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Patricia J. Y. Wong

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Asymptotic Behavior of Solutions of Partial Difference Inequalities

Patricia J. Y. Wong

Division of Mathematics
Nanyang Technological University
469, Bukit Timah Road, Singapore 259756
Singapore
Email: wongjyp@nievax.nie.ac.sg

Abstract. We offer sufficient conditions for the oscillation of all solutions of the partial difference equations

$$y(m-1,n)+\beta(m,n)y(m,n-1)-\delta(m,n)y(m,n)+ \\ +P(m,n,y(m+k,n+\ell))=Q(m,n,y(m+k,n+\ell))$$

and

$$y(m-1,n)+\beta(m,n)y(m,n-1)-\delta(m,n)y(m,n)+ \\ +\sum_{i=1}^{\tau}P_i(m,n,y(m+k_i,n+\ell_i))=\sum_{i=1}^{\tau}Q_i(m,n,y(m+k_i,n+\ell_i)).$$

Several examples which dwell upon the importance of our results are also included.

AMS Subject Classification. 39A10

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1 Introduction

The theory of difference equations, the methods used in their solutions, and their wide applications have been and still are drawing numerous attention. In fact, in the last few years several monographs and hundreds of research papers have been written, e.g., see [1,2,3,4,5,6,13,14,15,16,17,19,20,22,23,24,25,26,27,28,29] and the references therein. In contrast, relatively few studies have been focused on the qualitative theory of partial difference equations, for instance, refer to [7,8,9,10,11,12,18,30,31,32,33,34]. Partial difference equations are not less important than difference equations - their significance is well illustrated in applications involving population dynamics with spatial migrations, chemical reactions, control systems, combinatorics and also finite difference schemes [14,18,21].

This is the final form of the paper.

Hence, to further the qualitative theory of partial difference equations, in this paper we shall consider the partial difference equations

$$y(m-1,n)+\beta(m,n)y(m,n-1)-\delta(m,n)y(m,n)+P(m,n,y(m+k,n+\ell)) \\ = Q(m,n,y(m+k,n+\ell)), \quad m\ge m_0,\ n\ge n_0 \qquad (1)$$

and

$$y(m-1,n)+\beta(m,n)y(m,n-1)-\delta(m,n)y(m,n)+\sum_{i=1}^{\tau}P_i(m,n,y(m+k_i,n+\ell_i)) \\ = \sum_{i=1}^{\tau}Q_i(m,n,y(m+k_i,n+\ell_i)), \quad m\ge m_0,\ n\ge n_0, \qquad (2)$$

where $k,\ell,k_i,\ell_i,\ 1\le i\le\tau$ are nonnegative integers, and $\beta(m,n),\ \delta(m,n)$ are functions of m and n such that for all large m and n,

$$\beta(m,n)\ge\beta>0 \qquad \text{and} \qquad \delta(m,n)\le\delta\ (>0).$$

It is noted that $\delta(m,n)$ is *not* required to be positive eventually.

Recently, Zhang and Liu [33] have discussed particular cases of (1) and (2), namely,

$$y(m-1,n)+y(m,n-1)-y(m,n)+a(m,n)y(m+k,n+\ell)=0 \qquad (3)$$

and

$$y(m-1,n)+y(m,n-1)-y(m,n)+\sum_{i=1}^{\tau}a_i(m,n)g_i(y(m+k_i,n+\ell_i))=0, \qquad (4)$$

where $a(m,n),\ a_i(m,n),\ 1\le i\le\tau$ are positive, and $g_i,\ 1\le i\le\tau$ are nondecreasing functions with $ug_i(u)>0$ for all $u\ne0$. Our results not only generalize and extend their work, but also complement several other oscillation criteria given in [7,8,9,10,11,12,30,31,32,34].

By a *solution* of (1) ((2)), we mean a nontrivial double sequence $\{y(m,n)\}$ satisfying (1) ((2)) for $m\ge m_0,\ n\ge n_0$. A sequence $\{y(m,n)\}$ is *eventually positive* (*negative*) if $y(m,n)>(<)\ 0$ for all large m and n. A solution of (1) ((2)) is said to be *oscillatory* if it is neither eventually positive nor negative, and *nonoscillatory* otherwise.

