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On the Symmetric Solutions to a Class of Nonlinear PDEs

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Abstract. The existence and uniqueness of symmetric solutions to the boundary value problem of nonlinear partial differential equations are established. The Dirichlet boundary condition is given on the ball in R^N.

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1 Introduction

We consider the following boundary value problem

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)^{\overset{*}{p}} + f(u, \left|\operatorname{grad}_p u\right|_p) = 0 \qquad \text{in} \qquad B_p, \tag{1}$$

$$u = 0 \qquad \text{on} \qquad \partial B_p, \tag{2}$$

where $0 < p < \infty$ and the function $u^{\overset{*}{p}}$ is defined as follows:

$$u^{\overset{*}{p}} = |u|^{p-1}\, u,$$

and the domain $B_p \in R^N$ is an open unit "ball" centered at the origin and ∂B_p means the boundary of the domain $B_{p.}$ In (1) $\operatorname{grad}_p u$ denotes the expression

$$\operatorname{grad}_p u = (u_{x_1}^{\overset{*}{p}}, u_{x_2}^{\overset{*}{p}}, .., u_{x_N}^{\overset{*}{p}}), \qquad u = u(x_1, x_2, \ldots, x_N)$$

and $\quad \left|(x_1, x_2, \ldots, x_N)\right|_p = \left(\sum\limits_{i=1}^{N} |x_i|^{\frac{1}{p}+1}\right)^{\frac{p}{p+1}}.$

If $p = 1$, the operator $\sum\limits_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right)^{\overset{*}{p}}$ in the equation (1) is reduced to Δu.

This is the final form of the paper.

For the problem (1)–(2) we shall define the distance ρ between the point and the origin in R^N as follows:

$$\rho^{\frac{1}{p}+1} = \sum_{i=1}^{N} |x_i|^{\frac{1}{p}+1} . \tag{3}$$

In the case $\rho = 1$ the equation (3) gives the equation of the unit "ball" B_p in R^N. We mention that the curve $\rho = 1$ in R^2 is a central symmetric convex curve which plays the same role in the case of nonlinear differential equation (1) as the unit circle in the case of linear ($p = 1$) partial differential equation.

For the "ball" B_p we introduce now instead of rectangular coordinates $x_1, x_2, x_3, \dots, x_N$ a new type of polar coordinates $\rho, \varphi_1, \dots, \varphi_{N-1}$ as follows

$$\begin{aligned} x_1 &= \rho \prod_{i=1}^{N-1} [S'(\varphi_i)] , \\ x_k &= \rho \, [S(\varphi_{k-1})] \prod_{i=k}^{N-1} [S'(\varphi_i)] \qquad \text{if} \qquad 1 < k \le N, \end{aligned} \tag{4}$$

where $S = S(\varphi_i)$, $1 < i \le N-1$ is the generalized sine function given by Á. Elbert [6]. The Pythagorean relation for this generalized sine function has the form

$$|S|^{\frac{1}{p+1}} + |S'|^{\frac{1}{p+1}} = 1, \qquad \text{where} \qquad S' = \frac{dS(\varphi)}{d\varphi}.$$

The unit "ball" B_p in R^N is defined by

$$B_p = \Big\{ (x_1, x_2, \dots, x_N) : \sum_{i=1}^{N} |x_i|^{\frac{1}{p}+1} \le 1 \Big\}, \qquad 0 < p < \infty.$$

