ be arbitrary. Assuming the existence of two different solutions of (18) satisfying (37) we obtain a contradiction to (38) with $\vartheta_1 = \vartheta_2 = \vartheta = \underline{\vartheta} = \overline{\vartheta}$, $\xi_1 = \xi_2 = \xi$. □

Because of (17), there is a number $T_* > 0$ with the following three properties

$$\begin{aligned} &\tilde{S}(t)u_0 \in \overline{\mathfrak{Q}} && (t \in [0, T_*]), \\ &|\pi_1 \tilde{S}(t)u_0|_\alpha > \hat{r}_1 && (|\pi_1 u_0|_\alpha = \hat{r}_1,\ t \in [0, T_*]) \text{ if } \hat{r}_1 < \infty, \\ &|\pi_2 \tilde{S}(t)u_0|_\alpha < \hat{r}_2 && (|\pi_2 u_0|_\alpha = \hat{r}_2,\ t \in [0, T_*]) \end{aligned} \tag{40}$$

for any $u_0 \in \operatorname{cl} \hat{\mathfrak{Q}}$.

Lemma 19. *Let $T_* > 0$ satisfy (40) and let $\varphi_0 \in \Phi_0$. Then*

$$\operatorname{cl} \mathcal{W}_0 \subset \mathcal{W}_\vartheta(\varphi_0) \qquad (\vartheta \in]0, T_*])$$

Proof. Let $\varphi_0 \in \Phi_0$, $\vartheta \in]0, T_*]$ be arbitrary. We define the continuous mapping $H : [0,1] \times \operatorname{cl} \mathcal{W}_0 \to \mathcal{U}_1^\alpha$ by

$$H(\tau\vartheta, \zeta) := \pi_1 \tilde{S}(\tau\vartheta)(\zeta + \varphi_0(\zeta)) \qquad (\zeta \in \operatorname{cl} \mathcal{W}_0).$$

By definition of T_* and $\mathcal{W}_\vartheta(\varphi_0)$ and by means of Lemma 18 there is a unique $U_0(\cdot, \vartheta, \xi, \varphi_0)$ for $\xi \in \mathcal{W}_\vartheta(\varphi_0)$. Hence we can define the inverse $H^{-1}(1, \cdot)$ of $H(1, \cdot)$ by

$$H^{-1}(1, \xi) := \pi_1 U_0(0, \vartheta, \xi, \varphi_0) \qquad (\xi \in \operatorname{cl} \mathcal{W}_\vartheta(\varphi_0)).$$

Because of (38) with $\vartheta = \vartheta_1 = \vartheta_2 = \underline{\vartheta} = \overline{\vartheta}$, $t = 0$, $\varphi_0 = \varphi_0'$, the function $H^{-1}(1, \cdot)$ is continuous, too. Thus $H(1, \cdot)$ is a homeomorphism from $\mathcal{W}_0$ onto $\mathcal{W}_\vartheta(\varphi_0)$. If $\hat{r}_1 = \infty$, i.e. $\mathcal{W}_0 = \mathcal{U}^\alpha$, the domain invariance theorem implies $\mathcal{W}_\vartheta(\varphi_0) = \mathcal{W}_0$.

If $\hat{r}_1 < \infty$ then $\mathcal{W}_0$ is an open and bounded set. Because of (40) we have

$$\{H(\tau, \xi) : \tau \in [0,1], \xi \in \partial \mathcal{W}_0\} \cap \mathcal{W}_0 = \emptyset.$$

Using an arbitrary base in the finite dimensional Banach space $\mathcal{U}_1^\alpha$, the homotopy theorem implies

$$\deg(H(1, \cdot), \mathcal{W}_0, \xi) = \deg(H(0, \cdot), \mathcal{W}_0, \xi) = \deg(I, \mathcal{W}_0, \xi) = 1$$

for any $\xi \in \mathcal{W}_0$. Thus for any $\xi \in \mathcal{W}_0$ there exists $\zeta \in \mathcal{W}_0$ with $\xi = H(1, \zeta)$. Therefore, $\operatorname{cl} \mathcal{W}_0 \subseteq \operatorname{cl} \mathcal{W}_\vartheta(\varphi_0)$. Because of (40), we have $\partial \mathcal{W}_\vartheta(\varphi_0) \cap \operatorname{cl} \mathcal{W}_0 = \emptyset$.