Throughout, with respect to equation (1) we shall assume that there exists a function $f:\mathbb{R}\to\mathbb{R}$ and double sequences $\{p(m,n)\}$, $\{p'(m,n)\}$, $\{q(m,n)\}$, $\{q'(m,n)\}$ such that

(A1) for $u\ne0$, $uf(u)>0$, $\dfrac{f(u)}{u}\ge\gamma\in(0,\infty)$;

(A2) for $u \neq 0$,

$$p(m,n) \leq \frac{P(m,n,u(m+k,n+\ell))}{f(u(m+k,n+\ell))} \leq p'(m,n),$$

$$q(m,n) \leq \frac{Q(m,n,u(m+k,n+\ell))}{f(u(m+k,n+\ell))} \leq q'(m,n); \quad \text{and}$$

(A3) $\limsup_{m,n\to\infty}[p(m,n) - q'(m,n)] > 0.$

Further, with respect to equation (2) for each $1 \leq i \leq \tau$ it is assumed that there exists a function $f_i : \mathbb{R} \to \mathbb{R}$ and double sequences $\{p_i(m,n)\}$, $\{p'_i(m,n)\}$, $\{q_i(m,n)\}$, $\{q'_i(m,n)\}$ such that

(B1) for $u \neq 0$, $uf_i(u) > 0$, $\dfrac{f_i(u)}{u} \geq \gamma_i \in (0,\infty)$;

(B2) for $u \neq 0$,

$$p_i(m,n) \leq \frac{P_i(m,n,u(m+k_i,n+\ell_i))}{f_i(u(m+k_i,n+\ell_i))} \leq p'_i(m,n),$$

$$q_i(m,n) \leq \frac{Q_i(m,n,u(m+k_i,n+\ell_i))}{f_i(u(m+k_i,n+\ell_i))} \leq q'_i(m,n); \quad \text{and}$$

(B3) $p_i(m,n) > q'_i(m,n)$ eventually.

The plan of the paper is as follows. In Section 2 we shall present some preliminary results which are needed later. The oscillation criteria for equations (1) and (2) are respectively offered in Sections 3 and 4. To illustrate the results obtained, five examples are discussed in Section 5.

2 Preliminaries

Lemma 1. *Suppose that $\{y(m,n)\}$ is an eventually positive solution of (1). Then, for all large m,n and all $i \geq 0$,*

$$y(m-1,n) \leq \delta y(m,n), \qquad y(m,n-1) \leq \frac{\delta}{\beta}\, y(m,n), \tag{5}$$

$$\left(\frac{1}{\delta}\right)^i y(m-i,n) \leq y(m,n) \leq \delta^i y(m+i,n) \tag{6}$$

and

$$\left(\frac{\beta}{\delta}\right)^i y(m,n-i) \leq y(m,n) \leq \left(\frac{\delta}{\beta}\right)^i y(m,n+i). \tag{7}$$

Remark 2. It is obvious that (5)–(7) also hold if $\{y(m,n)\}$ is an eventually positive solution of any one of the following: equation (2), or either of the inequalities

$$y(m-1,n)+\beta(m,n)y(m,n-1)-\delta(m,n)y(m,n)+P(m,n,y(m+k,n+\ell)) \le Q(m,n,y(m+k,n+\ell)), \quad (8)$$

$$y(m-1,n)+\beta(m,n)y(m,n-1)-\delta(m,n)y(m,n)+\sum_{i=1}^{\tau} P_i(m,n,y(m+k_i,n+\ell_i)) \le \sum_{i=1}^{\tau} Q_i(m,n,y(m+k_i,n+\ell_i)), \quad (9)$$

where $m \ge m_0,\ n \ge n_0$.

Throughout, we shall use the equation number $(\cdot)'$ to denote $(\cdot)$ with the inequality sign(s) reversed.