When we study the radially symmetric solution $u(x) = \nu(\rho)$ of the nonlinear boundary value problem (1)–(2) the nonlinear partial differential equation (1) is reduced to the following nonlinear ordinary differential equation (5)

$$\frac{\partial}{\partial \rho} \left(\frac{\partial \nu}{\partial \rho} \right)^{\overset{*}{p}} + \frac{N-1}{\rho} \left(\frac{\partial \nu}{\partial \rho} \right)^{\overset{*}{p}} + \overline{f}(\nu, |\nu'|) = 0, \qquad \rho \in (0,1) , \tag{5}$$

where $f(u, \left|\operatorname{grad}_p u\right|_p) = \overline{f}(\nu, |\nu'|)$ since $\left|\operatorname{grad}_p u\right|_p = |\nu'|^{\frac{p^2}{p+1}}$. We note that the equation (5) can be written also in the form

$$(\rho^{N-1} \nu'^{\overset{*}{p}}(\rho))' + \rho^{N-1} \overline{f}(\nu, |\nu'|) = 0, \qquad \rho \in (0,1) .$$

Instead of the boundary condition (2) we shall consider the conditions

$$\nu(1) = 0, \tag{6}$$

$$\nu'(0) = 0. \tag{7}$$

Now let us take the boundary value problem of another nonlinear partial differential equation instead of (1)–(2)

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[|\nabla u|^{p-1} \nabla u \right] + f(u, |\mathrm{grad}\, u|) = 0 \qquad \text{in} \qquad B, \tag{8}$$

$$u = 0 \qquad \text{on} \qquad \partial B, \tag{9}$$

where the unit ball B in R^N is defined by

$$B = \left\{ (x_1, x_2, \ldots, x_N) : \sum_{i=1}^{N} x_i^2 \leq 1 \right\},$$

as it is usual in the Euclidean metric and $|\mathrm{grad}\, u| = \left(\sum_{i=1}^{N} u_{x_i}^2 \right)^{\frac{1}{2}}$ $(p = 1)$. The expression

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[|\nabla u|^{p-1} \nabla u \right]$$

in (8) is used to call p-Laplacian.This operator appears in many contexts in physics: non-Newtonian fluids, reaction-diffusion problems, non-linear elasticity, and glaceology, just to mention a few applications(see [3], [9], [10], [11], [12]). If $p = 1$ the equation (8) is also reduced to the semilinear problem

$$\Delta u + f(u, |\mathrm{grad}\, u|) = 0,$$

the existence of these problems are investigated in [1], [2], [7].The radially symmetric solutions of the Dirichlet problem of

$$\Delta u + f(u) = 0$$

were examined by Gidas, Wei-Ming Ni, and Nirenberg for the ball B [8]. If $p > 0$ then, applying the usual spherical transformation, the radially symmetric solutions of equation (8) has to satisfy formally the same equation as (5). So, if we examine the solutions of (5) we get results on the radially symmetric solutions both for the nonlinear partial differential equation (1) in the "ball" B_p and also for the nonlinear partial differential equation (8) in the Euclidean ball B. Here our aim is to show existence and uniqueness results of symmetric solutions for the problem

$$\frac{\partial}{\partial \rho} \left(\frac{\partial \nu}{\partial \rho} \right)^{\overset{*}{p}} + \frac{N-1}{\rho} \left(\frac{\partial \nu}{\partial \rho} \right)^{\overset{*}{p}} + e^{\lambda \nu + \kappa |\nu'|} = 0, \qquad \rho \in (0,1)\,,$$

$$\nu(1) = 0, \qquad \nu'(0) = 0,$$

where λ, κ are negative real numbers. In the case $p = 1$ the existence and uniqueness results of the problem

$$\frac{\partial}{\partial\rho}\left(\frac{\partial\nu}{\partial\rho}\right) + \frac{N-1}{\rho}\left(\frac{\partial\nu}{\partial\rho}\right) + e^{\lambda\nu+\kappa|\nu'|} = 0, \qquad \rho \in (0,1),$$
$$\nu(1) = 0, \qquad \nu'(0) = 0,$$

are established in [4].