Since $H(1, \cdot)|\partial \mathcal{W}_0$ is a bijection from $\partial \mathcal{W}_0$ onto $\partial \mathcal{W}_\vartheta(\varphi_0)$ we have $\operatorname{cl} \mathcal{W}_0 \subset \mathcal{W}_\vartheta(\varphi_0)$. □

Lemma 20. *Let $T_* > 0$ satisfy (40). Let $U_0(\cdot, \vartheta, \xi, \varphi_0)$ be a solution of (18) satisfying $U_0(t, \vartheta, \xi, \varphi_0) \in \overline{\mathfrak{Q}}$ for $t \in [0, \vartheta]$ where $\vartheta \in [0, T_*]$, $\xi \in \mathcal{W}_\vartheta(\varphi_0)$, $\varphi_0 \in \Phi_0$. Then*

$$|\pi_2 U_0(t, \vartheta, \xi, \varphi_0)|_\alpha \leq \hat{r}_2 \qquad (t \in [0, \vartheta]).$$

Proof. The claim follows from (40) and $|\pi_2 U_0(0, \vartheta, \xi, \varphi_0)|_\alpha \leq \hat{r}_2$. □

Lemma 21. *Let $T > 0$ and let u be a solution of* (18a) *with $u(t) \in \overline{\mathcal{Q}}$ for $t \in [0,T]$. Further let $\vartheta = T$, $\xi = \pi_1 u(T) \in \mathrm{cl}\,\mathcal{W}_0$, $\varphi_0 \in \Phi_0$ and let $U_0(\cdot,\vartheta,\xi,\varphi_0)$ be a solution of* (18) *satisfying $U_0(t,\vartheta,\xi,\varphi_0) \in \overline{\mathcal{Q}}$ for $t \in [0,T]$. Then*

$$|\pi_i[u(t) - U_0(t,\vartheta,\xi,\varphi_0)]|_\alpha \le |\pi_2 u(0) - \varphi_0(\pi_1 u(0))|_\alpha \psi_2^i(t) \qquad (t \in [0,T]).$$

Proof. The proof proceeds similarly to the proof of Lemma 18. □

4.1.2 The Graph Transformation $G_0(\vartheta)$. For simplicity let $\gamma = \alpha$. For $\vartheta \ge 0$ let $\Phi_0(\vartheta)$ be the set of all $\varphi_0 \in \Phi_0$ for which $U_0(\cdot,\vartheta,\xi,\varphi_0)$ satisfies $U_0(t,\vartheta,\xi,\varphi_0) \in \overline{\mathcal{Q}}$ for any $t \in [0,\vartheta]$ and any $\xi \in \mathrm{cl}\,\mathcal{W}_0$. We define $G_0(\vartheta) : \Phi_0(\vartheta) \to \mathbb{G}$ by

$$(G_0(\vartheta)\varphi_0)(\xi) := \pi_2 U_0(\vartheta,\vartheta,\xi,\varphi_0) \qquad (\varphi_0 \in \Phi_0(\vartheta),\ \xi \in \mathrm{cl}\,\mathcal{W}_0,\ \vartheta \ge 0).$$

Further let $T_* > 0$ satisfy (40).

Lemma 22. *G_0 possesses the following properties:*

1. *$\Phi_0(\vartheta) = \Phi_0$, $G_0(\vartheta)\Phi_0 \subseteq \Phi_0$ for $\vartheta \ge 0$.*
2. *$(\vartheta,\varphi_0) \mapsto G_0(\vartheta)\varphi_0$ is continuous in (ϑ,φ_0).*
3. *There are $T_0 > 0$ and $\kappa_0(T_0) \in]0,1[$ such that*

$$||G_0(\vartheta)\varphi_0 - G_0(\vartheta)\varphi_0'||_0 \le \kappa_0(T_0)||\varphi_0 - \varphi_0'||_0 \qquad (\vartheta \ge T_0,\ \varphi_0,\varphi_0' \in \Phi_0).$$

4. *We have*

$$G_0(\vartheta_2)G_0(\vartheta_1) = G_0(\vartheta_1 + \vartheta_2) \qquad (\vartheta_i \ge 0). \tag{41}$$

Proof. 1. The first claim will be proved by induction. First we note that

$$\Phi_0(\vartheta) = \Phi_0 \qquad (\vartheta \in [0,T_*]) \tag{42}$$

follows from the definition of T_* and from the Lemmata 18 and 19.