Remark 3. By a similar argument, it can be shown that $(5)'$–$(7)'$ hold if $\{y(m,n)\}$ is an eventually *negative* solution of any one of the following: (1), (2), $(8)'$ or $(9)'$.

Remark 4. Let $\{y(m,n)\}$ be an eventually positive solution of either (1), (2), (8) or (9). If $\beta \ge \delta$ and $\delta \le 1$, then (5) implies that

$$y(m-1,n) \le y(m,n) \quad \text{and} \quad y(m,n-1) \le y(m,n), \quad (10)$$

i.e., eventually positive solutions of (1), (2), (8) as well as of (9) are nondecreasing.

Remark 5. Let $\{y(m,n)\}$ be an eventually *negative* solution of either (1), (2), $(8)'$ or $(9)'$. If $\beta \ge \delta$ and $\delta \le 1$, then from $(5)'$ we get $(10)'$, i.e., eventually negative solutions of (1), (2), $(8)'$ as well as of $(9)'$ are nonincreasing.

Lemma 6. *The following identity holds*

$$\sum_{i=m-k}^{m}\sum_{j=n-\ell}^{n}[y(i-1,j)+\beta y(i,j-1)-\delta y(i,j)]$$
$$= (1+\beta-\delta)\sum_{i=m-k}^{m-1}\sum_{j=n-\ell}^{n-1} y(i,j)+\beta\sum_{i=m-k}^{m-1} y(i,n-\ell-1)+(1-\delta)\sum_{i=m-k}^{m-1} y(i,n)$$
$$+(\beta-\delta)\sum_{j=n-\ell}^{n-1} y(m,j)+\beta y(m,n-\ell-1)-\delta y(m,n)+\sum_{j=n-\ell}^{n} y(m-k-1,j).$$

3 Oscillation of equation (1)

For simplicity, we shall use the notation

$$\mu(m,n) = p(m,n) - q'(m,n).$$

Further, let

$$E = \{r > 0 \mid \delta - r\mu(m,n) > 0 \text{ eventually}\}.$$

Theorem 7. *Suppose that there exist integers $M \geq m_0$ and $N \geq n_0$ such that*

$$\sup_{r\in E, m\geq M, n\geq N} \frac{r}{\gamma\beta^\ell} \min\left\{\delta^\ell\theta^{1/\ell},\ \delta^k\theta^{1/k}\right\} < 1 \tag{11}$$

where

$$\theta = \prod_{i=1}^{k}\prod_{j=1}^{\ell}[\delta - r\mu(m+i,n+j)]. \tag{12}$$

Then,

(a) *the inequality* (8) *has no eventually positive solution;*
(b) *the inequality* (8)′ *has no eventually negative solution;*
(c) *all solutions of equation* (1) *are oscillatory.*

Corollary 8. *Suppose that $k,\ell \geq 1$ and*

$$\liminf_{m,n\to\infty} \frac{1}{k\ell}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\mu(m+i,n+j) > \frac{\delta^{k+\ell+1}}{\gamma\beta^\ell} \max\left\{\frac{k^k}{(1+k)^{1+k}},\ \frac{\ell^\ell}{(1+\ell)^{1+\ell}}\right\} = \frac{\delta^{k+\ell+1}}{\gamma\beta^\ell}\,\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}, \tag{13}$$

where $\alpha = \min\{k,\ell\}$. Then, the conclusion of Theorem 7 holds.

Theorem 9. *Suppose that there exist integers $M \geq m_0$ and $N \geq n_0$ such that if $\ell \geq k$,*

$$\sup_{r\in E, m\geq M, n\geq N} \frac{r}{\gamma\beta^\ell}\left(\frac{\delta}{2}\right)^k \prod_{i=1}^{k}[\delta - r\mu(m+i,n+i)] \times \prod_{j=k+1}^{\ell}[\delta - r\mu(m+k,n+j)] < 1; \tag{14}$$

and if $\ell < k$,

$$\sup_{r\in E, m\geq M, n\geq N} \frac{r}{\gamma\beta^{\ell}} \left(\frac{\delta}{2}\right)^{\ell} \prod_{i=1}^{\ell}[\delta - r\mu(m+i,n+i)] \times \prod_{j=\ell+1}^{k}[\delta - r\mu(m+j,n+\ell)] < 1. \quad (15)$$

Then, the conclusion of Theorem 7 holds.