2 Results

Let us consider the following boundary value problem

$$\frac{\partial}{\partial\rho}\left(\frac{\partial\nu}{\partial\rho}\right)^{\overset{*}{p}} + \frac{N-1}{\rho}\left(\frac{\partial\nu}{\partial\rho}\right)^{\overset{*}{p}} + e^{\lambda\nu+\kappa|\nu'|} = 0, \qquad \rho \in (0,1)$$
$$\nu(1) = a, \qquad a \in R^+, \qquad \nu'(0) = 0. \tag{10}$$

We shall say that the function is the positive solution of problem (10) if

i) $\nu(\rho)$ is continuous on $[0,1]$ and $\nu(\rho) > 0$ in the interval $(0,1]$;

ii) $\nu'(\rho)$ exists and is continuous, moreover $\nu'(\rho) \le 0$ in the interval $[0,1]$;

iii) $\nu(\rho)$ satisfies the boundary conditions: $\nu(1) = a$, for $a \ge 0$, $\nu'(0) = 0$;

iv) $\nu''(\rho)$ exists almost everywhere and locally Lebesgue integrable in the interval $[0,1]$;

v) $\nu(\rho)$ satisfies the differential equation

$$\frac{\partial}{\partial\rho}\left(\frac{\partial\nu}{\partial\rho}\right)^{\overset{*}{p}} + \frac{N-1}{\rho}\left(\frac{\partial\nu}{\partial\rho}\right)^{\overset{*}{p}} + e^{\lambda\nu+\kappa|\nu'|} = 0, \ \ \rho \in (0,1).$$

Theorem 1. *If $a \ge 0$ then the boundary value problem (10) has at most one positive radial solution.*

Proof. Let us denote by $\nu_1(\rho)$ and $\nu_2(\rho)$ two different positive solutions to the boundary value problem (10). Without loss of generality we may suppose, that there exists a point $\rho = \gamma$, $\gamma \in [0,1)$ such that $\nu_1(\rho) \ge \nu_2(\rho)$. If $\nu_1(\rho) - \nu_2(\rho) < 0$ in the interval $[0,1)$ then we change the notations of $\nu_1(\rho)$ and $\nu_2(\rho)$ for the opposite. Let us denote by $\delta \in (\gamma, 1]$, the first zero of the function $\nu_1(\rho) - \nu_2(\rho)$ which lays to the right from γ. By the Lagrange's theorem there exists $\beta \in (\gamma, \delta)$ for which $\nu_1'(\beta) - \nu_2'(\beta) < 0$ and $\nu_1(\beta) - \nu_2(\beta) > 0$. We shall denote by $\alpha \in [0,\beta)$ the zero of the function $\nu_1'(\rho) - \nu_2'(\rho)$. If there are more zeroes $\alpha_1, \alpha_2, \ldots, \alpha_k$ of $\nu_1'(\rho) - \nu_2'(\rho)$ in the interval $[0,\beta)$ then let us take the notation $\alpha = \max(\alpha_1, \alpha_2, \ldots, \alpha_k)$.

In this case we can summarize that $\nu_1(\rho) - \nu_2(\rho) > 0$ and $\nu_1'(\rho) - \nu_2'(\rho) < 0$, $\rho \in (\alpha, \beta]$, $\nu_1'(\alpha) - \nu_2'(\alpha) = 0$.

Since the functions $\nu_1(\rho)$ and $\nu_2(\rho)$ satisfy the nonlinear differential equation in (10) therefore substituting them into the differential equation and subtracting the two equations we get the equation

$$[\rho^{N-1}(\nu_1'^{\overset{*}{p}} - \nu_2'^{\overset{*}{p}})]' + \rho^{N-1}[e^{\lambda\nu_1+\kappa|\nu_1'|} - e^{\lambda\nu_2+\kappa|\nu_2'|}] = 0. \tag{11}$$

We introduce the following notations

$$J(\rho) = \nu_1^{\overset{*}{p}}(\rho) - \nu_2^{\overset{*}{p}}(\rho)\,,$$
$$K(\rho) = \nu_1'^{\overset{*}{p}}(\rho) - \nu_2'^{\overset{*}{p}}(\rho)\,,$$

moreover $J(\rho)$ and $K(\rho)$ have the properties

$$\begin{array}{cc} J(1) = 0, & K(0) = 0, \\ J(\gamma) > 0, & K(\beta) < 0, \\ J(\rho) > 0,\ \rho \in (\alpha, \beta], & K(\alpha) = 0, \\ & K(\rho) < 0,\ \rho \in (\alpha, \beta]. \end{array} \tag{12}$$