Moreover, using Lemma 18 with $t = \vartheta_1 = \vartheta_2 = \underline{\vartheta} = \overline{\vartheta} \in [0,T_*]$, the inequality

$$|(G_0(\vartheta)\varphi_0)(\xi_1) - (G_0(\vartheta)\varphi_0)(\xi_2)|_\alpha \le \chi_1|\xi_1 - \xi_2|_\alpha \qquad (\xi_i \in \mathrm{cl}\,\mathcal{W}_0,\ \varphi_0 \in \Phi_0)$$

follows. By means of Lemma 20 we have $|(G_0(\vartheta)\varphi_0)(\xi)|_\alpha \le \hat{r}_2$ for any $\xi \in \mathrm{cl}\,\mathcal{W}_0$, $\varphi_0 \in \Phi_0$. Therefore $G_0(\vartheta)\Phi_0 \subseteq \Phi_0$ for $\vartheta \in [0,T_*]$.

Let now

$$\Phi_0(\vartheta) = \Phi_0, \quad G_0(\vartheta)\Phi_0 \subseteq \Phi_0 \qquad (\vartheta \in [0,mT_*]) \tag{43}$$

for $m = m_0 \in \mathbb{N}$. We want show that (43) holds for $m = m_0 + 1$, too.

Let $\varphi_0 \in \Phi_0$ be arbitrary. Because of (43), we have

$$\varphi_0^{(m_0)} := G_0(m_0T_*)\varphi_0 \in \Phi_0.$$

Because of (42) for any $(\vartheta, \xi) \in [m_0 T_*, (m_0+1)T_*] \times \mathrm{cl}\, \mathcal{W}_0$, there is a unique

$$\xi_{m_0}(\vartheta, \xi, \varphi_0) := \pi_1 U_0(0, \vartheta - m_0 T_*, \xi, \varphi_0^{(m_0)}) \in \mathrm{cl}\, \mathcal{W}_0.$$

Moreover, there is a unique

$$\xi_0(\vartheta, \xi) := \pi_1 U_0(0, m_0 T_*, \xi_{m_0}(\vartheta, \xi), \varphi_0) \in \mathrm{cl}\, \mathcal{W}_0.$$

Thus $\tilde{S}(\cdot)(\xi_0(\vartheta, \xi) + \varphi_0(\xi_0(\vartheta, \xi)))$ solves (18). Since

$$\tilde{S}(t)(\xi_0(\vartheta, \xi) + \varphi_0(\xi_0(\vartheta, \xi))) \in \mathrm{cl}\,(\mathcal{B}_{\mathcal{U}_1^\alpha}(\hat{r}_1) + \mathcal{B}_{\mathcal{U}_2^\alpha}(\hat{r}_2)) \qquad (t \in [0, m_0 T_*]),$$

we have

$$\tilde{S}(t)(\xi_0(\vartheta, \xi) + \varphi_0(\xi_0(\vartheta, \xi))) \in \overline{\mathcal{Q}} \qquad (t \in [0, (m_0+1)T_*]).$$

By means of Lemma 18 we have a unique $U_0(\cdot, \vartheta, \xi, \varphi_0)$ in $\overline{\mathcal{Q}}$ and hence

$$U_0(t, \vartheta, \xi, \varphi_0) = \tilde{S}(t)(\xi_0(\vartheta, \xi) + \varphi_0(\xi_0(\vartheta, \xi))) \in \overline{\mathcal{Q}}$$

for $t \in [0, \vartheta]$, $(\vartheta, \xi) \in [m_0 T_*, (m_0+1)T_*] \times \mathrm{cl}\, \mathcal{W}_0$, too. Applying Lemma 18 once more (with $T = \vartheta = \vartheta_i = \underline{\vartheta} = \overline{\vartheta}$, $\xi_i = \xi$, $\varphi_0 = \varphi_0'$) and by means of Lemma 19, the relation $G_0(\vartheta)\varphi_0 \in \Phi_0$ follows. Thus (43) is true for $m = m_0 + 1$, too. By induction $\Phi_0(\vartheta) = \Phi_0$, $G_0(\vartheta)\Phi_0 \subseteq \Phi_0$ follow for any $\vartheta \geq 0$.