Corollary 10. *Suppose that*

$$\liminf_{m,n\to\infty} \mu(m,n) = \mu > \frac{\nu^{\nu}}{\gamma\beta^{\ell}} \left(\frac{\delta}{2}\right)^{\alpha} \left(\frac{\delta}{1+\nu}\right)^{1+\nu}, \quad (16)$$

where $\alpha = \min\{k,\ell\}$ *and* $\nu = \max\{k,\ell\}$. *Then, the conclusion of Theorem 7 holds.*

Theorem 11. *Suppose that there exist integers* $M \geq m_0$ *and* $N \geq n_0$ *such that*

$$\sup_{r\in E, m\geq M, n\geq N} \frac{r}{\gamma\beta^{\ell}} \prod_{i=1}^{k}[\delta - r\mu(m+i,n)] \prod_{j=1}^{\ell}[\delta - r\mu(m+k,n+j)] < 1. \quad (17)$$

Then, the conclusion of Theorem 7 holds.

Corollary 12. *Suppose that*

$$\mu(m,n) \geq c > \frac{\delta^{k+\ell+1}}{\gamma\beta^{\ell}} \frac{(k+\ell)^{k+\ell}}{(k+\ell+1)^{k+\ell+1}}. \quad (18)$$

Then, the conclusion of Theorem 7 holds.

4 Oscillation of equation (2)

Theorem 13. *Suppose that for each* $1 \leq i \leq \tau$,

$$\liminf_{m,n\to\infty} p_i(m,n) = p_i, \qquad \liminf_{m,n\to\infty} q_i'(m,n) = q_i', \qquad p_i > q_i'; \quad (19)$$

and

$$\sum_{i=1}^{\tau}(p_i - q_i')\gamma_i \frac{\beta^{\ell_i}}{\delta^{k_i+\ell_i+1}} (\alpha_i+1)^{\alpha_i+1} \left(\frac{2}{\alpha_i}\right)^{\alpha_i} > 1, \quad (20)$$

where $\alpha_i = \min\{k_i,\ell_i\}$, $1 \leq i \leq \tau$. *Then,*

(a) *the inequality (9) has no eventually positive solution;*

(b) *the inequality* (9)$'$ *has no eventually negative solution;*
(c) *all solutions of equation* (2) *are oscillatory.*

Theorem 14. *Suppose that for each* $1 \le i \le \tau$,

$$\limsup_{m,n\to\infty} \sum_{s=1}^{\tau} \sum_{i=m-k'}^{m} \sum_{j=n-\ell'}^{n} [p_s(i,j) - q'_s(i,j)]\gamma_s \, \frac{1}{\delta^{i+k_s-m}} \left(\frac{\beta}{\delta}\right)^{j+\ell_s-n} > w, \quad (21)$$

where $k' = \min_{1\le i\le\tau} k_i$, $\ell' = \min_{1\le i\le\tau} \ell_i$, *and*

$$\begin{aligned} w &= \delta, && \beta \ge \delta,\ \delta \le 1,\\ &= \delta^{k'+1}, && \beta \ge \delta,\ \delta \ge 1,\\ &= \delta\left(\frac{\delta}{\beta}\right)^{\ell'}, && \beta \le \delta,\ \delta \le 1,\\ &= \delta\left[\left(\frac{\delta}{\beta}\right)^{\ell'} - 1 + \delta^{k'}\right], && \beta \le \delta,\ \delta \ge 1,\ \delta-\beta \le 1,\\ &= \delta\left\{\frac{\beta + (\delta-\beta-1)\delta^{k'+1}}{(\delta-1)(\delta-\beta)}\left[\left(\frac{\delta}{\beta}\right)^{\ell'} - 1\right] + \delta^{k'}\right\}, && \beta \le \delta,\ \delta \ge 1,\ \delta-\beta \ge 1. \end{aligned}$$