Rearranging the differential equation (11) we obtain

$$\left[\rho^{N-1}K(\rho)\right]' + \rho^{N-1}K(\rho)A(\rho) - \rho^{N-1}J(\rho)B(\rho) = 0\,,$$

where the expressions $A(\rho)$ and $B(\rho)$ have the forms

$$A(\rho) = \frac{e^{\lambda\nu_1+\kappa|\nu_1'|} - e^{\lambda\nu_1+\kappa|\nu_2'|}}{\nu_1'^{\overset{*}{p}}(\rho) - \nu_2'^{\overset{*}{p}}(\rho)},$$
$$B(\rho) = \frac{e^{\lambda\nu_2+\kappa|\nu_2'|} - e^{\lambda\nu_1+\kappa|\nu_2'|}}{\nu_1^{\overset{*}{p}}(\rho) - \nu_2^{\overset{*}{p}}(\rho)}.$$

Using the properties of the function $e^{\lambda\nu+\kappa|\nu'|}$ we get that $A(\rho) \geq 0$ and $B(\rho) \geq 0$ when $\rho \in (\alpha, \beta]$. Thus from the equation (11) we obtain the inequality

$$\left[\rho^{N-1}K(\rho)\right]' + \rho^{N-1}K(\rho)\frac{e^{\lambda\nu_1+\kappa|\nu_1'|} - e^{\lambda\nu_1+\kappa|\nu_2'|}}{\nu_1'^{\overset{*}{p}}(\rho) - \nu_2'^{\overset{*}{p}}(\rho)} \geq 0\,, \qquad \rho \in (\alpha, \beta]. \tag{13}$$

If we multiply the inequality in (13) by the expression

$$\exp\left\{-\int_\rho^b eta\frac{e^{\lambda\nu_1+\kappa|\nu_1'|} - e^{\lambda\nu_1+\kappa|\nu_2'|}}{\nu_1'^{\overset{*}{p}}(\tau) - \nu_2'^{\overset{*}{p}}(\tau)}d\tau\right\},$$

and take the integral on the interval $[\delta, \beta]$ where $\delta \in (\alpha, \beta)$ we get the inequality

$$\beta^{N-1}K(\beta) - \delta^{N-1}K(\delta)\exp\left\{-\int_\delta^\gamma \frac{e^{\lambda\nu_1+\kappa|\nu_1'|} - e^{\lambda\nu_1+\kappa|\nu_2'|}}{\nu_1'^{\overset{*}{p}}(\tau) - \nu_2'^{\overset{*}{p}}(\tau)}d\tau\right\} \geq 0.$$

If we take $\delta \to \alpha$ then we get that

$$K(\beta) \geq 0,$$

since $K(\alpha) = 0$. This is contradiction with (12).

In the next theorem we establish the existence result:

Theorem 2. *If $a \geq 0$ then the boundary value problem (10) has a unique positive solution.*

In the following we need some subsidiary statements.

Lemma 3. *If $a \geq 0$ then there is a positive solution to problem (10).*

Proof. Let us define the mappings

$$(\Phi\mu)(t) = a + \int_t^1 \mu(\tau)d\tau,$$

$$(\Psi\mu)(t) = \left[\int_0^t \left(\frac{\tau}{t}\right)^{N-1} e^{\lambda(\Phi\mu)(\tau)+\kappa\mu(\tau,a)} d\tau\right]^{\frac{1}{p}},$$

$$H = \left\{\mu(\tau,a) \in C[0,1),\ 0 \leq \mu(\tau,a) \leq \left(\frac{e^{\lambda a}}{N}\right)^{\frac{1}{p}},\ t \in (0,1),\ \mu(0,a) = 0\right\}.$$

The functions which belong to the set ΦH are uniformly bounded and equicontinuous functions therefore H is compact. Since every Cauchy sequence being contained in the set H converges in H then H is closed. Thus the set H is bounded, convex, closed and compact in the Banach space $C[0,1)$.