2. The continuity properties of G_0 follows from Lemma 18 with $\xi_1 = \xi_2$.

3. Let $\vartheta \geq 0$, $\varphi_0,\ \varphi_0' \in \Phi_0$. Lemma 18 implies

$$||G_0(\vartheta)\varphi_0 - G_0(\vartheta)\varphi_0||_1 \leq \kappa_0(\vartheta)||\varphi_0 - \varphi_0'||_1$$

where

$$\kappa_0(\vartheta) := \frac{\psi_2^2(\vartheta)}{\chi_2 - \chi_1} = \mathrm{e}^{\vartheta \varrho_2} \frac{\chi_2}{\chi_2 - \chi_1}.$$

Because of (7), there is a number $T_0 > 0$ such that $\kappa_0(\vartheta) \leq \kappa_0(T_0) < 1$ for any $\vartheta \geq T_0$.

4. The solution u of (18a) with initial value $\xi + (G_0(\vartheta_2)G_0(\vartheta_1)\varphi_0)(\xi)$ at $\vartheta_1 + \vartheta_2$ satisfies (18b) with $\vartheta = \vartheta_1 + \vartheta_2$. Lemma 18 implies (41). □

Lemma 23. *For any $\vartheta > 0$ there is $\varphi_0^* \in \Phi_0$ being the unique fixed-point of*

$$\varphi_0 = G_0(\vartheta)\varphi_0 \qquad (\varphi_0 \in \Phi_0).$$

Moreover, φ_0^* is independent of ϑ.

Proof. Let $\vartheta \geq T_0$. By means of the first three claims of Lemma 22 the operator $G_0(\vartheta)$ is a continuous, $\kappa_0(T_0)$-contractive self-mapping of the closed set Φ_0 in the Banach space $\mathbb{G}_0$. Thus for any $\vartheta \geq T_0$ there is a unique fixed-point $\varphi_0^*(\vartheta)$ of $G_0(\vartheta)$ in Φ_0.

Let $m \in \mathbb{N} \setminus \{0\}$, $T \geq T_0$ be arbitrary. Because of

$$G_0(T/m)\varphi_0^*(T) = G_0(T/m)G_0(T)\varphi_0^*(T) = G_0(T)(G_0(T/m)\varphi_0^*(T)$$

and the uniqueness of the fixed-point φ_0^*T of $G_0(T)$, the point $\varphi_0^*(T)$ is the unique fixed-point of $G_0(T/m)$, too. Thus for any $\vartheta \geq 0$ there is a unique fixed-point $\varphi_0^*(\vartheta)$ of $G_0(\vartheta)$ in Φ_0. Moreover

$$\varphi_0^*(\vartheta) = \varphi_0^*(\frac{k}{m}\vartheta) \qquad (\vartheta > 0,\ k, m \in \mathbb{N} \setminus \{0\}).$$

Using this property and the continuity of G_0, i.e. the continuous dependence of $\varphi_0^*(\vartheta)$ on ϑ, the independence of $\varphi_0^*(\vartheta)$ of ϑ follows. Thus $\varphi_0^* := \varphi_0^*(T_0)$ is the unique fixed-point of $G_0(\vartheta)$ in Φ_0 for any $\vartheta > 0$. □

4.2 Invariance and Exponential Tracking Properties of $\overline{\mathcal{M}}^*$

Let $\mathcal{M}^* = \mathrm{graph}(\varphi_0^*|\mathcal{W}_0)$ where φ_0^* is the function as described in Lemma 23.

Let $u_0 \in \mathcal{M}^*$ be arbitrary. There is $\tau > 0$ such that $\pi_1\tilde{S}(\vartheta)u_0 \in \mathcal{W}_0$ for $\vartheta \in [0, \tau]$. Thus there exists $U_0(\cdot, \vartheta, \pi_1\tilde{S}(\vartheta)u_0, \varphi_0^*)$ and we have

$$\varphi_0^*(\pi_1\tilde{S}(\vartheta)u_0) = (G_0(\vartheta)\varphi_0^*)(\pi_1\tilde{S}(\vartheta)u_0) = \pi_2U(\vartheta, \vartheta, \pi_1\tilde{S}(\vartheta)u_0, \varphi_0^*) = \pi_2\tilde{S}(\vartheta)u_0,$$

i.e. $\tilde{S}(\vartheta)u_0 \in \mathcal{M}^*$ for $\vartheta \in [0, \tau]$. Thus $\mathcal{M}^*$ is locally positively invariant.