Then, the conclusion of Theorem 13 *holds.*

5 Examples

Example 15. Consider the partial difference equation

$$y(m-1,n) + \frac{n+1}{n}\,y(m,n-1) - \frac{n+1}{n-1}\,y(m,n) + \frac{n+4}{n}\,y(m+6,n+4) = 0, \qquad m \ge 1,\ n \ge 21. \quad (22)$$

Here, $k = 6$, $\ell = 4$,

$$\beta(m,n) = \frac{n+1}{n} \ge 1 \equiv \beta$$

and

$$\delta(m,n) = \frac{n+1}{n-1} = 1 + \frac{2}{n-1} \le 1 + \frac{2}{20} = 1.1 \equiv \delta.$$

Choosing $f(u) = u$, we have $\gamma = 1$. Further, since

$$\frac{P(m,n,y(m+6,n+4))}{f(y(m+6,n+4))} = \frac{n+4}{n}, \qquad \frac{Q(m,n,y(m+6,n+4))}{f(y(m+6,n+4))} = 0,$$

we may take

$$p(m,n) = p'(m,n) = \frac{n+4}{n}, \qquad q(m,n) = q'(m,n) = 0.$$

Thus, (A1)–(A3) are fulfilled.

Case (a) : Corollary 8

The left side of (13) is

$$\liminf_{m,n\to\infty} \frac{1}{24}\sum_{i=1}^{6}\sum_{j=1}^{4}\frac{n+j+4}{n+j}=1,$$

which is more than the right side $(=0.234)$.

Case (b) : Corollary 10

We find that

$$\liminf_{m,n\to\infty} \mu(m,n)=1>\frac{\nu^\nu}{\gamma\beta^\ell}\left(\frac{\delta}{2}\right)^\alpha\left(\frac{\delta}{1+\nu}\right)^{1+\nu}=0.0101$$

and so (16) is satisfied.

Case (c) : Corollary 12

We have

$$\mu(m,n)\ge 1\equiv c>\frac{\delta^{k+\ell+1}}{\gamma\beta^\ell}\ \frac{(k+\ell)^{k+\ell}}{(k+\ell+1)^{k+\ell+1}}=0.1.$$

Hence, (18) is fulfilled.

Case (d) : Theorem 13

Here, $\tau=1,\ p_1=1$ and $q_1'=0$. The left side of (20) is 68.5, which is more than 1.

Case (e) : Theorem 14

This is the case when $\beta\le\delta,\ \delta\ge 1,\ \delta-\beta\le 1$. We see that (21) holds as

$$\limsup_{m,n\to\infty}\sum_{i=m-6}^{m}\sum_{j=n-4}^{n}\frac{j+4}{j}\ \frac{1}{(1.1)^{i+6-m}}\ \frac{1}{(1.1)^{j+4-n}}$$
$$=\left(\sum_{i=0}^{6}\frac{1}{1.1^i}\right)\left(\sum_{i=0}^{4}\frac{1}{1.1^i}\right)=22.3>w=2.46.$$

Hence, it follows from Corollaries 8–12, Theorems 13 and 14 that equation (22) is oscillatory. In fact, (22) has an oscillatory solution given by $\{y(m,n)\}=\left\{(-1)^m\frac{1}{n}\right\}$.

Example 16. Consider the partial difference equation

$$y(m-1,n)+\frac{n}{n+1}\,y(m,n-1)-y(m,n)+\frac{n}{n+1}\,y(m+4,n+3)=0,\qquad m\ge 1,\ n\ge 1. \tag{23}$$

In this example,

$$\beta(m,n)=\frac{n}{n+1}\ge\frac{1}{2}\equiv\beta\quad\text{ and }\quad\delta(m,n)=1\equiv\delta.$$

Taking $f(u)=u$, we have $\gamma=1$. Subsequently, we may choose

$$p(m,n)=p'(m,n)=\frac{n}{n+1},\qquad q(m,n)=q'(m,n)=0.$$

Clearly, (A1)–(A3) are satisfied. Further,

$$\lim_{m,n\to\infty}\mu(m,n)=1\quad\text{ and }\quad\mu(m,n)\ge\frac{1}{2}\equiv c.$$

It can be checked that all the conditions of Corollaries 8–12, Theorems 13 and 14 (the cases $\beta\le\delta,\ \delta\le 1$ or $\beta\le\delta,\ \delta\ge 1,\ \delta-\beta\le 1$) are fulfilled. Therefore, we conclude that (23) is oscillatory. In fact, (23) has an oscillatory solution given by $\{y(m,n)\}=\{(-1)^m n\}$.