The mapping Ψ is a continuous mapping from H to H. Applying the Schauder fixed point theorem the mapping Ψ has a fixed point.

Using notation $\mu(\rho,a) = -\nu'(\rho,a)$ the positive solution to problem (10) has the form

$$\nu(t,a) = a + \int_t^1 \mu(\tau)d\tau = a + \int_t^1 \left[\int_0^\tau \left(\frac{\rho}{\tau}\right)^{N-1} e^{\lambda\nu(\rho,a)+\kappa\mu(\rho,a)} d\rho\right]^{\frac{1}{p}} d\tau. \quad (14)$$

Lemma 4. *Let $\nu(t,a)$ be the unique positive solution to the problem (10). If $0 \leq a_2 < a_1$, then $\nu(t,a_1) \geq \nu(t,a_2)$ and $\nu'(t,a_1) \geq \nu'(t,a_2)$ for all $t \in [0,1)$.*

Proof. Let $\nu(t,a_1)$ and $\nu(t,a_2)$ be the unique positive solution to the problem (10) for $a_1$and a_2, respectively. Let us take the notation

$$j(t) = \nu^{\overset{*}{p}}(t,a_1) - \nu^{\overset{*}{p}}(t,a_2) \text{ and } k(t) = \nu'^{\overset{*}{p}}(t,a_1) - \nu'^{\overset{*}{p}}(t,a_2).$$

Clearly $j(1) = a_1^{\overset{*}{p}} - a_2^{\overset{*}{p}} > 0$ and $k(0) = 0$. Hence there exists at least one point $t = \alpha,\quad \alpha \in [0,1)$ such that $k(\alpha) = 0$ and $j(t) > 0$ in the interval $(\alpha, 1]$. If there are more values of α $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ for which $k(\alpha) = 0$ in the interval $[0, 1)$ then let us take the notation $\alpha = \max(\alpha_1, \alpha_2, \ldots, \alpha_k)$. We may assume that there exists a point $\beta \in [\alpha, 1)$ where $\nu(t, a_1) > \nu(t, a_2)$ and $\nu'(t, a_1) < \nu'(t, a_2)$, that is $j(t) > 0$, and $k(t) < 0$ in the interval $(\alpha, \beta]$. In an analogous way as in the proof of Theorem 1 one can obtain that $k(\beta) \geq 0$. It is contradiction since we supposed that $k(\beta) < 0$.

The inequality $\nu(t, a_1) \geq \nu(t, a_2)$ we get in a similar way as in the proof of Theorem 1.

Proof of Theorem 2. When $a \to 0$ we get that $\nu(t, a)$ and $\nu'(t, a)$ converges uniformly to $\nu(t, 0)$ and $\nu'(t, 0)$ in the interval $[0, 1]$, respectively. Taking $a \to 0$ in the expression (14) we shall get the positive solution to the problem

$$\frac{\partial}{\partial \rho}\left(\frac{\partial \nu}{\partial \rho}\right)^{\overset{*}{p}} + \frac{N-1}{\rho}\left(\frac{\partial \nu}{\partial \rho}\right)^{\overset{*}{p}} + e^{\lambda\nu + \kappa|\nu'|} = 0, \qquad \rho \in (0, 1),$$
$$\nu(1) = 0, \qquad \nu'(0) = 0,$$

in the following form

$$\nu(t, 0) = \int_t^1 \left[\int_0^\tau \left(\frac{\rho}{\tau}\right)^{N-1} e^{\lambda\nu(\rho,0) - \kappa\nu'(\rho,0)} d\rho\right]^{\frac{1}{p}} d\tau.$$

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