Because of (17), the vector field of (15) is pointing strictly outward and is nonzero on the boundary $\partial\mathcal{M}^*$ if $\partial\mathcal{M}^* \neq \emptyset$. Thus $\overline{\mathcal{M}}^*$ is overflowing invariant.

Now we shall prove the exponential tracking property. For it let $u_0 \in \overline{\mathcal{Q}}$ with $\pi_1\tilde{S}(t)u_0 \in \mathcal{W}_0$ for $t \geq 0$. Further let $\tau > 0$. Since $\tilde{S}(m\tau)u_0 \in \mathcal{W}_0$, we may define the sequence $(\eta_m)_{m=0}^{\infty}$ by $\eta_m = \pi_1U_0(0, m\tau, \tilde{S}(m\tau)u_0, \varphi_0^*)$ for $m \in \mathbb{N}$. Note that $\eta_m \in \mathcal{W}_0$.

Applying Lemma 21 with $T = m\tau$, $\xi = \tilde{S}(m\tau)u_0$, $\varphi_0 = \varphi_0^*$, $u(t) = \tilde{S}(t)u_0$ for $[0, T]$ we get

$$|\pi_i[\tilde{S}(t)u_0 - U_0(t, m\tau, \tilde{S}(m\tau)u_0, \varphi_0^*)]|_\alpha \leq |\pi_2u_0 - \varphi_0^*(\pi_1u_0)|_\alpha\psi_2^i(t) \tag{44}$$

for $t \in [0, \vartheta_m]$. Especially we have

$$|\pi_1u_0 - \eta_m|_\alpha \leq |\pi_2u_0 - \varphi_0^*(\pi_1u_0)|_\alpha.$$

Because of the compactness of $\mathrm{cl}\,\mathcal{W}_0 \cap \{\eta \in \mathcal{U}_1^\alpha : |\pi_1u_0 - \eta|_\alpha \leq |\pi_2u_0 - \varphi_0^*(\pi_1u_0)|_\alpha\}$, there is a subsequence $(\eta_{m_j})_{j=0}^{\infty}$ converging to some $\hat{\eta} \in \mathrm{cl}\,\mathcal{W}_0$. We choose $\hat{u}(\cdot, u_0) = \tilde{S}(\cdot)(\hat{\eta} + \varphi_0^*(\hat{\eta}))$. Let $T > 0$ and $\delta > 0$ be arbitrary. Because of the continuous dependence on the initial data, there is $j_0 = j_0(\delta, T) \in \mathbb{N}$ such that $m_j\tau \geq T$ and

$$|\pi_i[\hat{u}(t, u_0) - U_0(t, m_j\tau, \tilde{S}(m_j\tau)u_0, \varphi_0^*)]|_\alpha \leq \delta|\pi_2u_0 - \varphi_0^*(\pi_1u_0)|_\alpha\psi_2^i(t)$$

for $t \in [0,T]$ and $j \geq j_0$. Combining this inequality with (44) we find

$$|\pi_i[\tilde{S}(t)u_0 - \hat{u}(t,u_0)]|_\alpha \leq (1+\delta)|\pi_2 u_0 - \varphi_0^*(\pi_1 u_0)|_\alpha \psi_2^i(t) \qquad (t \in [0,T])$$

for any $T > 0$ and any $\delta > 0$ and hence, letting $\delta \to 0$, $T \to \infty$

$$|\pi_i[\tilde{S}(t)u_0 - \hat{u}(t,u_0)]|_\alpha \leq |\pi_2 u_0 - \varphi_0^*(\pi_1 u_0)|_\alpha \psi_2^i(t) \qquad (t \geq 0).$$

Moreover, we have $\hat{u}(t,u_0) \in \overline{\mathcal{Q}}$ and $\pi_1\hat{u}(t,u_0) \in \operatorname{cl}\mathcal{W}_0$ for any $t \geq 0$. Thus $\pi_1\hat{u}(0,u_0) \in \mathcal{W}_0$ and $\hat{u}(t,u_0) \in \mathcal{M}^*$ for $t \geq 0$.