Example 17. Consider the partial difference equation

$$y(m-1,n)+\frac{n+2}{n+1}\,y(m,n-1)-\frac{1}{2}\,y(m,n)+\frac{(n-4)(n+2)}{2n(n+1)}\,y(m+2,n+1)=0,\qquad m\ge 1,\ n\ge 5. \tag{24}$$

Here,

$$\beta(m,n)=\frac{n+2}{n+1}\ge 1\equiv\beta\quad\text{ and }\quad\delta(m,n)=\frac{1}{2}\equiv\delta.$$

Choosing $f(u)=u$, we have $\gamma=1$. Let

$$p(m,n)=p'(m,n)=\frac{(n-4)(n+2)}{2n(n+1)},\qquad q(m,n)=q'(m,n)=0.$$

Then, it follows that

$$\lim_{m,n\to\infty}\mu(m,n)=\frac{1}{2}\quad\text{ and }\quad\mu(m,n)\ge\frac{(5-4)(5+2)}{2(5)(5+1)}=\frac{7}{60}\equiv c.$$

We check that all the conditions of Corollaries 8–12, Theorems 13 and 14 (the case $\beta\ge\delta,\ \delta\le 1$) are satisfied. Hence, all solutions of (24) are oscillatory. One such solution is given by $\{y(m,n)\}=\left\{(-1)^m\frac{1}{n+1}\right\}$.

Example 18. Consider the partial difference equation

$$y(m-1,n)+\frac{2m+1}{m}\,y(m,n-1)-\frac{3}{2}\,y(m,n)+\frac{(3m-2)(m+3)}{2m(m-1)}\,y(m+3,n+4)$$
$$+\frac{(2m+1)(m-2)(m+1)}{2m^2(m-1)}\,y(m+1,n+2)=0,\quad m\ge 3,\ n\ge 1. \tag{25}$$

In this example, $\tau=2,\ k_1=3,\ \ell_1=4,\ k_2=1,\ \ell_2=2,$

$$\beta(m,n)=\frac{2m+1}{m}\ge 2\equiv\beta \quad\text{ and }\quad \delta(m,n)=\frac{3}{2}\equiv\delta.$$

Taking $f_1(u)=f_2(u)=u$, we have $\gamma_1=\gamma_2=1$. Let

$$p_1(m,n)=p_1'(m,n)=\frac{(3m-2)(m+3)}{2m(m-1)},$$
$$p_2(m,n)=p_2'(m,n)=\frac{(2m+1)(m-2)(m+1)}{2m^2(m-1)},$$
$$q_i(m,n)=q_i'(m,n)=0,\ i=1,2.$$

Then,

$$p_1=\frac{3}{2},\qquad p_2=1,\qquad q_1'=q_2'=0.$$

It can be easily computed that the right side of (20) is more than 1.

Further, condition (21) also holds as

$$\limsup_{m,n\to\infty}\sum_{s=1}^{2}\sum_{i=m-1}^{m}\sum_{j=n-2}^{n}[p_s(i,j)-q_s'(i,j)]\frac{1}{\delta^{i+k_s-m}}\left(\frac{\beta}{\delta}\right)^{j+\ell_s-n}$$
$$=\limsup_{m,n\to\infty}\sum_{i=m-1}^{m}\sum_{j=n-2}^{n}\frac{(3i-2)(i+3)}{2i(i-1)}\ \frac{1}{\delta^{i+3-m}}\left(\frac{\beta}{\delta}\right)^{j+4-n}$$
$$+\limsup_{m,n\to\infty}\sum_{i=m-1}^{m}\sum_{j=n-2}^{n}\frac{(2i+1)(i-2)(i+1)}{2i^2(i-1)}\ \frac{1}{\delta^{i+1-m}}\left(\frac{\beta}{\delta}\right)^{j+2-n}$$
$$=15.0>w=\frac{9}{4}\qquad(\text{the case }\beta\ge\delta,\ \delta\ge 1).$$

Hence, by Theorems 13 and 14 equation (25) is oscillatory. In fact, one such solution is given by $\{y(m,n)\}=\left\{(-1)^n\frac{1}{m}\right\}$.

Example 19. Consider the partial difference equation

$$y(m-1,n)+\frac{n-1}{n}\,y(m,n-1)-\frac{n+1}{n}\,y(m,n)+\frac{n+1}{2n}\,y(m+2,n+1)$$
$$+\frac{1}{2}\,y(m+4,n+4)=0,\quad m\ge 1,\ n\ge 3. \tag{26}$$

We have

$$\beta(m,n) = \frac{n-1}{n} \geq \frac{2}{3} \equiv \beta \quad \text{and} \quad \delta(m,n) = \frac{n+1}{n} \leq \frac{4}{3} \equiv \delta.$$

By letting $f_1(u) = f_2(u) = u$ and

$$p_1(m,n) = p_1'(m,n) = \frac{n+1}{2n}, \quad p_2(m,n) = p_2'(m,n) = \frac{1}{2},$$

$$q_i(m,n) = q_i'(m,n) = 0,\ i = 1,2,$$

we check that the hypotheses of Theorem 13 are satisfied. Therefore, all solutions of equation (26) are oscillatory. In fact, (26) has an oscillatory solution given by $\{y(m,n)\} = \{(-1)^m(n+1)\}$.

It is, however, noted that this example does not fulfill the condition of Theorem 14 (the case $\beta \leq \delta,\ \delta \geq 1,\ \delta - \beta \leq 1$). This illustrates well the difference in nature of the criteria developed.

References

[1] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, 1992.

[2] R. P. Agarwal, Difference equations and inequalities: A survey, In Proceedings of the First World Congress on Nonlinear Analysts 1992, ed. V. Lakshmikantham, Walter de Gruyter and Co., 1091–1108.

[3] R. P. Agarwal, M. Maria Susai Manuel and E. Thandapani, Oscillatory and nonoscillatory behavior of second order neutral delay difference equations, *Mathl. Comput. Modelling*, **24** (1), (1996), 5–11.

[4] R. P. Agarwal, S. Pandian and E. Thandapani, Oscillatory property for second order nonlinear difference equations via Lyapunov second method, *Advances in Nonlinear Dynamics*, to appear.

[5] R. P. Agarwal, E. Thandapani and P. J. Y. Wong, Oscillations of higher order neutral difference equations, *Applied Mathematics Letters*, **10** (1), (1997), 71–78.

[6] S. S. Cheng and W. T. Patula, An existence theorem for a nonlinear difference equation, *Nonlinear Analysis*, **20**, (1993), 193–203.

[7] S. S. Cheng and B. G. Zhang, Qualitative theory of partial difference equations (I): Oscillation of nonlinear partial difference equations, *Tamkang J. Math.*, **25**, (1994), 279–288.

[8] S. S. Cheng, S. L. Xie and B. G. Zhang, Qualitative theory of partial difference equations (II): Oscillation criteria for direct control systems in several variables, *Tamkang J. Math.*, **26**, (1995), 65–79.

[9] S. S. Cheng, S. L. Xie and B. G. Zhang, Qualitative theory of partial difference equations (III): Forced oscillations of parabolic type partial difference equations, *Tamkang J. Math.*, **26**, (1995), 177–192.

[10] S. S. Cheng, S. L. Xie and B. G. Zhang, Qualitative theory of partial difference equations (IV): Forced oscillations of hyperbolic type nonlinear partial difference equations, *Tamkang J. Math.*, **26**, (1995), 337–360.