4.3 Existence and Properties of $G_1(\vartheta)$, $G_2(\vartheta)$

Let $k \geq 2$ and let $\gamma \in]\alpha, \beta+1[$ be fixed. For $\vartheta > 0$ let $\mathbb{U}_\vartheta := C([0,\vartheta], \mathcal{U}^\alpha)$ be the Banach space equipped with the norm $||\cdot||$ defined by $||u|| := \max_{t\in[0,\vartheta]} |\pi_1 u(t)|_\alpha + \max_{t\in[0,\vartheta]} |\pi_2 u(t)|_\alpha$. Let $\mathcal{F}_\vartheta$ be the open set of all continuous functions $u \in \mathbb{U}_\vartheta$ with $\pi_1 u(0) \in \mathcal{W}_0$ and $u(t) \in \overline{\mathcal{Q}}$.

For $\vartheta > 0$, $(\varphi_0, \varphi_1) \in \Phi_0 \times \Phi_1$, $U \in \mathbb{U}_\vartheta$ we introduce the integral operators $F_{\vartheta,\varphi_0}(\cdot,\cdot) : \mathcal{F}_\vartheta \times \mathcal{W}_\vartheta(\varphi_0) \to \mathbb{U}_\vartheta$ and $P(\vartheta,\varphi_1,U) : \mathbb{U}_\vartheta \to \mathbb{U}_\vartheta$ defined by

$$\begin{aligned} F_{\vartheta,\varphi_0}(u,\xi)(t) = &\int_\vartheta^t \mathrm{e}^{\pi_1 A(t-\tau)}\pi_1 \tilde{f}(u(\tau))\,d\tau + \int_0^t \mathrm{e}^{(t-\tau)\pi_2 A}\pi_2\tilde{f}(u(\tau))\,d\tau \\ &+\mathrm{e}^{(t-\vartheta)\pi_1 A}\xi + \mathrm{e}^{t\pi_2 A}\varphi_0(\pi_1 u(0)), \end{aligned}$$

$$\begin{aligned} (P(\vartheta,\varphi_1,U)u)(t) = &\int_\vartheta^t \mathrm{e}^{\pi_1 A(t-\tau)}\pi_1 D\tilde{f}(U(\tau))u(\tau)\,d\tau + \\ &+\int_0^t \mathrm{e}^{(t-\tau)\pi_2 A}\pi_2 D\tilde{f}(U(\tau))u(\tau)\,d\tau + \mathrm{e}^{t\pi_2 A}\varphi_1(\pi_1 U(0))\pi_1 u(0) \end{aligned}$$

for $(u,\xi) \in \mathcal{F}_\vartheta \times \mathcal{W}_\vartheta(\varphi_0)$, $t \in [0,\vartheta]$. Then the solution $u = U_0(\cdot,\vartheta,\xi,\varphi_0) \in \mathcal{F}_\vartheta$ of (18) is a fixed-point of $F_{\vartheta,\varphi_0}(\cdot,\xi)$ in $\mathcal{F}_\vartheta$ and inversely. Moreover, a solution $u = U_1(\cdot,\vartheta,\xi,\varphi_0,\varphi_1,h_1)$ of (19) is a solution of the fixed-point problem

$$u = P(\vartheta,\varphi_1,U_0(\cdot,\vartheta,\xi,\varphi_0))u + Q \tag{45}$$

with $Q = Q_1(\vartheta,\xi,\varphi_0,\varphi_1,h_1)$ defined by

$$Q_1(\vartheta,\xi,\varphi_0,\varphi_1,h_1) = \mathrm{e}^{(t-\vartheta)\pi_1 A}h_1,$$

and a solution $u = U_2(\cdot,\vartheta,\xi,\varphi_0,\varphi_1,\varphi_2,h_1,h_2)$ of (20) is a solution of (45) with $Q = Q_2(\vartheta,\xi,\varphi_0,\varphi_1,\varphi_2,h_1,h_2)$ defined by

$$\begin{aligned} &Q_2(\vartheta,\xi,\varphi_0,\varphi_1,\varphi_2,h_1,h_2) = \\ &\quad\int_\vartheta^t \mathrm{e}^{\pi_1 A(t-\tau)}\pi_1 D^2\tilde{f}(U_0(*))(U_1(\tau,\vartheta,\xi,\varphi_0,\varphi_1,h_1),U_1(\tau,\vartheta,\xi,\varphi_0,\varphi_1,h_2))\,d\tau \\ &\quad+\int_0^t \mathrm{e}^{(t-\tau)\pi_2 A}\pi_2 D^2\tilde{f}(U_0(*))(U_1(\tau,\vartheta,\xi,\varphi_0,\varphi_1,h_1),U_1(\tau,\vartheta,\xi,\varphi_0,\varphi_1,h_2))\,d\tau \\ &\quad+\mathrm{e}^{t\pi_2 A}\varphi_2(\pi_1 U_0(0,\vartheta,\xi,\varphi_0))(U_1(0,\vartheta,\xi,\varphi_0,\varphi_1,h_1),U_1(0,\vartheta,\xi,\varphi_0,\varphi_1,h_2)) \end{aligned}$$

where $U_0(*)$ stands for $U_0(\tau,\vartheta,\xi,\varphi_0)$.