[11] S. S. Cheng, S. L. Xie and B. G. Zhang, Qualitative theory of partial difference equations (V): Sturmian theorems for a class of partial difference equations, *Tamkang J. Math.*, **27**, (1996), 89–97.

[12] S.-C. Fu and L.-Y. Tsai, Oscillation in nonlinear difference equations, In Advances in Difference Equations II, *Computers Math. Appl.*, to appear.
[13] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.
[14] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, New York, 1991.
[15] V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer, Dordrecht, 1993.
[16] V. Lakshmikantham and D. Trigiante, *Difference Equations with Applications to Numerical Analysis*, Academic Press, New York, 1988.
[17] J. J. Li and C. C. Yeh, Existence of positive nondecreasing solutions of nonlinear difference equations, *Nonlinear Analysis*, **22**, (1994), 1271–1284.
[18] X. P. Li, Partial difference equations used in the study of molecular orbits, *Acta Chimica Sinica*, **40**, (1982), 688–698.
[19] J. Popenda, The oscillation of solutions of difference equations, *Computers Math. Appl.*, **28**, (1994), 271–279.
[20] A. N. Sharkovsky, Y. L. Maistrenko and E. Y. Romanenko, *Difference Equations and Their Applications*, Kluwer, Dordrecht, 1993.
[21] J. C. Strikwerda, *Finite Difference Schemes and Partial Differential Equations*, Wadsworth and Brooks, California, 1989.
[22] E. Thandapani, Oscillation theorems for perturbed nonlinear second order difference equations, *Computers Math. Appl.*, **28**, (1994), 309–316.
[23] P. J. Y. Wong and R. P. Agarwal, Oscillation theorems and existence of positive monotone solutions for second order nonlinear difference equations, *Mathl. Comput. Modelling*, **21**, (1995), 63–84.
[24] P. J. Y. Wong and R. P. Agarwal, Oscillation theorems for certain second order nonlinear difference equations, *J. Math. Anal. Appl.*, **204**, (1996), 813–829.
[25] P. J. Y. Wong and R. P. Agarwal, Oscillation and monotone solutions of second order quasilinear difference equations, *Funkcialaj Ekvacioj*, **39**, (1996), 491–517.
[26] P. J. Y. Wong and R. P. Agarwal, On the oscillation of an mth order perturbed nonlinear difference equation, *Archivum Mathematicum*, **32**, (1996), 13–27.
[27] P. J. Y. Wong and R. P. Agarwal, Oscillation and nonoscillation of half-linear difference equations generated by deviating arguments, In Advances in Difference Equations II, *Computers Math. Appl.*, to appear.
[28] P. J. Y. Wong and R. P. Agarwal, Summation averages and the oscillation of second order nonlinear difference equations, *Mathl. Comput. Modelling*, **24** (9), (1996), 21–35.
[29] P. J. Y. Wong and R. P. Agarwal, Comparison theorems for the oscillation of higher order difference equations with deviating arguments, *Mathl. Comput. Modelling*, **24** (12), (1996), 39–48.
[30] P. J. Y. Wong and R. P. Agarwal, Oscillation criteria for nonlinear partial difference equations with delays, *Computers Math. Appl.*, **32** (6), 57–86.
[31] P. J. Y. Wong and R. P. Agarwal, Nonexistence of unbounded nonoscillatory solutions of partial difference equations, *J. Math. Anal. Appl.*, **214**, (1997), 503–523.
[32] P. J. Y. Wong and R. P. Agarwal, On the oscillation of partial difference equations generated by deviating arguments, *Acta Mathematical Hungarica*, to appear.
[33] B. G. Zhang and S. T. Liu, Oscillation of partial difference equations, *PanAmerican Math. J.*, **5**, (1995), 61–70.
[34] B. G. Zhang and S. T. Liu, Oscillation of partial difference equations with variable coefficients, In Advances in Difference Equations II, *Computers Math. Appl.*, to appear.
[35] B. G. Zhang, S. T. Liu and S. S. Cheng, Oscillation of a class of delay partial difference equations, *J. Difference Eqns. Applic.*, **1**, (1995), 215–226.