Lemma 24. *The operator* $I - P(\vartheta, \varphi_1, U) : \mathbb{U}_\vartheta \to \mathbb{U}_\vartheta$ *is a linear homeomorphism from* $\mathbb{U}_\vartheta$ *onto itself for* $\vartheta > 0$, $\varphi_1 \in \Phi_1$, $U \in \mathcal{F}_\vartheta$.

Proof. Let $\vartheta > 0$, $\varphi_1 \in \Phi_1$, $U \in \mathcal{F}_\vartheta$ be given. Obviously $P(\vartheta, \varphi_1, U)$ is linear. Since $\gamma > \alpha$ one can show that $P(\vartheta, \varphi_1, U)$ is completely continuous. Let u be a solution of $u = P(\vartheta, \varphi_1, U)u$. Then u is a solution of (19) with $h_1 = 0$, and the Lemma 13 and Theorem 16 imply $u = 0$ since $A_1 = A_2 = B_1 = B_2 = 0$ and $w = 0$ is the solution of (25). Therefore, $I - P(\vartheta, \varphi_1, U)$ is injective. Since $P(\vartheta, \varphi_1, U)$ is completely continuous, $I - P(\vartheta, \varphi_1, U)$ is surjective, too. Thus $I - P(\vartheta, \varphi_1, U)$ is a linear, continuous bijection from Banach space $\mathbb{U}_\vartheta$ onto itself which has a continuous inverse by means of Banach's Theorem. □

Lemma 25. *Let* $k = 2$ *and let* $\vartheta > 0$, $\varphi_0 \in \Phi$, $\xi \in \mathcal{W}_\vartheta(\varphi)$. *If* φ_0 *is* C^2 *then* $U_0(\cdot, \vartheta, \xi, \varphi_0)$ *is* C^2 *in* $\xi \in \operatorname{cl} \mathcal{W}_0$. *Moreover,*

$$\begin{aligned} D_3 U_0(t, \vartheta, \xi, \varphi_0) h_1 &= U_1(t, \vartheta, \xi, \varphi_0, D\varphi_0, h_1) \\ D_{3,3} U_0(t, \vartheta, \xi, \varphi_0)(h_1, h_2) &= U_2(t, \vartheta, \xi, \varphi_0, D\varphi_0, D^2\varphi_0, h_1, h_2) \end{aligned} \tag{46}$$

for $t \in [0, \vartheta]$, $\xi \in \operatorname{cl} \mathcal{W}_0$, h_1, $h_2 \in \mathcal{U}_1^\alpha$.

Proof. Let $\vartheta > 0$ and let $\varphi_0 \in \Phi_0$ be twice continuously differentiable.

We note that F_{ϑ,φ_0} belongs to $C^2(\mathcal{F}_\vartheta \times \mathcal{W}_\vartheta(\varphi_0), \mathbb{U}_\vartheta)$. Since $D_1 F_{\vartheta,\varphi_0}(U, \xi) = P(\vartheta, \varphi_0, U)$, Lemma 24 implies that $I - D_1 F_{\vartheta,\varphi_0}(U, \xi)$ is a linear homeomorphism of $\mathbb{U}_\vartheta$ into itself. Since $U_0(\cdot, \vartheta, \xi, \varphi_0)$ solves $u - F_{\vartheta,\varphi_0}(u, \xi) = 0$ we can apply the implicit function theorem in order to conclude the C^2-smoothness of $U_0(\cdot, \vartheta, \xi, \varphi_0)$ in $\xi \in \mathcal{W}_\vartheta(\varphi_0)$. Moreover, (46) follows from the implicit function theorem. Since $\operatorname{cl} \mathcal{W}_0 \subseteq \mathcal{W}_\vartheta(\varphi_0)$, the lemma is proved. □

Similar to Lemma 22 but using some more technical estimates (since $\gamma > \alpha$) one can show

Lemma 26. *There are* $T_2 > 0$ *and closed sets* $\tilde{\Phi}_j \subseteq \Phi_j$ *with* $0 \in \tilde{\Phi}_j$ *for* $j = 0, 1, 2$ *such that:*

1. $G_0(T_2)$, $G_1(T_2)(\varphi_0, \cdot)$, $G_2(T_2)(\varphi_0, \varphi_1, \cdot)$ *are uniformly contractive on* Φ_0, Φ_1, Φ_2, *respectively, for* $(\varphi_0, \varphi_1) \in \Phi_0 \times \Phi_1$.
2. $G_0(T_2)\tilde{\Phi}_0 \subseteq \tilde{\Phi}_0$, $G_1(T_2)(\tilde{\Phi}_0 \times \tilde{\Phi}_1) \subseteq \tilde{\Phi}_1$, $G_2(T_2)(\tilde{\Phi}_0 \times \tilde{\Phi}_1 \times \tilde{\Phi}_2) \subseteq \tilde{\Phi}_2$.
3. $G_1(T_2)(\cdot, \varphi_1)$, $G_2(T_2)(\cdot, \cdot, \varphi_2)$ *are continuous for* $(\varphi_1, \varphi_2) \in \tilde{\Phi}_1 \times \tilde{\Phi}_2$.

Because of (46), we have

$$DG_0(T_2)(\varphi_0) = G_1(T_2)(\varphi_0, D\varphi_0), \quad D^2 G_0(T_2)(\varphi_0) = G_2(T_2)(\varphi_0, D\varphi_0, D^2\varphi_0)$$

for twice continuously differentiable $\varphi_0 \in \tilde{\Phi}_0$. Choosing $\varphi_0 = 0$ and applying the fiber contraction principle, the C^2 smoothness of the manifold follows.

Thus Theorem 11 is proved. □

References

[CLS92] Chow, S.-N., Lu, K., Sell, G.R.: Smoothness of inertial manifolds. Journal of Mathematical Analysis and Applications **169** (1992) 283–312

[CFNT89] Constantin, P., Foias, C., Nicolaenko, B., Témam, R.: Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations, volume 70 of Applied Mathematical Sciences. Springer 1989

[EL75] Eisenfeld, J., Lakshmikantham, V.: Comparison principle and nonlinear contractions in abstract spaces. Journal of Mathematical Analysis and Applications **49** (1975) 504–511

[FST88] Foias, C., Sell, G.R., Temam, R.: Inertial manifolds for nonlinear evolutionary equations. J. of Differential Equations **73** (1988) 309–353

[FST89] Foias, C., Sell, G.R., Titi, E.S.: Exponential tracking and approximation of inertial manifolds for dissipative nonlinear equations. J. of Dynamics and Differential Equations **1** (1989) 199–244

[Hen81] Henry, D.: Geometric Theory of Semilinear Parabolic Equations, volume 850 of Lecture Notes in Mathematics. Springer, 1981

[MPS88] Mallet-Parret, J., Sell, G.R.: Inertial manifolds for reaction diffussion equations in higher space dimension. J. Amer. Math. Soc. **1** (1988) 805–866

[Nin92] Ninomiya, H.: Some remarks on inertial manifolds. J. Math. Kyoto Univ. **32** (1992) 667–688

[Rob93] Robinson, J.C.: Inertial manifolds and the cone condition. Dyn. Syst. Appl. **2** (1993) 311–330

[Rob96] Robinson, J.C.: The asymptotic completeness of inertial manifolds. Nonlinearity **9** (1996) 1325–1340

[Rom94] Romanov, A.V.: Sharp estimates of the dimension of inertial manifolds for nonlinear parabolic equations. Russ. Acad. Sci., Izv., Math. **43** (1994) 31–47

[Tem88] Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Springer, New York, 1988

[Tem97] Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics. 2nd ed. Springer, New York, 1988

[Van89] Vanderbauwhede, A.: Centre Manifolds, Normal Forms and Elementary Bifurcations, pages 89 – 169. Dynamics Reported, Volume 2. John Wiley & Sons, 1989

[Wig94] Wiggins, S.: Normally Hyperbolic Invariant Manifolds in Dynamical Systems. Applied Mathematical Sciences 105. Springer, 1994